International Journal of Applied Mathematics

Volume 30 No. 6, 2017

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version

OPERATOR-VALUED MULTIPLICATION OPERATORS ON WEIGHTED BANACH FUNCTION SPACES

Ajay Kumar Sharma

Department of Mathematics, Government Degree College, Udhampur-182101, Jammu and Kashmir, India

Abstract:

In this paper, we study multiplication operators induced by strongly measurable operatorvalued maps on weighted Banach function spaces. We characterize the boundedness, invertibility, closed range, and compactness of such multiplication operators. We also investigate the Fredholm properties of these operators. The results are established in the general setting of vector-valued weighted Banach function spaces with absolutely continuous norms.

2010 Mathematics Subject Classification: Primary 47B38; Secondary 46E30.

Keywords: Multiplication operators, weighted Banach function spaces, compact operators, Fredholm operators, composition operators.

1 Introduction

Let (Ω,Σ,μ) be a σ-finite measure space. Let $w : \Omega \to (0,\infty)$ be a measurable weight function which is locally integrable with respect to μ. We say that a normed linear space $X = (X,w)$ is a weighted Banach function space over (Ω,Σ,μ) if:

(P1) the norm $\|f\|_X = \|f\|_{X,w}$ is defined for every μ-measurable function f on Ω, and $f \in X$ if and only if $\|f\|_X < \infty$; $\|f\|_X = 0$ if and only if $f = 0$ μ-a.e.;

(P2) $\|f\|_X = \||f|\|_X$ for all $f \in X$;

(P3) if $0 \le f \le g$ μ-a.e., then $\|f\|_X \le \|g\|_X$;

(P4) if $0 \le f_n \uparrow f$ μ-a.e., then $\|f_n\|_X \uparrow \|f\|_X$;

(P5) if $E \in \Sigma$ is a measurable subset of Ω such that

$$w(E) := \int_E w \, d\mu < \infty,$$

then $\|\chi_E\|_X < \infty$;

(P6) for all measurable sets $E \in \Sigma$ with $w(E) < \infty$, there is a constant $C_E > 0$ such that

$$\int_E fw \, d\mu \le C_E\|f\|_X$$

for all $f \in X$.

International Journal of Applied Mathematics

Volume 30 No. 6, 2017

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version

Given a Banach function space $X = (X,w)$ over (Ω,Σ,μ), its associate space $X' = (X',w)$ is given by

$$X' = \left\{ f : f \text{ is } \mu\text{-measurable on } \Omega \text{ and } \|f\|_{X'} := \sup\left\{ \int_\Omega fgw\, d\mu : \|g\|_X \le 1 \right\} < \infty \right\},$$

which is also a Banach function space in the sense of (P1)–(P6).

The spaces X and X'' are complete normed linear spaces and $X'' = X$. Holder's inequality¨

$$\int_\Omega fgw\, d\mu \le \|f\|_X \|g\|_{X'}$$

holds for all $f \in X$ and $g \in X'$ and is sharp. Since $X = X'' = (X')'$,

and

$$\|f\|_X = \sup\left\{ \int_\Omega fgw\, d\mu : \|g\|_{X'} \le 1 \right\}$$

$$\|g\|_{X'} = \sup\left\{ \int_\Omega gfw\, d\mu : \|f\|_X \le 1 \right\}$$

$f \in X$ and $g \in X'$.

for every A function f in a Banach function space X is said to have absolutely continuous norm if

$\|f\chi_{E_n}\|_{X,w} \to 0$ for each sequence $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma$ satisfying $E_n \downarrow \emptyset$ μ-a.e. If each function in X has absolutely continuous norm, then X is called a Banach function space with absolutely continuous norm.

A weighted rearrangement invariant Banach function space is a weighted Banach function space such that whenever $f \in X$ and g is equi-measurable with f, then $g \in X$ and $\|f\|_X = \|g\|_X$. For a Banach function space with absolutely continuous norm, the Banach dual coincides with its associate space. For details on Banach function spaces and rearrangement invariant Banach function spaces, one can refer to [2, 3, 6, 7, 8] and references therein.

Let B be a separable Banach space and for a strongly measurable function $f : \Omega \to B$, define a function $\|f\| : \Omega \to [0,\infty)$ as:

$$\|f\|(x) = \|f(x)\|_B \qquad \text{for all } x \in \Omega.$$

All the notations make sense for f by replacing the modulus by norm. This leads to the natural definition of the Banach function space $X(\Omega,B)$ of vector-valued functions, with norm $\|\cdot\|_X$ and the associate space $X'(\Omega,B)$ with norm $\|\cdot\|_{X'}$ given by

,

And

International Journal of Applied Mathematics

Volume 30 No. 6, 2017

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version

$$\|g\|_{X'} = \sup\left\{\int_\Omega \|f(x)\|_B\|g(x)\|_B w(x)\,d\mu(x) : \|f\|_X \leq 1\right\}, \quad g \in X'$$

$$\|f\|_X = \sup\left\{\int_\Omega \|f(x)\|_B\|g(x)\|_B w(x)\,d\mu(x) : \|g\|_{X'} \leq 1\right\}, \quad f \in X.$$

The Banach function space $X(\Omega,B)$ is a Banach space and for a Banach function space with absolutely continuous norm, the Banach dual coincides with its associate space, that is,

$$X^*(\Omega,B) = X'(\Omega,B^*),$$

provided B has the Radon-Nikodym property.´

For a strongly measurable function $\theta : \Omega \to L(B)$, the class of all bounded linear operators on a Banach space B, the multiplication transformation $M_\theta : X(\Omega,B) \to X(\Omega,B)$ is defined as

$$(M_\theta f)(x) = \theta(x)f(x),$$

for all $x \in \Omega$.

Let $T : \Omega \to \Omega$ be a measurable transformation inducing a composition operator C_T on

$X(\Omega,B)$ defined by

$$(C_T f)(x) = f(T(x)),$$

for all $x \in \Omega$.

The composite multiplication operator is defined as

$$(M_\theta C_T f)(x) = \theta(x)f(T(x)),$$

for all $x \in \Omega$.

2 Operator-ValuedMultiplicationOperatorsonWeightedBanach Function Spaces

In this section, we study multiplication operators with some of their properties like boundedness, invertibility, range, and compactness. Let θ be a strongly measurable operator-valued map on Ω. Then the boundedness, invertibility, and compactness of the multiplication operator M_θ induced by θ are characterized in terms of θ.

Theorem 2.1. *The multiplication transformation* $M_\theta : X(\Omega,B) \to X(\Omega,B)$ *is bounded if and only if* $\theta \in L^\infty(\Omega,L(B))$. *Moreover,*

$$\|M\theta\| = \|\theta\|L\infty(\Omega,L(B)) = \text{esssup}_{x\in\Omega}\ \|\theta(x)\|L(B),$$

where $\|\theta(x)\|_{L(B)}$ *is the operator norm of* $\theta(x)$ *in* $L(B)$ *induced by* $\|\cdot\|_B$.

Proof. Assume that $\theta \in L^\infty(\Omega,L(B))$. For $f \in X(\Omega,B)$, we have

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version

$$\begin{aligned}\|M_\theta f\|_X &= \sup\left\{\int_\Omega \|M_\theta f(x)\|_B\|g(x)\|_B w(x)\,d\mu(x) : \|g\|_{X'} \le 1,\ g\in X'(\Omega,B)\right\}\\ &= \sup\left\{\int_\Omega \|\theta(x)f(x)\|_B\|g(x)\|_B w(x)\,d\mu(x) : \|g\|_{X'} \le 1\right\}\\ &\le \|\theta\|_{L^\infty(\Omega,L(B))}\sup\left\{\int_\Omega \|f(x)\|_B\|g(x)\|_B w(x)\,d\mu(x) : \|g\|_{X'} \le 1\right\}\\ &= \|\theta\|_{L^\infty(\Omega,L(B))}\|f\|_X.\end{aligned}$$

Consequently, M_θ is a bounded operator on $X(\Omega,B)$ and $\|M_\theta\| \le \|\theta\|_{L^\infty(\Omega,L(B))}$.

Conversely, suppose that M_θ is a bounded operator on $X(\Omega,B)$. In case θ is not in $L^\infty(\Omega,L(B))$, then we can find a sequence $\{E_{ni}\}$ of disjoint measurable subsets with finite measure such that $n_1 = 1$, $n_i \to \infty$ as $i \to \infty$, and

$$\|\theta(x)\|_{L(B)} > n_i \qquad \text{for all } x \in E_{ni}.$$

Since B is separable, choose a countable dense subset $\{b_j\}_{j=1}^\infty$ of the unit sphere of B. For each i, since $\|\theta(x)\|_{L(B)} > n_i$ on E_{ni}, there exists j_i such that $\|\theta(x)b_{ji}\|_B > n_i/2$ on a subset of positive measure. For each i, define f_i as

$$f_i(x) = \begin{cases} b_{j_i}, & x \in F_i,\\ 0, & \text{otherwise,}\end{cases}$$

where $F_i \subset E_{ni}$ has positive finite measure.

Then each $f_i \in X(\Omega,B)$ and $\|M_\theta f_i\|_X \ge (n_i/2)\|f_i\|_X$. This contradicts the boundedness of M_θ on $X(\Omega,B)$. Moreover, one can easily find that $\|M_\theta\| = \|\theta\|_{L^\infty(\Omega,L(B))}$. □

Theorem 2.2. *Let M_θ be a bounded multiplication operator on $X(\Omega,B)$. Then M_θ is bounded below if and only if there exists $\delta > 0$ such that*

$$\|\theta(x)t\|_B \ge \delta\|t\|_B$$

for all $t \in B$ and for almost all $x \in \Omega$.

Proof. Suppose there exists $\delta > 0$ such that $\|\theta(x)t\|_B \ge \delta\|t\|_B$ for all $t \in B$ and for almost all $x \in \Omega$. Then for each $f \in X(\Omega,B)$,

$$\|M_\theta f\|_X = \|\theta f\|_X \ge \delta\|f\|_X.$$

Thus, M_θ is bounded below.

Conversely, assume that M_θ is bounded below. Then there exists $\delta > 0$ such that

$$\|M_\theta f\|_X \ge \delta\|f\|_X$$

for all $f \in X(\Omega,B)$. Suppose there exists a set $E \in \Sigma$ of positive measure such that for every $x \in E$, there exists $t_x \in B$ with $\|t_x\|_B = 1$ and $\|\theta(x)t_x\|_B < \delta/2$. Since B is separable, choose a countable dense subset $\{b_j\}_{j=1}^\infty$ of the unit sphere. Then $E = \bigcup_{j=1}^\infty E_j$, where

Received: November 15, 2017

International Journal of Applied Mathematics

Volume 30 No. 6, 2017

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version

$$E_j = \{x \in E : \|\theta(x)b_j\|_B < \delta/2\}.$$

Some E_j has positive measure. Choose $F \subset E_j$ with $0 < \mu(F) < \infty$ and define $f_F(x) = b_j$ on F, 0 otherwise. Then $f_F \in X(\Omega,B)$ and $\|M_\theta f_F\|_X < \delta\|f_F\|_X$, a contradiction. Therefore, $\|\theta(x)t\|_B \geq \delta\|t\|_B$ for almost all $x \in \Omega$ and for all $t \in B$. □

Theorem 2.3. *Let* (Ω,Σ,μ) *be a σ-finite measure space and let B be separable. Then* $M_\theta : X(\Omega,B) \to X(\Omega,B)$ *is invertible with inverse* M_ψ *for some* $\psi \in L^\infty(\Omega,L(B))$ *if and only if:*

(i) $\theta(x)$ *is invertible in* $L(B)$ *for μ-almost all* $x \in \Omega$, *and*

(ii) $\operatorname{esssup}_{x\in\Omega} \|\theta(x)^{-1}\|_{L(B)} < \infty$.

Proof. Suppose that M_θ is invertible and $M_\theta^{-1} = M_\psi$ for some $\psi \in L^\infty(\Omega,L(B))$. For each $t \in B$, define $E_t : \Omega \to B$ as

$$E_t(x) = t, \qquad \text{for all } x \in \Omega.$$

This gives us that $\|E_t\|_X = \|t\|_B\|\chi_\Omega\|_X < \infty$, and hence, $E_t \in X(\Omega,B)$. As $M_\theta^{-1} = M_\psi$, we have

$$\theta(x)\psi(x)t = \psi(x)\theta(x)t = t$$

for all $t \in B$ and for all $x \in \Omega$.

This implies that $\theta(x)$ is invertible for almost all $x \in \Omega$. Also, for each $t \in B$ and $x \in \Omega$, we get

$$\|t\|_B = \|\psi(x)\theta(x)t\|_B \leq \|\psi(x)\|_{L(B)}\|\theta(x)t\|_B \leq \|\psi\|_{L^\infty(\Omega,L(B))}\|\theta(x)t\|_B.$$

Thus,

$$\|\theta(x)t\|_B \geq \epsilon\|t\|_B,$$

with $\epsilon = \|\psi\|^{-1}_{L^\infty(\Omega,L(B))}$. This implies $\|\theta(x)^{-1}\|_{L(B)} \leq \epsilon^{-1}$ a.e., so $\operatorname{esssup}_x \|\theta(x)^{-1}\|_{L(B)} < \infty$. Conversely, assume that conditions (i) and (ii) are true. If $\psi(x)$ is the inverse of $\theta(x)$, then we can write $\theta(x)\psi(x)f(x) = \psi(x)\theta(x)f(x) = f(x)$

for all $x \in \Omega$ and $f \in X(\Omega,B)$. Therefore,

$$M_\psi M_\theta f = M_\theta M_\psi f = f$$

for all $f \in X(\Omega,B)$. By condition (ii), $\psi \in L^\infty(\Omega,L(B))$ and M_θ is invertible. □

Theorem 2.4. *Let* $M_\theta \in L(X(\Omega,B))$. *Assume that* $\theta(x)$ *is injective for μ-almost every* $x \in \Omega$. *Then* M_θ *has closed range if and only if there exists* $\epsilon > 0$ *such that*

$$\epsilon\|t\|_B \leq \|\theta(x)t\|_B$$

for all $t \in B$ *and for almost all* $x \in \Omega$.

Proof. Suppose there exists $\epsilon > 0$ such that $\|\theta(x)t\|_B \geq \epsilon\|t\|_B$ for all $t \in B$ and for almost all $x \in \Omega$. Then by Theorem 2.2, M_θ is bounded below and hence has closed range.

Conversely, assume that M_θ has closed range. Since $\theta(x)$ is injective a.e., M_θ is injective. A bounded injective operator with closed range has a bounded inverse from its range onto $X(\Omega,B)$;

International Journal of Applied Mathematics

Volume 30 No. 6, 2017

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version

hence it is bounded below. By Theorem 2.2, there exists $\epsilon > 0$ such that $\|\theta(x)t\|_B \geq \epsilon\|t\|_B$ for almost all $x \in \Omega$ and for all $t \in B$. □

For $B = \mathbb{C}^2$ and $\theta = \mathrm{diag}(1,0)$, M_θ has closed range but $m(\theta) = 0$. Thus closed range must not be identified with bounded-below behavior on the whole domain when a kernel is present.

Theorem 2.5. *Let M_θ be any bounded operator on $X(\Omega,B)$. Then M_θ is a compact operator if $X(N,\epsilon,B)$ is finite dimensional for each $\epsilon > 0$, where*

$$N = N(\theta,\epsilon) = \{x \in \Omega : \|\theta(x)t\|_B \geq \epsilon\|t\|_B \textit{ for all } t \in B\}$$

and

$$X(N,\epsilon,B) = \{f \in X(\Omega,B) : f(x) = 0 \textit{ for } x \notin N\}.$$

Proof. Assume that $X(N,\epsilon,B)$ is finite dimensional for each $\epsilon > 0$. In particular, $X(N, \frac{1}{n}, B)$ is finite dimensional for each $n \geq 1$. For each $n \in \mathbb{N}$, we define

$$\theta_n(x) = \begin{cases} \theta(x), & \text{if } x \in A_n, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$A_n = \left\{x \in \Omega : \|\theta(x)t\|_B \geq \frac{1}{n}\|t\|_B \text{ for all } t \in B\right\}.$$

Since $X(A_n, \frac{1}{n}, B)$ is finite dimensional, the range of M_{θ_n} is contained in this finite-dimensional subspace, so M_{θ_n} has finite rank and hence is compact. Then for each $f \in X(\Omega,B)$, we have

$$\begin{aligned} \|(M_{\theta_n} - M_\theta)f\|_X &= \sup\left\{\int_{\Omega\setminus A_n} \|\theta(x)f(x)\|_B\|g(x)\|_B w(x)\,d\mu(x) : \|g\|_{X'} \leq 1\right\} \\ &\leq \frac{1}{n}\|f\|_X. \end{aligned}$$

Thus $M_{\theta_n} \to M_\theta$ as $n \to \infty$ in the operator norm and so, M_θ is a compact operator. □

Theorem 2.6. *If θ is a strongly measurable operator-valued map such that for some $k > 0$, $\|\theta(x)t\|_B \geq k\|t\|_B$ for all $t \in B$ whenever $\|\theta(x)\|_{L(B)} \geq k$, then M_θ is a compact operator if and only if $X(N,\epsilon,B)$ is finite dimensional for each $\epsilon > 0$.*

Proof. Assume that M_θ is compact. Then $M_\theta|_{X(N,\epsilon,B)}$ is also a compact operator and

$$\|M_\theta \chi_{X(N,\epsilon,B)} f\|_X \geq \epsilon\|\chi_{X(N,\epsilon,B)} f\|_X$$

for each $f \in X(\Omega,B)$. Thus, $M_\theta|_{X(N,\epsilon,B)}$ is invertible on its range and being compact, we find that $X(N,\epsilon,B)$ is finite dimensional.

The converse follows from Theorem 2.6. □

Received: November 15, 2017

International Journal of Applied Mathematics

Volume 30 No. 6, 2017

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version

Theorem 2.7. *Suppose* (Ω,Σ,μ) *is a non-atomic measure space and* $M_\theta \in L(X(\Omega,B))$*, where* $X(\Omega,B)$ *is a weighted Banach function space having absolutely continuous norm and* B *is separable. Then* M_θ *is compact if and only if* $\theta = 0$ μ*-almost everywhere.*

Proof. The zero operator is compact.

Conversely, suppose θ is not zero a.e. Then there exists $\epsilon > 0$ such that the set

$$E = \{x \in \Omega : \|\theta(x)\|_{L(B)} > 2\epsilon\}$$

has positive measure. Choose a unit vector $b \in B$ and a positive-measure subset $F \subset E$ on which

$$\|\theta(x)b\|_B > \epsilon.$$

By σ-finiteness, take F of finite weighted measure. Since μ is non-atomic, F contains pairwise disjoint measurable subsets F_n of positive measure. Put

$$f_n = \frac{\chi_{F_n} b}{\|\chi_{F_n}\|_X}.$$

Then $\|f_n\|_{X(B)} = 1$ and $\|M_\theta f_n\|_{X(B)} \geq \epsilon$. For $n \neq m$, disjointness and the lattice property imply

$$\|M\theta fn - M\theta fm\|X(B) \geq \epsilon.$$

Hence the image of the unit sequence has no convergent subsequence, so M_θ is not compact.

□

Theorem 2.8. *Suppose* (Ω,Σ,μ) *is a non-atomic measure space and* $M_\theta \in L(X(\Omega,B))$*, where* $X(\Omega,B)$ *is a weighted Banach function space having absolutely continuous norm and* B *is separable. If* M_θ *is Fredholm, then* M_θ *is invertible. Consequently,*

$$\mathrm{ind}(M_\theta) = 0.$$

Proof. A Fredholm operator has finite-dimensional kernel and closed range with finite codimension.

If $\ker M_\theta$ is nonzero, choose $f \neq 0$ in the kernel. Because multiplication operators commute with scalar characteristic multipliers, $\chi_E f$ belongs to $\ker M_\theta$ for every measurable E for which $\chi_E f \in X$. Non-atomicity allows infinitely many pairwise disjoint positive-measure subsets of a set on which f is nonzero, producing infinitely many linearly independent kernel elements, a contradiction. Hence $\ker M_\theta = \{0\}$.

A similar localization argument applied to the quotient $X(\Omega,B)/\mathrm{R}(M_\theta)$ shows that a proper closed range would have infinite codimension on a non-atomic space: if $g \notin \mathrm{R}(M_\theta)$, its restrictions to infinitely many disjoint positive-measure subsets yield independent quotient classes. Therefore the finite-codimension range of a Fredholm multiplication operator must be the whole space. Thus M_θ is bijective. By the bounded inverse theorem it is invertible, and its Fredholm index is zero. □

International Journal of Applied Mathematics

Volume 30 No. 6, 2017

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version

3 Composite Multiplication Operators

In this section, we extend our study to composite multiplication operators, which arise naturally in the study of dynamical systems and weighted composition operators.

Theorem 3.1. *Let $M_\theta C_T$ be a composite multiplication operator on $X(\Omega,B)$ with X having absolutely continuous norm. Suppose $T : \Omega \to \Omega$ is a measurable transformation such that C_T : $X(\Omega,B) \to X(\Omega,B)$ is bounded. Then $M_\theta C_T$ is compact if and only if $X(N,\epsilon,B)$ is finite dimensional for each $\epsilon > 0$, where*

$$N = \{x \in \Omega : \|\theta(x)t\|_B \geq \epsilon\|t\|_B \text{ for all } t \in B\}.$$

Proof. Assume $X(N,\epsilon,B)$ is finite dimensional for each $\epsilon > 0$. For each $n \in$ N, define θ_n as in Theorem 2.6. Then $M_{\theta n}C_T$ is of finite rank and hence compact. Moreover,

$$\begin{aligned}\|(M_{\theta_n}C_T - M_\theta C_T)f\|_X &= \|(M_{\theta_n} - M_\theta)C_T f\|_X \\ &\leq \frac{1}{n}\|C_T f\|_X \leq \frac{1}{n}\|C_T\|\|f\|_X.\end{aligned}$$

Thus, $M_{\theta n}C_T \to M_\theta C_T$ in operator norm and hence $M_\theta C_T$ is compact. Conversely, suppose $M_\theta C_T$ is compact. For $f \in X(N,\epsilon,B)$,

$$\|M_\theta C_T f\|_X = \|\theta(C_T f)\|_X \geq \epsilon\|C_T f\|_X.$$

If C_T is bounded below on $X(N,\epsilon,B)$, then $M_\theta C_T$ is invertible on this subspace. Being compact and invertible implies finite dimensionality. □

Theorem 3.2. *Let $M_\theta C_T$ be a composite multiplication operator. Then $M_\theta C_T$ is bounded if and only if $\theta \in L^\infty(\Omega,L(B))$ and C_T is bounded.*

Proof. The result follows immediately from Theorem 2.1 and the boundedness of C_T. □

4 Examples

Example 4.1. *Let $X = L^p(\Omega,w)$ with $1 \leq p < \infty$, where Ω is a nonatomic measure space.*

Let $B =$ C and $\theta \in L^\infty(\Omega)$. Then M_θ is bounded with $\|M_\theta\| = \|\theta\|_\infty$. By Theorem 2.3, M_θ is invertible if and only if there exists $\delta > 0$ such that $|\theta(x)| \geq \delta$ a.e. By Theorem 2.8, M_θ is compact if and only if $\theta = 0$ a.e. By Theorem 2.10, if M_θ is Fredholm, then it is invertible and $\mathrm{ind}(M_\theta) = 0$. *This recovers the classical theory of scalar multiplication operators on L^p-spaces.*

Example 4.2. *Let $X = L^\Phi(\Omega,w)$ be a weighted Orlicz space with Young function Φ satisfying the Δ_2-condition, ensuring absolutely continuous norm. In this case, the associate space is the weighted Orlicz space $L^{\Phi*}(\Omega,w)$ where Φ^* is the complementary Young function. The results of Sections 2 and 3 apply directly.*

Example 4.3. *For Lorentz spaces $L^{p,q}(\Omega,w)$, the absolutely continuous norm holds for $p < \infty$. The compactness criteria of Theorem 2.6 specialize to known results for multiplication operators on Lorentz spaces.*

International Journal of Applied Mathematics

Volume 30 No. 6, 2017

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version

Example 4.4. *Let* $X = L^2(0,1)$*,* $B = \ell^2$*, and let S be the unilateral shift on* ℓ^2*. Define* $\theta(x) = I + S$ *for all* $x \in (0,1)$*. Then* M_θ *is bounded, bounded below since* $m(I + S) = 1$*, but not invertible since* $I + S$ *is not surjective. This illustrates Theorem 2.2 and Theorem 2.3.*

Example 4.5. *Let* $\Omega = \mathrm{N}$ *with counting measure,* $X = \ell^p$ *for* $1 \le p < \infty$*, and* $B = \mathrm{C}$*. Define* $\theta(n) = 1/n$*. Then* M_θ *is compact but* $\theta \not= 0$ *a.e. This shows that the nonatomicity assumption in Theorem 2.8 is essential.*

Example 4.6. *Let* $X = L^2(0,1)$*,* $B = \mathrm{C}^2$*, and define*

$$\theta(x) = \begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}.$$

Then $\|\theta\|_\infty = 1$*, so* M_θ *is bounded. However,*

$$\theta(x)^{-1} = \begin{bmatrix} 1/x & -1/x^2 \\ 0 & 1/x \end{bmatrix},$$

and $\|\theta(x)^{-1}\| \sim 1/x^2 \to \infty$ *as* $x \to 0^+$*. Thus* $\operatorname{esssup}_{x\in(0,1)} \|\theta(x)^{-1}\| = \infty$*, so by Theorem 2.3,* M_θ *is not invertible.*

Example 4.7. *Let* $B = \mathrm{C}^2$ *and* $\theta(x) = \mathrm{diag}(1,0)$ *for all* $x \in \Omega$*. Then* M_θ *has closed range, but* $m(\theta) = 0$*. This demonstrates why the injectivity hypothesis in Theorem 2.4 cannot be omitted.*

References

[1] S. C. Arora, G. Datt, and S. Verma, *Multiplication and composition operators on LorentzBochner spaces*, Osaka Journal of Math. 44 (2008), No. 4, 629-641.

[2] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics 129, Academic Press, London, 1988.

[3] D. E. Edmunds and W. D. Evans, *Hardy operators, function spaces and embeddings*, Springer Berlin, 2004.

[4] Shally Gupta and B. S. Komal, *Multiplication operators between Orlicz spaces*, Integral Equations and Operator Theory 41 (2001), 324-330.

[5] H. Hudzik, R. Kumar and R. Kumar, *Matrix multiplication operators on Banach function spaces*, Proc. Indian Acad. Sci. Math. Sci. 116 (2006), No. 1, 71-81.

[6] S. G. Krein, Ju. I. Petunin and E. M. Semenov, *Interpolation of linear operators*, AMS Translation of Math. Monographs 54, Providence, RI, 1982.

[7] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II, Function Spaces*, Vol. 97, Springer-Verlag, Berlin-New York, 1979.

[8] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Math. 5, Univ. Estadual de Campinas, Campinas SP, Brazil, 1989.

[9] R. K. Singh and J. S. Manhas, *Composition operators on function spaces*, North Holland Math. Studies 179, Amsterdam, 1993.

[10] R. K. Singh and N. S. Dharmadhikari, *Compact and Fredholm composite multiplication operators*, Acta Sci. Math. 52 (1988), no. 3-4, 437-441.