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SENSITIVITIES OF INTEREST RATE SWAPS UNDER THE G2++ MODEL

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Abstract: The two-additive-factor Gaussian model G2++ (which encompasses the famous two-factor Hull-White model) is a stochastic model which describes the instantaneous short rate dynamic. It has functional qualities required in various practical purposes as in Asset Liability Management and in Trading of interest rate derivatives.

Recently we derived analytic expressions for the price sensitivities of zero-coupon bonds, coupon-bearing bonds and the portfolio with respect to the shocks linked to the unobservable two-uncertainty factors underlying the G2++ model.

Interest Rate Swaps (IRS) are instruments largely used by market participants for many purposes. It appears that sounding analyzes related to the hedging of portfolios made by swaps is not clear in the financial literature.

Our main goal here is to provide analytic expressions for the price sensitivities for the IRS with respect to the G2++ model of the interest rate. Our present results might provide a support for practitioners, using portfolio of swaps in their hedge decision-making.

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1. Introduction

We will derive here suitable price/value sensitivities related to interest rate swap (IRS) and portfolio of IRSs.The IRS deserve particularly our attention because it appears to be instruments largely used by market participants (companies, local governments, financial institutions, traders, ...). Despite this market importance played by IRS, it appears that sounding analyzes related to swaps is not clear in the financial literature. To partially fill this lack, we provide here the analysis corresponding to a G2++ model of the interest rate. Recently in (see [9]), the authors provide same analysis about the zero coupon bonds and the coupon bearing bonds and its portfolio.

Among our main motivations in this work is to provide suitable tools for hedging a given position (a portfolio of IRSs) sensitive to the interest rate. Very often in the literature, the systematic study of a hedging operation is essentially done by using just one-type of financial instrument, as for example the underlying asset in the case of an equity option. However in practice, the market participant really makes use of a portfolio to perform the hedge. The G2++ model should be considered as a main contribution of this paper. In Section 2 this model is detailed. We explain it also in details in our previous work [9]. The introduction of interest rate sensitivities with respect to the two uncertainty shock factors associated with the considered G2++ model allows us to fulfill this practice requirement. Our approach is inspired from [17], where the study is focused on the case of one-factor uncertainty models for the interest rates (as the Vasicek and CIR models). Our results related to the IRS sensitivities and the associated price change decomposition are presented in Sections 5 and 6. The zero coupon bonds (ZCB) are keys for deriving price changes for IRS which are linear functions of ZCBs. As a consequence, we are able to get directly the IRS change value by an explicit expression depending essentially on two shock factors, which are actually realizations of independent standard random Gaussian variables. Here a decomposition of any IRS price change into three parts is derived. Therefore it is found that the IRS change may be approximated by the sum of a constant residual term and a polynomial term whose coefficients are made by the various sensitivities with respect to the two underlying shock factors (see Theorem 1). The point here is that the change value of IRS instruments may be seen as linear combination of change values of various zero-coupon bonds. In Section 10 we apply all the sensitivities findings in the previous sections in order to derive the three parts decompositions for the change value of IRS portfolio which is considered as a linear combination of IRSs (see Theorem 2). As we plan to show in a next project, all the results found in this section may be used as starting points for performing the hedging operation for a portfolio sensitive to the interest rate. It would be emphasized that the main results obtained in this paper are general enough in the sense that they do not rely on series of particular data. The basic and starting point is that the model should be already appropriately calibrated. It is one of the reasons why we have chosen to mix our result statements with different numerical calibration examples in the numerical work. Numerical illustrations are given in Section 11, and the conclusion is presented in Section 13.

2. The G2++ model

The two-additive-factor Gaussian model G2++ (see [2]) describes the short rate r_t as

$$r_t = \varphi(t) + x_{t;1} + x_{t;2}, \tag{1}$$

where $t \longmapsto \varphi(t)$ is a (deterministic) function which allows the model to fit the current observed interest rates. In (1), $x_{t;1}$ and $x_{t;2}$ may be viewed as state variables whose dynamics are assumed to be given by

$$dx_{t;1}(\cdot) = -\kappa_1 x_{t;1} dt + \sigma_1 dW_{t;1}(\cdot) \tag{2}$$

and

$$dx_{t;2}(\cdot) = -\kappa_2 x_{t;2} dt + \sigma_2 dW_{t;2}(\cdot). \tag{3}$$

All of these dynamics are given under an adjusted risk-neutral measure Q. Here $W_{t;1}(\cdot)$ and $W_{t;2}(\cdot)$ are two correlated standard Brownian motions with a (constant) correlation $\rho \equiv \rho_{x_1,x_2}$, with $-1 < \rho < 1$. In (2) and (3), κ_1, κ_2, σ_1 and σ_2 are nonnegative real numbers which represent the model parameters.

3. IRS value

A plain vanilla Interest Rate Swap (IRS) is an OTC contract between two counter parties to exchange interest payments. The first of them agrees to pay to the second one, a fixed interest rate on the contractual notional principal. In return, it receives interest at floating rate on the same notional principal for the same period of time. IRSs generally help to control instability in interest rates, but if markets change in a surprising way, they can also lead to losses.

To be explicit, let us consider

$$0 \le t_0 < t < t_1 < \ldots < t_i < \ldots < t_M. \tag{4}$$

Here t is the future time such that the remaining cash-flow times payment of the considered IRS, with maturity t_M, should take place at times

$$\mathcal{T} = \big(t_1, \ldots, t_i, \ldots, t_M\big). \tag{5}$$

The time t-value of the zero-coupon bond $P(t, t_i)$ under G2++, having maturity t_i is defined by

$$P(t,t_i) = \exp\Bigg[-\Big\{b\big(\kappa_1(t_i - t)\big)x_{t;1} + b\big(\kappa_2(t_i - t)\big)x_{t;2} - \frac{1}{(t_i - t)}c\big(t, t_i; \Upsilon\big)\Big\}(t_i - t)\Bigg], \tag{6}$$

where the future values $x_{t;1}(\cdot)$ and $x_{t;2}(\cdot)$ of the underlying state variables of G2++ (given their current values $x_{0;1}$ and $x_{0;2}$) are needed and are clearly explictly in Proposition 1:

$$\Upsilon \equiv \Big(\kappa_1, \kappa_2, \sigma_1, \sigma_2, \rho\Big), \tag{7}$$

$$b(u) \equiv \Big(\frac{1}{u}\Big)\big\{1 - \exp[-u]\big\} \quad \text{for} \quad u \neq 0, \quad \text{otherwise} \quad b(0) = 1, \tag{8}$$

$$c\big(t, t_i; \Upsilon\big) \equiv \ln\left[\frac{P^{mkt}(0,t_i)}{P^{mkt}(0,t)}\right] + \frac{1}{2}\Big(V^2(t_i - t; \Upsilon) - V^2(t_i; \Upsilon) + V^2(t; \Upsilon)\Big) \tag{9}$$

and

$$\frac{1}{u}V^2(u;\Upsilon) = \Big(\frac{\sigma_1}{\kappa_1}\Big)^2\Big[1 - b\big(\kappa_1 u\big) - \frac{\kappa_1 u}{2}b^2\big(\kappa_1 u\big)\Big] + \Big(\frac{\sigma_2}{\kappa_2}\Big)^2\Big[1 - b\big(\kappa_2 u\big) - \frac{\kappa_2 u}{2}b^2\big(\kappa_2 u\big)\Big] + 2\rho\Big(\frac{\sigma_1\sigma_2}{\kappa_1\kappa_2}\Big)\Big[1 - b\big(\kappa_1 u\big) - b\big(\kappa_2 u\big) + b\big(\{\kappa_1 + \kappa_2\}u\big)\Big]. \tag{10}$$

A proof of the validity of relation (10) is available from the textbook (see [2]). The form used in [2] is slightly different since it is seen there that

$$V^2(u;\Upsilon) =$$

$$\Big(\frac{\sigma_1}{\kappa_1}\Big)^2\Big[u+\Big(\frac{2}{\kappa_1}\Big)\exp[-\kappa_1 u]-\Big(\frac{1}{2\kappa_1}\Big)\exp[-2\kappa_1 u]-\Big(\frac{3}{2\kappa_1}\Big)\Big]$$
$$+\Big(\frac{\sigma_2}{\kappa_2}\Big)^2\Big[u+\Big(\frac{2}{\kappa_2}\Big)\exp[-\kappa_2 u]-\Big(\frac{1}{2\kappa_2}\Big)\exp[-2\kappa_2 u]-\Big(\frac{3}{2\kappa_2}\Big)\Big]$$
$$+2\rho\Big(\frac{\sigma_1\sigma_2}{\kappa_1\kappa_2}\Big)\Big[1-b\big(\kappa_1 u\big)-b\big(\kappa_2 u\big)+b\big(\{\kappa_1+\kappa_2\}u\big)\Big]u. \tag{11}$$

Actually, (10) and (11) are same since

$$\frac{1}{u}\exp[-u]=\frac{1}{u}-b(u) \qquad \text{and} \qquad \frac{1}{u}\exp[-2u]=ub^2(u)+\frac{1}{u}-2b(u).$$

With (6), the time-t yield-to-maturity for the maturity $T=t+\tau$, with $0<\tau$, is readily given by

$$y(t;\tau)\equiv b(\kappa_1\tau)x_{t;1}+b(\kappa_2\tau)x_{t;2}-\frac{1}{\tau}c\big(t,t+\tau;\Upsilon\big). \tag{12}$$

Therefore the yield curve defined by the mapping

$$\tau\in(0,\infty)\longmapsto y(t;\tau)$$

can be easily plotted by making use of (12).

The IRS marked-to-market value at time t is defined by:

$$\begin{aligned}&\textbf{value\_Swap}_t=\\&\textbf{value\_Swap}\Big(t,\mathcal{T};\textbf{notional};\textbf{rate\_Swap};P(t,t_1),\ldots,P(t,t_M)\Big)\\&=\textbf{notional}\times\Bigg(P(t,t_1)\Big\{y(t_0,t_1)-\textbf{rate\_Swap}\Big\}\tau(t,t_1)\\&+\sum_{i=2}^{M}P(t,t_i)\Big\{F(t;t_{i-1},t_i)-\textbf{rate\_Swap}\Big\}\tau(t_{i-1},t_i)\Bigg).\end{aligned} \tag{13}$$

The yield-rate $y(t_0,t_1)$ for the time-period (t_0,t_1) is given by

$$y(t_0,t_1)=\frac{1}{\tau(t_0,t_1)}\bigg(\frac{1}{P(t_0,t_1)}-1\bigg). \tag{14}$$

In (13), **rate_Swap** denotes the contractual predetermined swap rate defined such that at the contract time issuance t^*, with $t^*\le t_0$, the IRS has a zero market value, that is $\textbf{value\_Swap}_{t^*}=0$.

When time passes, the IRS market value $\mathbf{value\_Swap}_t$ at any time t before the maturity is given as in expression (13) and has no reason to be equal to zero.

At time t_1 the payments are just related to the reduced time-period (t, t_1), such that the reference floating rate is the yield-to-maturity $y(t_0, t_1)$. The value of this last is well-known at time t. For the next times t_i, with $2 \le i \le M$, the reference rate at time t_i is the forward rate $F(t; t_{i-1}, t_i)$ defined by

$$F(t; t_{i-1}, t_i) \equiv \frac{1}{\tau(t_{i-1}, t_i)} \left(\frac{P(t, t_{i-1})}{P(t, t_i)} - 1 \right), \tag{15}$$

which corresponds to the interest rate applies to the time-period (t_{i-1}, t_i) seen at time t. Remind that the forward rate here is useful to determine the IRS market value since the yield-rate $y(t_{i-1}, t_i)$ remains unknown at the future time t.

When using expression (15) then it appears that

$$\begin{aligned}
&\mathbf{value_Swap}_t \\
&\equiv \mathbf{notional} \times \Bigg\{ P(t, t_1) \Big\{ y(t_0, t_1) - \mathbf{rate_Swap} \Big\} \tau(t, t_1) \\
&\qquad + \Big(P(t, t_1) - P(t, t_M) \Big) \\
&\qquad - \mathbf{rate_Swap} \times \sum_{i=2}^{M} P(t, t_i) \tau(t_{i-1}, t_i) \Bigg\}.
\end{aligned} \tag{16}$$

4. IRS change value

The IRS market value as in (16) is one-thing, but for the position managing or hedging the market value change matters. Therefore for the time-period $(0, t)$ let us set

$$\mathbf{change\_value\_Swap}_{0,t}(\cdot) \equiv \mathbf{value\_Swap}_t(\cdot) - \mathbf{value\_Swap}_0. \tag{17}$$

From (16), we will have

$$\begin{aligned}
&\mathbf{change_value_Swap}_{0,t}(\cdot) \\
&\equiv \mathbf{notional} \times \Bigg\{ - \Big\{ y(t_0, t_1) - \mathbf{rate_Swap} \Big\} P(0, t_1) t
\end{aligned}$$

$$+\Big(1+\big\{y(t_0,t_1)-\textbf{rate\_Swap}\big\}\tau(t,t_1)\Big)\Big(P(t,t_1)(\cdot)-P(0,t_1)\Big)$$
$$-\Big(P(t,t_M)(\cdot)-P(0,t_M)\Big)$$
$$-\textbf{rate\_Swap}\times\sum_{i=2}^{M}\Big(P(t,t_i)(\cdot)-P(0,t_i)\Big)\tau(t_{i-1},t_i)\Bigg\}. \tag{18}$$

With this last expression, the IRS market value change during the time-period $(0,t)$ arises as a linear combination of changes of zero-coupon bonds $P(t,t_i)(\cdot)-P(0,t_i)$ with some various maturities t_i's. So one has mainly led to the expansion of the zero-coupon bond price change. Each of the zero-coupon bond price change has been analyzed in [9].

For this, we start to observe more information about the future values $x_{t;1}(\cdot)$ and $x_{t;2}(\cdot)$ of the underlying state variables (given their current values $x_{0;1}$ and $x_{0;2}$) are needed.

We assumed in this part that the G2++ model is calibrated, such that $x_{0,1}, x_{0,2}$ and Υ are known. For this purpose, the dynamic equations (2) and (3) are explored in order to get the following proposition.

Proposition 1. *Under the G2++ model, the future time-t values of the state variables $x_{\cdot;1}$ and $x_{\cdot;2}$, conditionally on their current values $x_{0;1}$ and $x_{0;2}$, are given by*

$$x_{t;1}(\cdot)=\mathcal{E}(t;\kappa_1)x_{0;1}+\sigma_1\mathcal{F}^{\frac{1}{2}}(t;\kappa_1,\kappa_1)\varepsilon_1(\cdot) \tag{19}$$

and

$$x_{t;2}(\cdot)=\mathcal{E}(t;\kappa_2)x_{0;2}+\sigma_2\mathcal{F}^{\frac{1}{2}}(t;\kappa_2,\kappa_2)\Big\{\omega\varepsilon_1(\cdot)+\sqrt{1-\omega^2}\varepsilon_2(\cdot)\Big\}, \tag{20}$$

where $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$ are two independent standard normal random variables,

$$\omega\equiv\omega(t;\rho,\kappa_1,\kappa_2)=\rho\frac{\mathcal{F}(t;\kappa_1,\kappa_2)}{\mathcal{F}^{\frac{1}{2}}(t;\kappa_1,\kappa_1)\mathcal{F}^{\frac{1}{2}}(t;\kappa_2,\kappa_2)}, \tag{21}$$

$$\mathcal{E}(u;k)=\exp\big[-ku\big] \tag{22}$$

and the quantity $\mathcal{F}(u;k_i,k_j)$ is defined as

$$\mathcal{F}(u;\kappa_i,\kappa_j)\equiv\frac{1}{(\kappa_i+\kappa_j)}\big\{1-\exp\big[-(\kappa_i+\kappa_j)u\big]\big\}. \tag{23}$$

Actually, $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$ are given by

$$\varepsilon_1(\cdot)\equiv\varepsilon_1(\cdot;t,\kappa_1)$$

$$= \exp[-\kappa_1 t]\mathcal{F}^{-\frac{1}{2}}(t;\kappa_1,\kappa_1)\int_0^t \exp[\kappa_1 s]dW_{s;1}(\cdot) \tag{24}$$

and

$$\begin{aligned}
&\varepsilon_2(\cdot) \equiv \varepsilon_2(\cdot;t,\rho,\kappa_1,\kappa_2)\\
&= \frac{1}{\sqrt{1-\omega^2}}\Big\{-\omega\exp[-\kappa_1 t]\mathcal{F}^{-\frac{1}{2}}(t;\kappa_1,\kappa_1)\int_0^t \exp[\kappa_1 s]dW_{s;1}(\cdot)\\
&\qquad + \exp[-\kappa_2 t]\mathcal{F}^{-\frac{1}{2}}(t;\kappa_2,\kappa_2)\int_0^t \exp[\kappa_2 s]dW_{s;2}(\cdot)\Big\}.
\end{aligned} \tag{25}$$

Before starting the next section, let us denote by

$$\widetilde{\Upsilon} \equiv \Big(P^{mkt}(0,T), P^{mkt}(0,t); x_{0;1}, x_{0;2}; \Upsilon\Big). \tag{26}$$

5. Sensitivities of the IRS

In order to make a decomposition for the IRS value change (which is useful for the hedging purpose) we are lead to introduce sensitivities of order k, for all non negative integers k. Under the G2++ model for the short interest rate, a decomposition for the zero-coupon bond change, as stated in [10] is ready to be used in the IRS decomposition part.

The Residual term of IRS which represent the sensitivity of order 0 is defined by the expression

$$\begin{aligned}
&\mathbf{Res_Swap}(t,\mathcal{T};\widetilde{\Upsilon})\\
&\equiv \mathbf{Res_Swap}\Big(t,\mathcal{T};\mathbf{notional};\mathbf{rate_Swap};\widetilde{\Upsilon}\Big)\\
&\equiv \mathbf{notional}\times\Big\{-\big\{y(t_0,t_1)-\mathbf{rate_Swap}\big\}P(0,t_1)t\\
&\quad +\Big(1+\big\{y(t_0,t_1)-\mathbf{rate_Swap}\big\}\tau(t,t_1)\Big)\mathbf{Res_ZC}(t,t_1;\widetilde{\Upsilon})\\
&\qquad -\mathbf{Res_ZC}(t,t_M;\widetilde{\Upsilon})\\
&\quad -\mathbf{rate_Swap}\times\sum_{i=2}^{M}\mathbf{Res_ZC}(t,t_i;\widetilde{\Upsilon})\tau(t_{i-1},t_i)\Big\},
\end{aligned} \tag{27}$$

where

$$\mathbf{Res\_ZC}\big(t,t_i;\widetilde{\Upsilon}\big) \equiv \Theta\big(t,t_i\big) - P(0,t_i)$$

$$\Theta\big(t,t_i\big) \equiv \Theta\Big(t,t_i; P^{mkt}(0,t), P^{mkt}(0,t_i); x_{0,1}, x_{0,2}; \Upsilon\Big)$$
$$= \exp\bigg[-\Big\{b\big(\kappa_1(t_i-t)\big)\mathcal{E}(t;\kappa_1)x_{0;1} + b\big(\kappa_2(t_i-t)\big)\mathcal{E}(t;\kappa_2)x_{0;2}$$
$$- \frac{1}{(t_i-t)}c\big(t,t_i;\Upsilon\big)\Big\}(t_i-t)\bigg]. \tag{28}$$

Now for the sensitivity of order k, it is defined as follows:

$$\mathbf{Sens\_Swap}(k;t,\mathcal{T};\widetilde{\Upsilon})$$
$$\equiv \mathbf{Sens\_Swap}\Big(k;t,\mathcal{T};\mathbf{notional};\mathbf{rate\_Swap};\widetilde{\Upsilon}\Big)$$
$$\equiv \mathbf{notional}\times$$
$$\bigg\{\Big(1+\big\{y(t_0,t_1)-\mathbf{rate\_Swap}\big\}\widetilde{\tau_1}\Big)\mathbf{Sens\_ZC}(k;t,t_1;\widetilde{\Upsilon})$$
$$-\mathbf{Sens\_ZC}(k;t,t_M;\widetilde{\Upsilon})$$
$$-\mathbf{rate\_Swap}\times\sum_{i=2}^{M}\mathbf{Sens\_ZC}(k;t,t_i;\widetilde{\Upsilon})\tau(t_{i-1},t_i)\bigg\}, \tag{29}$$

where the sensitivities of the zero coupon are expressed by

$$\mathbf{Sens\_ZC}\big(k;t_i,T;\widetilde{\Upsilon}\big)$$
$$\equiv \bigg(\Theta(t,t_i)\mathbf{c}(j-1,k)\big[\lambda_1(t,t_i)\big]^{k+1-j}\big[\lambda_2(t,t_i)\big]^{j-1}\bigg)_{j\in\{1,\ldots,k+1\}}. \tag{30}$$

Where $\Theta(t,t_i)$ as defined as in (28)

$$\lambda_1 \equiv \lambda_1\big(t,t_i;\Upsilon\big) = \bigg\{\sigma_1 b\big(\kappa_1(t_i-t)\big)\mathcal{F}^{\frac{1}{2}}(t;\kappa_1,\kappa_1)+$$
$$\omega\sigma_2 b\big(\kappa_2(t_i-t)\big)\mathcal{F}^{\frac{1}{2}}(t;\kappa_2,\kappa_2)\bigg\}(t_i-t) \tag{31}$$

$$\lambda_2 \equiv \lambda_2\big(t,t_i;\Upsilon\big) = \sqrt{1-\omega^2}\sigma_2 b\big(\kappa_2(t_i-t)\big)\mathcal{F}^{\frac{1}{2}}(t;\kappa_2,\kappa_2)\big(t_i-t\big), \tag{32}$$

$\mathbf{c}(j-1,k)$ is the binomial coefficient notation.

It appears here that $\mathbf{Sens\_Swap}\big(k;t,\mathcal{T};\widetilde{\Upsilon}\big)$ is a $(k+1)$-th dimensional vectors, as it is a linear combination of M number of $(k+1)$-th dimensional vectors $\mathbf{Sens\_ZC}\big(k;t,t_i;\widetilde{\Upsilon}\big)$'s as defined in (30).

Then the result related to the swap market value change is contained in the following three-parts decomposition as defined in Section 6.

6. Decomposition for IRS change value

As is seen in (see [9]), under the G2++ model, the price at a future horizon t for any zero coupon bond is the result of two shocks $\varepsilon_1(\cdot) \equiv \varepsilon_1(\cdot; t, \kappa_1)$ and $\varepsilon_2(\cdot) \equiv \varepsilon_2(\cdot; t, \kappa_1, \kappa_2, \rho)$ corresponding to the two uncertainty risk/opportunity linked to the model under consideration.

These two shocks should belong to some domain $\mathcal{D}^{\text{swap}}(t, \mathcal{T})$ where all the zero-coupon bonds involved in the considered position have model prices having economical sense.

Our result, related to the analysis of the Swap price change under the G2++ model, is included in the following three-parts decomposition result.

Theorem 1. *Assume that at a future time horizon t satisfying (4) the yield curve, under the G2++ model, has moved as a consequence of shocks $\varepsilon_1(\cdot) \equiv \varepsilon_1(\cdot; t; \kappa_1)$ and $\varepsilon_2(\cdot) \equiv \varepsilon_2(\cdot; t; \kappa_1, \kappa_2, \rho)$. Consider an IRS maturing at time t_M and let p be a nonnegative integer. Then real numbers $\eta_1 = \eta_1(\varepsilon_1, p)$ and $\eta_2 = \eta_2(\varepsilon_2, p)$ do exist such that the IRS change during the time-period $(0, t)$ is given by*

$$\begin{aligned}&\mathbf{change\_value\_Swap}_{0,t}(\cdot) = \\ &\mathbf{Res\_Swap}(t, \mathcal{T}; \widetilde{\Upsilon}) + \sum_{k=1}^{p} \frac{(-1)^k}{k!} \mathbf{Sens\_Swap}'\Big(k; t, \mathcal{T}; \widetilde{\Upsilon}\Big) \bullet \varepsilon^{[k]}(\cdot) \\ &+ \mathbf{Rem\_Swap}'\Big(p+1; t, \mathcal{T}; \widetilde{\Upsilon}, \eta(\cdot)\Big) \bullet \varepsilon^{[p+1]}(\cdot),\end{aligned} \tag{33}$$

where $\eta = (\eta_1, \eta_2)$, not clearly known explicitly, is contained in an open set $\Delta \equiv \Delta(\varepsilon_1, \varepsilon_2)$ which is one among the following four sets:

$$]0, \varepsilon_1[\times]0, \varepsilon_2[\tag{34}$$

$$]0, \varepsilon_1[\times]\varepsilon_2, 0[\tag{35}$$

$$]\varepsilon_1, 0[\times]0, \varepsilon_2[\tag{36}$$

and

$$]\varepsilon_1, 0[\times]\varepsilon_2, 0[. \tag{37}$$

Because of the big equation, let us denote by

$$\exp_{t_i} = \exp\Big[-\Big(\lambda_1(t, t_i)\eta_1 + \lambda_2(t, t_i)\eta_2\Big), rs = rate\_Swap.$$

Then the last term in (33) is really defined by

$$\begin{aligned}
&\mathbf{Rem_Swap}'\Big(p+1; t, \mathcal{T}; \widetilde{\Upsilon}, \eta(\cdot)\Big) \bullet \varepsilon^{[p+1]} \\
&\equiv \frac{(-1)^{p+1}}{(p+1)!}\mathbf{notional} \times \Bigg\{\Big(1+\Big\{y(t_0,t_1)-rs\Big\}\widetilde{\tau}_1\Big) \times \\
&\qquad\qquad \exp_{t_1} \mathbf{Sens_ZC}'(p+1; t, t_1; \widetilde{\Upsilon}) \bullet \varepsilon^{[p+1]} \\
&- \exp_{t_M} \mathbf{Sens_ZC}'(p+1; t, t_M; \widetilde{\Upsilon}) \bullet \varepsilon^{[p+1]} \\
&-rs \times \sum_{i=2}^{M} \exp_{t_i} \mathbf{Sens_ZC}(p+1; t, t_i; \widetilde{\Upsilon}) \bullet \varepsilon^{[p+1]} \tau(t_{i-1}, t_i)\Big)\Bigg\}.
\end{aligned} \tag{38}$$

With (33), it may be said that we have the approximation

$$\begin{aligned}
&\mathbf{change_value_Swap}_{0,t}(\cdot) \approx \\
&\qquad \mathbf{Res_Swap}(t, \mathcal{T}; \widetilde{\Upsilon}) + \\
&\qquad \sum_{k=1}^{p} \frac{(-1)^k}{k!} \mathbf{Sens_Swap}'\Big(k; t, \mathcal{T}; \widetilde{\Upsilon}\Big) \bullet \varepsilon^{[k]}(\cdot).
\end{aligned} \tag{39}$$

The error related to this approximation is defined as

$$\begin{aligned}
&\mathbf{error_Swap_change_approx}(0,t)(\cdot) \\
&\quad = \mathbf{change_value_Swap}_{0,t}(\cdot) - \Bigg\{\mathbf{Res_Swap}(t, \mathcal{T}; \widetilde{\Upsilon}) \\
&\qquad + \sum_{k=1}^{p} \frac{(-1)^k}{k!} \mathbf{Sens_Swap}'\Big(k; t, \mathcal{T}; \widetilde{\Upsilon}\Big) \bullet \varepsilon^{[k]}(\cdot)\Bigg\} \\
&\quad \equiv \mathbf{Rem_Swap}'\Big(p+1; t, \mathcal{T}; \widetilde{\Upsilon}, \eta(\cdot)\Big) \bullet \varepsilon^{[p+1]}(\cdot).
\end{aligned} \tag{40}$$

A main point here is that a deterministic estimates of the error related to the IRS change approximation is obtained by

$$\begin{aligned}
\mathcal{R}_Swap \equiv \max\Bigg\{&\Big|\mathbf{Rem_Swap}'\big(p+1; t, T; \widetilde{\Upsilon}, \eta\big) \bullet \varepsilon^{[p+1]}\Big|; \\
&\eta = (\eta_1, \eta_2) \in \triangle(\varepsilon_1, \varepsilon_2) \text{ and } (\varepsilon_1, \varepsilon_2) \in \mathcal{D}^{\text{swap}}(t, \mathcal{T})\Bigg\},
\end{aligned} \tag{41}$$

where $\mathcal{D}^{\text{swap}}(t, \mathcal{T})$ is some suitable domain detailed in [9].

7. Portfolio of Interest Rate Swaps

Let us denote by $\mathcal{S}_t$ the time t value of a portfolio made by I^{**} types of payer IRSs $S^{**}_{\cdot;i^{**}}$ (with a notional of $N^{**}_{i^{**}}$, swap rate $r^{**}_{i^{**}}$ and maturity $t_{M^{**}(i^{**})}$) and I^{*} types of receiver IRSs $S^{*}_{\cdot;i^{*}}$ (resp. $N^{*}_{i^{*}}$, $r^{*}_{i^{*}}$ and $t_{M^{*}(i^{*})}$). Of course, I^{**} and I^{*} are positive integer numbers. For $i^{**} \in \{1,\ldots,I^{**}\}$ and $i^{*} \in \{1,\ldots,I^{*}\}$, the IRSs

$$S^{**}_{\cdot;i^{**}} \quad \text{and} \quad S^{*}_{\cdot;i^{*}},$$

are assumed respectively to have the notional

$$\textbf{notional}(i^{**}) \quad \text{and} \quad \textbf{notional}(i^{*}),$$

rate swaps

$$\textbf{rate\_Swap}(i^{**}) \quad \text{and} \quad \textbf{rate\_Swap}(i^{*})$$

maturities

$$t^{**}_{M^{**}(i^{**})}(i^{**}) \quad \text{and} \quad t^{*}_{M^{*}(i^{*})}(i^{*})$$

and have the ordered payment times

$$\mathcal{T}^{**}_{\cdot;i^{**}} = \Big(t^{**}_1(i^{**}),\ldots,t^{**}_{j^{**}}(i^{**}),\ldots,t^{**}_{M^{**}(i^{**})}(i^{**})\Big)$$

$$\mathcal{T}^{*}_{\cdot;i^{*}} = \Big(t^{*}_1(i^{*}),\ldots,t^{*}_{j^{*}}(i^{*}),\ldots,t^{*}_{M^{*}(i^{*})}(i^{*})\Big)$$

Where $j^{**} \in \big\{1,2,\cdots,M^{**}(i^{**})\big\}$ and $j^{*} \in \big\{1,2,\cdots,M^{*}(i^{*})\big\}$. For the whole portfolio, it is also suitable to introduce

$$\mathcal{T}^{**} = \Big(\mathcal{T}^{**}_{\cdot;i^{**}}\Big)_{i^{**}\in\{1,\ldots,I^{**}\}} \qquad \text{and} \qquad \mathcal{T}^{*} = \Big(\mathcal{T}^{*}_{\cdot;i^{*}}\Big)_{i^{*}\in\{1,\ldots,I^{*}\}}.$$

The time-t value of such a portfolio may be written as

$$\mathcal{S}_t = \sum_{i^{**}=1}^{I^{**}} n^{**}_{i^{**}} S^{**}_{t;i^{**}} - \sum_{i^{*}=1}^{I^{*}} n^{*}_{i^{*}} S^{*}_{t;i^{*}}. \tag{42}$$

Therefore there are $n^{**}_{i^{**}}$ IRSs of type i^{**} each worth $S^{**}_{t;i^{**}}$, and $n^{*}_{i^{*}}$ IRSs of type i^{*} each worth $S^{*}_{t;i^{*}}$.

It is supposed that the holder of such IRS portfolio has just the purpose of limiting the risk resulting from an adverse movement of the yield curve at some time-horizon t. The shape of the yield curve at this future time t is assumed to be suitably governed by the G2++ model. In practice t corresponds to the horizon for which she has a more and less clear view about a possible movement of the market. To simplify the situation, we will just focus on the case where t is sufficiently close to the present time 0, in the sense that it satisfies a condition like (4).

8. IRS portfolio change value

As usual, the future portfolio value $\mathcal{S}_t(\cdot)$ is unknown at time 0, and depends on the variation of the yield curve at this time t. The change value of the considered portfolio for the period $(0, t)$ is given by

$$\begin{aligned}&\mathbf{change\_value\_port\_Swap}_{0,t}(\cdot) \equiv \mathcal{S}_t(\cdot) - \mathcal{S}_0\\&= \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} \left\{ S_{t;i^{**}}^{**}(\cdot) - S_{0;i^{**}}^{**} \right\} - \sum_{i^{*}=1}^{I^{*}} n_{i^{*}}^{*} \left\{ S_{t;i^{*}}^{*}(\cdot) - S_{0;i^{*}}^{*} \right\}.\end{aligned} \tag{43}$$

It appears here that the IRS portfolio change value depends on

$$S_{t;i^{**}}^{**}(\cdot) - S_{0;i^{**}}^{**} \qquad \text{and} \qquad S_{t;i^{*}}^{*}(\cdot) - S_{0;i^{*}}^{*}$$

which correspond to the price changes for the payer and receiver IRSs having the maturities $t_{M^{**}(i^{**})}^{**}(i^{**})$ and $t_{M^{*}(i^{*})}^{*}(i^{*})$ respectively. It means that we can benefit from the above finding related to the IRS single position. It is seen in (18) that each of these two changes may be represented by a linear combination of changes of zero-coupon bonds prices. As a result, the IRS portfolio appears to be a linear combination of various zero coupon bonds.

As in the case of a single IRS position, these two shocks should belong to some domain for which all the zero-coupon bonds involved in the considered position have model prices having economical sense.

9. Sensitivities for the IRS portfolio

As the IRS portfolio change is a linear function of IRSs changes, then we can benefit from the expression in (18) related to change of a single IRS when defining the sensitivities related to the considered portfolio. Therefore the portfolio sensitivity of order zero or residual term, measuring the passage of time when the yield curve remains unchanged, is given by

$$\mathbf{Res}(t, \mathcal{S}; \widetilde{\Upsilon}) \equiv \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**} \mathbf{Res\_Swap}(t, \mathcal{T}_{\cdot;i^{**}}^{**}; \widetilde{\Upsilon})$$

$$-\sum_{i^*=1}^{I^*} n^*_{i^*}\mathbf{Res\_Swap}(t,\mathcal{T}^*_{\cdot;i^*};\widetilde{\Upsilon}). \tag{44}$$

As in (27), the zero order sensitivities for the i^{**th} single IRS position is

$$\begin{aligned}
&\mathbf{Res_Swap}(t,\mathcal{T}^{**}_{\cdot;i^{**}};\widetilde{\Upsilon})\\
&\equiv N^{**}_{i^{**}}\times\Big\{-\{y^{**}_{01}(i^{**})-r^{**}_{i^{**}}\}P\big(t,t^{**}_1(i^{**})\big)t\\
&\quad+\Big(1+\{y^{**}_{01}(i^{**})-r^{**}_{i^{**}}\}\widetilde{\tau}^{**}_{j^{**}}(i^{**})\Big)\mathbf{Res_ZC}\big(t,t^{**}_1(i^{**});\widetilde{\Upsilon}\big)\\
&\quad-\mathbf{Res_ZC}\big(t,t^{**}_{M^{**}(i^{**})}(i^{**});\widetilde{\Upsilon}\big)\\
&\quad-r^{**}_{i^{**}}\times\sum_{j^{**}=2}^{M^{**}(i^{**})}\mathbf{Res_ZC}(t,t_{j^{**}}(i^{**});\widetilde{\Upsilon})\tau^{**}_{j^{**}}\Big\}.
\end{aligned}\tag{45}$$

Recall that the zero-order sensitivity $\mathbf{Res\_ZC}\big(t,t^{**}_{j^{**}}(i^{**});\widetilde{\Upsilon}\big)$ for the zero coupon bond having a maturity of $t^{**}_{j^{**}}(i^{**})$ is given by

$$\mathbf{Res\_ZC}\big(t,t^{**}_{j^{**}}(i^{**});\widetilde{\Upsilon}\big)\equiv\Theta\big(t,t^{**}_{j^{**}}(i^{**})\big)-P\big(0,t^{**}_{j^{**}}(i^{**})\big)$$

and where $\Theta\big(t,t^{**}_{j^{**}}(i^{**})\big)$ is defined explicitly as in (28). The IRS zero-order sensitivity $\mathbf{Res\_Swap}\big(t,\mathcal{T}^*_{\cdot;i^*};\widetilde{\Upsilon}\big)$ is similarly defined as (45) by changing the double star by one star.

In order to make the three-parts decomposition of the IRS portfolio change, useful for performing the hedging operation, we are also lead to introduce the k-th order sensitivity defined as

$$\begin{aligned}
\mathbf{Sens}(k;t,\mathcal{S};\widetilde{\Upsilon})\equiv&\sum_{i^{**}=1}^{I^{**}} n^{**}_{i^{**}}\mathbf{Sens_Swap}(k;t,\mathcal{T}^{**}_{\cdot;i^{**}};\widetilde{\Upsilon})\\
&-\sum_{i^*=1}^{I^*} n^*_{i^*}\mathbf{Sens_Swap}(k;t,\mathcal{T}^*_{\cdot;i^*};\widetilde{\Upsilon}).
\end{aligned}\tag{46}$$

Here the k-th order sensitivity $\mathbf{Sens\_Swap}(k;t,\mathcal{T}^{**}_{\cdot;i^{**}};\widetilde{\Upsilon})$ for the i^{**}-th IRS associated with the tenor $\mathcal{T}^{**}_{\cdot;i^{**}}$ is defined as in (29) by

$$\mathbf{Sens\_Swap}\big(k;t,\mathcal{T}^{**}_{\cdot;i^{**}};\widetilde{\Upsilon}\big)=N^{**}_{i^{**}}\Big\{$$

$$\Big(1+\{y_{01}^{**}(i^{**})-r_{i^{**}}^{**}\}\widetilde{\tau}_1^{**}(i^{**})\Big)\mathbf{Sens\_ZC}\big(k;t,t_1^{**}(i^{**});\widetilde{\Upsilon}\big)$$
$$-\mathbf{Sens\_ZC}\big(k;t,t_{M^{**}(i^{**})}^{**}(i^{**});\widetilde{\Upsilon}\big)$$
$$-r_{i^{**}}^{**}\times\sum_{j^{**}=2}^{M^{**}(i^{**})}\mathbf{Sens\_ZC}\big(k;t,t_{j^{**}}(i^{**});\widetilde{\Upsilon}\big)\times\tau_{j^{**}}^{**}(i^{**})\Big\}. \tag{47}$$

The k-th sensitivity $\mathbf{Sens\_ZC}\big(k;t,t_{j^{**}}^{**}(i^{**});\widetilde{\Upsilon}\big)$ for the zero coupon with the maturity $t_{j^{**}}^{**}(i^{**})$ is defined by

$$\mathbf{Sens\_ZC}\big(k;t,t_{j^{**}}^{**}(i^{**});\widetilde{\Upsilon}\big)$$
$$\equiv\Bigg(\Theta\big(t,t_{j^{**}}^{**}(i^{**})\big)\times$$
$$\mathbf{c}(l-1,k)\Big[\lambda_1\big(t,t_{j^{**}}^{**}(i^{**})\big)\Big]^{k+1-l}\Big[\lambda_2\big(t,t_{j^{**}}^{**}(i^{**})\big)\Big]^{l-1}\Bigg)_{l\in\{1,\dots,k+1\}}.$$

The expression for $\mathbf{Sens\_Swap}(k;t,\mathcal{T}_{\cdot;i^*}^*;\widetilde{\Upsilon})$ is similarly defined by replacing each double star by a one star. From the above expressions, to get the $\mathbf{Res}(t,\mathcal{S};\widetilde{\Upsilon})$ and $\mathbf{Sens}(k,t,\mathcal{S};\widetilde{\Upsilon})$'s and reduce the computation efforts, it appears to be useful to calculate in advance and store all zero-coupon sensitivities

$$\mathbf{Res\_ZC}\big(t,t_{j^{**}}^{**}(i^{**});\widetilde{\Upsilon}\big),\quad \mathbf{Res\_ZC}\big(t,t_{j^*}^{*}(i^{*});\widetilde{\Upsilon}\big)$$

$$\mathbf{Sens\_ZC}\big(k;t,t_{j^{**}}^{**}(i^{**});\widetilde{\Upsilon}\big),\quad \mathbf{Sens\_ZC}\big(k;t,t_{j^*}^{*}(i^{*});\widetilde{\Upsilon}\big)$$

for all $j^{**}\in\{1,\dots,M^{**}(i^{**})\}$, $j^*\in\{1,\dots,M^*(i^*)\}$,
$i^{**}\in\{1,\dots,I^{**}\}$ and $i^*\in\{1,\dots,I^*\}$.

It should be emphasized that the IRS portfolio sensitivities $\mathbf{Res}(t,\mathcal{S};\widetilde{\Upsilon})$ and $\mathbf{Sens}(k,t,\mathcal{S};\widetilde{\Upsilon})$ are deterministic quantities computed at the present time 0 and expected to capture the (random) value $\mathcal{S}_t(\cdot)$ of the IRS portfolio at the future time-horizon t. The fact to account for the horizon, in the sensitivities computations, is among our main contributions when compared with the existing sensitivities commonly known and used in the financial industry. By so doing, the change value of the swap portfolio is better captured and moreover it is possible to monitor the resulting error approximation.

10. Decomposition of the IRS portfolio change value

As for the case of a single IRS, the main key is to introduce a decomposition of the position change value over the following three-part decomposition result.

Theorem 2. *The G2++ model is defined as in (1), (2) and (3). Assume that at a future time horizon t satisfying (4) the yield curve has moved as a consequence of shocks $\varepsilon_1(\cdot) \equiv \varepsilon_1(\cdot;t;\kappa_1)$ and $\varepsilon_2(\cdot) \equiv \varepsilon_2(\cdot;t;\kappa_1,\kappa_2,\rho)$. Consider an IRS portfolio as defined in (42) and let p be a nonnegative integer.*

Then real numbers $\eta_1 = \eta_1(\varepsilon_1, p)$ and $\eta_2 = \eta_2(\varepsilon_2, p)$ do exist such that the IRS portfolio change value during the time-period $(0,t)$ is given by the sum

$$\mathbf{change\_port\_value\_Swap}_{0,t} = \mathbf{Res}(t;\mathcal{S};\widetilde{\Upsilon}) + \sum_{k=1}^{p} \frac{(-1)^k}{k!}\mathbf{Sens}'\big(k;t;\mathcal{S};\widetilde{\Upsilon}\big) \bullet \varepsilon^{[k]}(\cdot) + \mathbf{Rem}'\big(p+1;t;\mathcal{S};\eta(\cdot);\widetilde{\Upsilon}\big) \bullet \varepsilon^{[p+1]}(\cdot), \tag{48}$$

where $\mathbf{Res}(t,;\mathcal{S};\widetilde{\Upsilon})$ and $\mathbf{Sens}\big(k;t;\mathcal{S};\widetilde{\Upsilon}\big)$ are respectively defined in (44), (46).

Here $\eta(\cdot) = \big(\eta_1(\cdot), \eta_2(\cdot)\big)$, not clearly known explicitly is contained in an open set $\Delta \equiv \Delta(\varepsilon_1, \varepsilon_2)$ as defined in (34) to (37).

In (48), the remainder term of the IRS portfolio is defined by

$$\mathbf{Rem}\big(p+1;t,\mathcal{S};\eta;\widetilde{\Upsilon}\big) \equiv \sum_{i^{**}=1}^{I^{**}} n_{i^{**}}^{**}\mathbf{Rem\_Swap}\big(p+1;t,\mathcal{T}_{\cdot;i^{**}}^{**};\eta;\widetilde{\Upsilon}\big) - \sum_{i^{*}=1}^{I^{*}} n_{i^{*}}^{*}\mathbf{Rem\_Swap}\big(p+1;t,\mathcal{T}_{\cdot;i^{*}}^{*};\eta;\widetilde{\Upsilon}\big). \tag{49}$$

For shortness, the expressions for

$$\mathbf{Rem\_Swap}\big(p+1;t,\mathcal{T}_{\cdot;i^{**}}^{**};\eta;\widetilde{\Upsilon}\big) \text{ and } \mathbf{Rem\_Swap}\big(p+1;t,\mathcal{T}_{\cdot;i^{*}}^{*};\eta;\widetilde{\Upsilon}\big)$$

are not reported since it is sufficient to mimic things from those of $\mathbf{Res\_Swap}\big(t, \mathcal{T}_{\cdot;i^{**}}^{**};\widetilde{\Upsilon}\big)$ and $\mathbf{Sens\_Swap}\big(k;t,\mathcal{T}_{\cdot;i^{**}}^{**};\widetilde{\Upsilon}\big)$ in the equation (38) for the single swap.

The first term $\mathbf{Res}(t,\mathcal{S};\widetilde{\Upsilon})$ in the swap portfolio change value decomposition (48), corresponds to the passage of time in the sense that

$$\mathbf{change\_value\_port\_Swap}_{0,t}(\cdot)\Big|_{\varepsilon_1=0,\ \varepsilon_2=0} = \mathbf{Res}(t,\mathcal{S};\widetilde{\Upsilon}).$$

It may be noted that $\mathbf{Sens}\big(k;t;\mathcal{S};\widetilde{\Upsilon}\big)$ is a $(k+1)$-th dimensional vector, as being a linear combination of a number M of $(k+1)$-th dimensional vectors $\mathbf{Sens\_ZC}\big(k;t,t_i;\widetilde{\Upsilon}\big)$'s.

From (48), the swap portfolio change value decomposition may be approximated as

$$\mathbf{change\_value\_port\_Swap}_{0,t}(\cdot) \approx \left(\mathbf{Res}(t;\mathcal{S};\widetilde{\Upsilon}) + \sum_{k=1}^{p} \frac{(-1)^k}{k!}\mathbf{Sens}'\big(k;t;\mathcal{S};\widetilde{\Upsilon}\big) \bullet \varepsilon^{[k]}(\cdot)\right) \tag{50}$$

such that the error approximation is

$$\begin{aligned}&\mathbf{error\_approx\_port}_{0,t}(\cdot) \equiv \mathbf{Rem}'\Big(p+1;t,\mathcal{S},\eta;\widetilde{\Upsilon}\Big) \bullet \varepsilon^{[p+1]}(\cdot)\\ &= \mathbf{change\_value\_port\_Swap}_{0,t}(\cdot)\\ &-\left(\mathbf{Res}(t;\mathcal{S};\widetilde{\Upsilon}) + \sum_{k=1}^{p} \frac{(-1)^k}{k!}\mathbf{Sens}'\big(k;t;\mathcal{S};\widetilde{\Upsilon}\big) \bullet \varepsilon^{[k]}(\cdot)\right).\end{aligned} \tag{51}$$

The approximation (50) has only a sense whenever the error approximation (51) may be neglected from the perspective of the IRS portfolio holder. The challenge here is that the components of the vector $\varepsilon^{[k]}(\cdot)$ are made by realizations of two independent standard normal Gaussian variables and consequently have no reason to be of small sizes. The introduction of high order sensitivities as $\mathbf{Sens}\big(k;t;\mathcal{S};\widetilde{\Upsilon}\big)$ and the corresponding term $\frac{1}{k!}$ are expected to contribute in the reduction of the size of approximation error.

The size of such a remainder term is of importance as being involved in the hedging error. Less small is this size, more the approximation of the IRS portfolio change by the two term in the decomposition is better. It means that the change is essentially driven by a polynomial function. Then this fact can be explored in the hedging purpose by matching the various sensitivities. The challenge in our approach is that this size (when expressed in term of loss) should be very small from the perspective of the hedger.

It should be emphasized that replacing the future change of the IRS portfolio by a polynomial function (whose the coefficient are the initial portfolio sensitivities) is also interesting in the perspective of risk measurement and management. Indeed instead of re-evaluating the portfolio for each scenario of shocks considered (which is time and memory consuming) one has just to calculate the value of the polynomial function. Of course this is meaningful whenever the remainder term is negligible.

As in case of a single swap, it is possible to get a high bound for the remainder of the IRS portfolio under a view on the shocks at the future horizon t. For doing, let us recall that the suitable domain for the shocks $(\varepsilon_1, \varepsilon_2)$ in the case of the IRS portfolio is

$$\begin{aligned}&\mathcal{D}^{\text{port.swap}}(t, \mathcal{S}) \equiv \\ &\quad \mathcal{D}^{\text{port.swap}}(t, \mathcal{S})\Big(t, \mathcal{T}^{**}, \mathcal{T}^{*}; \\ &\qquad\qquad P^{mkt}(0,t), P^{**}_{mkt}, P^{*}_{mkt}, x_{0,1}, x_{0,2}, \Upsilon; \varepsilon_1^{\bullet}, \varepsilon_1^{\bullet\bullet}, \varepsilon_2^{\bullet}, \varepsilon_2^{\bullet\bullet}\Big)\end{aligned}$$

for some fixed real constants $\varepsilon_1^{\bullet}$, $\varepsilon_1^{\bullet\bullet}$, $\varepsilon_2^{\bullet}$ and $\varepsilon_2^{\bullet\bullet}$. A high bound '$\mathcal{R}\_port\_swap$' for the remainder IRS portfolio is given by

$$\begin{aligned}\mathcal{R}\_port\_swap \equiv \max\Big\{ &\Big|\mathbf{Rem}\Big(p+1; t, \mathcal{S}; \widetilde{\Upsilon}; \eta\Big) \bullet \varepsilon^{[p+1]}\Big|; \\ &\eta = (\eta_1, \eta_2) \in \triangle(\varepsilon_1, \varepsilon_2) \text{ and } (\varepsilon_1, \varepsilon_2) \in \mathcal{D}^{\text{port.swap}}(t, \mathcal{B})\Big\}.\end{aligned} \tag{52}$$

11. Numerical Illustration

The results derived in this paper are general enough in the sense that they do not lean on data. However they depend on a model which is assumed to be already suitably calibrated. In this section part, we will focus on giving numerical illustrations for sensitivities of the IRS, portfolio of IRSs. In a first step, we consider the sensitivities for IRSs which are the mains of our approach. Then in a next step, we try to show how useful is the introduction of high order sensitivities when one has to deal with IRS and the associated portfolios.

• The model considered

The G2++ takes as input the market term structure of interest rate at some point 0 of time. To avoid to choose a particular market data, we assume that the present interest rate structure

$$P^{mkt}(0,T) = \exp\big(-y(0;T)\tau(0,T)\big) \tag{53}$$

is obtained from the initial yield-curve

$$y(0;T) = L_0 + S_0 b_2(\lambda T) + C_0 b_3(\lambda T) \qquad \text{for } \; T > 0 \tag{54}$$

with $b_2(u) = \frac{1}{u}((1 - \exp[-u])$ and $b_3(u) = b_2(u) - \exp(-u)$ and $\lambda > 0$. Here λ, L_0, S_0 and C_0 are used to denote respectively the level, slope and convexity associated with the used static model.These values are given in Table 1. Actually (54) was introduced by Diebold, Ji and Li (see [5]) as a parsimonious reformulation of an original interpolation expression for the yield curve pioneered to Nelson and Siegel. We consider the parameter values as displayed in Table 1. We assume here that the interest rate model at the initial time 0 is driven by a calibrated G2++ model as in [2] with the parameters presented in Table 2. Another calibration in Table 3 are used. Note that for the two considered calibrations, we examine various time-horizon t = 90 days to show that the passage time matters. The state variables $x_{0;1}$ and $x_{0;2}$ are initially chosen as $x_{0;1} = x_{0;2} = 0$ in order to have all the zero-coupon prices generated by the G2++ to perfectly fit those observed in the market as given in (54).

• **Numerical illustrations for IRS**

To better visualize our above results on the IRS, especially as those stated in Theorem 1, it is suitable to consider some numerical illustrations corresponding to different types of calibration. The results derived in this paper are general enough in the sense that they do not lean on data. However they depend on a model which is assumed to be already suitably calibrated. In this section part, we will focus on giving numerical illustrations for change value, sensitivities and Error approximations of the IRS.

• **IRS change value**

In this part, we consider an IRS with the notional 10 000 000, 7 years maturity and the frequency payment of 6 months. All of these characteristics are summarized in Table 4. Always the time horizon of 3 months is considered, though in practice for the hedging perspective it is considered shorter horizon, as one month or, one or two weeks. The large horizon considered here is motivated by the willing to analyze the behavior of our sensitivities under extreme situations. We first analyze the IRS price changes at the considered horizon t under the presence of the shocks $(\varepsilon_1, \varepsilon_2)$ and with the different Set calibrations (Tables 2 and 3). It means that the values obtained represent the exact ones which may be obtained whenever the interest rate really follows the G2++ model. Table 6 show these changes. In Table 6 the exact IRS values change for the different calibrations are displayed in the third and fourth columns. They are obtained by computing $S_t(\cdot) - S_0$ according to (17), and where S_t is obtained from the definition of IRS value as in (16).

Under the shocks $\varepsilon_1 = 2$ and $\varepsilon_2 = 2$ then $\{S_{90} - S_0\}_{calib1} = 76\,982.36$ and

$\{S_{90} - S_0\}_{calib2} = -403\,421.92$. This confirms the well-known fact that when calibration changes then the market can greatly change, so that it appears useful not to wait too long time to hedge the position. For the shocks $\varepsilon_1 = -2$, $\varepsilon_2 = 2$ the exact change values $\{S_{90} - S_0\}_{calib1} = -534\,059.15$ (resp. $\{S_{90} - S_0\}_{calib2} = -571\,066.85$) which represents 5.34% (resp. 5.71%) of the IRS notional value. If instead of considering just one IRS we deal with a position of 1 000 of such IRS, then the values becomes $-534\,059\,150$ (resp. -571 066 850). Clearly a loss with such a magnitude size may be unacceptable for the hedger point of view.

• IRS sensitivities for different order

To get a good understanding about the role of the high order sensitivities, we provide some results in Tables 8 and 9. Table 8 uses the set calibration 1 (resp. Table 9 uses the set calibration 2.) The IRS sensitivity of order one is a two dimensional vector whose components are presented in the first column of Table 8 (resp. 9). It should be read here that the first order sensitivity with respect to the factor shock ε_1 is $-150\,709.95$ (resp. $-39\,654.64$), while the first order sensitivity with respect to the factor shock ε_2 is 78 065.42 (resp. 205 647.24). For the third column, 1.82 (resp. 135.57) is the IRS third order sensitivity with respect to the shock factor ε_2^3, -14.94 (resp. -81.76) is the third order sensitivity with respect to the shock $\varepsilon_1\varepsilon_2^2$, 53.27 (resp. 16.44) is the third order sensitivity with respect to $\varepsilon_1^2\varepsilon_2$ and -60.88 (resp. -1.10) is the third order sensitivity with respect to the shock factor ε_1^3. The other columns and rows can be similarly interpreted. In the fifth column, the numbers represent the sensitivities of order 6 multiplied by 10^3, for example, the 6th order sensitivity with respect to the shock $\varepsilon_1^3\varepsilon_2^3$ is 0.21×10^{-3} (resp. 0.33×10^{-3}).

When comparing Tables 8 and 9 it appears that the sensitivities of the IRS using set calibration of Table 2 and set calibration of Table 3 at the same order have same sizes. For example, Table 8 with seventieth order of sensitivities, sizes are of order 10^5 same for Table 9.

• Error approximation of the IRS

In the sequel for the two considered calibration situations, the illustrations are just limited to long horizon $t = 90$ days. We are interested to derive numerical values of the error approximation. For Tables 12 and 13, the first and second columns are given some possible values of the shocks ε_1 and ε_2 that belong the specific domain. In the next columns, the corresponding error approximations are given for different orders $p = \{1, 2, 6, 9, 12\}$. The first order approximation for the IRS value change for the horizon t, as described in equation (39), are presented in the first columns of Tables 12 and 13. They are the difference between the exact value and the approximation of order 1.

It may be observed from Table 12 (resp. Table 13), that the error is exactly equal to zero when there is no shock (i.e. $\varepsilon_1 = \varepsilon_2 = 0$). This is the consequence that the IRS price change is reduced to the residual term. Under the shocks $\varepsilon_1 = 2$ and $\varepsilon_2 = 2$ then $\{Er1\}_{calib1} = -2\,732.88$ and $\{Er1\}_{calib2} = -7\,105.54$. If instead of considering just one IRS we deal with a position of 1 000 of such IRS, then the values becomes $-2\,732\,880$ (resp. -7 105 540). Clearly a loss with such a magnitude size may be unacceptable for the hedger point of view. It means that limiting to a first order approximation should not be sufficient in practice. This is the reason why we introduce and consider high order approximations, whose the numerical results are displayed in Tables 12 and 13. For the same shocks $\varepsilon_1 = 3.5$, $\varepsilon_2 = 3.5$, a time-horizon $t = 90$ days and the Set calibration of Table2, then the error approximation of order 6 is $\{Er6\}_{calib1} = 0.01 \times 10^{-4}$ as given in Table 12. It means that when considering a position of 1 000 of such IRSs the error approximation is reduced to 0.001. Of course an amount loss with such a magnitude size is negligible for the hedger point of view giving a notional of 10 000 000. For the set calibration of Tables 3, as is seen in Table 13, for the same levels of shocks, then the error approximation of order 6 is just $\{Er6\}_{calib2} = -0.01 \times 10^{-3}$.

Observe that under the set calibration of Table2, then in general the error approximations are lesser than the one obtained under the setting of Table3. For example with the shocks $\varepsilon_1 = -3$, $\varepsilon_2 = -3$ and time-horizon $t = 90$ $\{Er2\}_{calib1} = -94.29$ and $\{Er2\}_{calib2} = 306.62$.

With these latter tables our intention is to illustrate the fact that the calibration of model is very determining in the risk measurement and position management. But we can result that when the high order level is here than the error order of this sensitivity approximation tend to be zero. And this is what we are searching.

12. Numerical illustrations for the IRS portfolio

This part is devoted to numerical illustrations of the results obtained to the portfolio of IRS. Same to the examples presented for single IRS, we will focus on the numerical illustrations for the portfolio of IRSs. Here the emphasis is put on the hedging aspect, which is one of our main purpose in the future work. We start on the study of the exact change of the IRS portfolio, under some view on the possible shocks affecting the interest rate. As mentioned in the previous Section 10, the sensitivities and the error approximations of this decomposition are numerically stated here.

To illustrate the fact that the model calibration is very determining for the hedging result, we consider both the set calibrations as in Tables 2 and 3. The time-horizons considered for the hedge are again 90 days.

The portfolio to study is made by three types of payer swaps ${S_1}^{**}$, ${S_2}^{**}$, and ${S_3}^{**}$, and one type of receiver swaps ${S_1}^{*}$. Each swap has a face value equal 1 000 000 and differ by annual or semi-annual coupon payment. Their characteristics are summarized in Table 5. As we assume that the present time corresponds to the time-inceptions of all of these swaps, then the considered portfolio has an initial zero value (**portf_value_swap** =0).

- **IRS portfolio change value**

The considered portfolio change values under the scenarios for $\varepsilon_1, \varepsilon_2$ are displayed in Table 7. In the third column we report the portfolio change value under set calibration of Table 2 and in the fourth column resp. to Table 3. Observe that the portfolio change value when the curve remains unchanged (which corresponds to $\varepsilon_1 = \varepsilon_2 = 0$) at the considered horizon (90 days) for the set calibration of Table 2 is equal to 164.49 and 237.21 for the set calibration of Table 3. These values results due to the passage of time. These two change values are quietly different, while it corresponds to different interest rate environments (as reflected by the model parameters). As a general view, it may be observed that under the set calibration of Table 2 the portfolio change is less than under the set calibration of Table 3. Next we move to numerically visualize the portfolio IRSs-sensitivities.

- **IRS portfolio sensitivities for different order**

This table reports some sensitivities of the IRS portfolio given in Table 5 for the two calibration and for some $p = \{1, 2, 3, 5, 6, 7, 12\}$. All these sensitivities are interpreted as in Tables 8 and 9. As example, in the first column of Table 10 (resp. Table 11), the first order sensitivity with respect to ε_1 is equal to $-12\,308.38$ (resp. $-3\,234.92$) and with respect to ε_2 is equal to $2\,576.80$ (resp. $15\,723.55$). The other columns are interpreted in the same way of the IRS sensitivities in Tables 8 and 9. Here we remark that the sensitivity for the set calibration of Table 2 and the set calibration of Table 3 vary in same way but it is different in size. This lead us to conclude that for the two calibrations converge to same way. It is sufficient to have good result for $p \geq 6$ in the approximation using sensitivities. As a result, the choice of a high level order of p can give a good sensitivities approximation which avoids the effect of any choice calibration.

For a best visualization of the error of approximation of a IRS portfolio, we present the following part.

• **Error approximation of the IRS portfolio**

It is remarkable here that when using the Set calibration of Tables 2 and 3, the error approximation for the order $p = 6$ is sufficient as written in column six. For this same order and for the set calibration 2, the error is not too far as shown in the column six. So when using a high level order $p \geq 6$, the two sets vary in the same way and the error approximation is decreasing.

13. Conclusion

In this paper we have introduced the interest rate swap (IRS) price sensitivities with respect to the underlying uncertainty two-factor related to the G2++ model which govern the interest rate structure. Then these sensitivities are used in order to obtain a three-parts decomposition of the IRS price change. Similar results for the portfolio of interest rate swaps are also derived.

In contrast with the usual Greek parameters (delta, gamma, duration, convexities, etc.), the sensitivities introduced in this work have the feature of taking into account the horizon where one wish to measure the position value change.

This paper is only devoted to the introduction and analysis of various instruments sensitivities. The results obtained here may be used as tools for hedging a position linked to the interest rates by a portfolio of IRS. The full details for that purpose will be performed in our next project.

Here we have opened a way for the development of sensitivities and hedging under a general Gaussian Affine Term Structure Model for the interest rate.

There are also important and basic interest rate instruments, as the options on zero-coupon bonds for which do not appear as linear combination of various ZCBs. It means that the sensitivities and three-parts decomposition for these nonlinear instruments would require more technical difficulties than the ones used in this paper. The details and corresponding results may be a subject of a future investigation.

As for the well established available prices of IRS, the sensitivities introduced in this work involve unobservable state variables underlying the considered G2++ model. Proxies for these last can be obtained by using observable zero-coupon prices. The common way in the literature is to make use of any filtering approach as the Kalman one or some of its variants. When considering the price or value changes at a future time, once the sensitivities are computed, it may arises the question about the views on shock levels associated with these unobservable variables at this horizon. To have views on these variables, as assumed in the robustness tests of the sensitivities introduced in this paper, is not so intuitive. As these variables follow the standard Gaussian laws, then conservative values for these shocks should be inside the interval $(-5, 5)$. But statistical considerations and numerical tests can help in forming views on extreme values for the shocks. However it should be emphasized that in the hedging application having view on shock values is not so useful, since what really matters is just the offset between the sensitivities of the position to hedge with those of the hedging instruments.

When considering the above mentioned constraints, we have avoided to use generated model ZCB prices without economical sense. But actually the weakness of the G2++ model Contents remains an issue, due to the fact that the model calibration is done on an improper manner as not discarding these ZCB prices higher than 1. Always staying in the hedging perspective, two questions can be raised: 1) what does happen if the calibration is done in a consistent manner? 2) how to modify the standard G2++ model in order to get a new one which is consistent with the possibility to deal with interest rates near zero or below?

14. Tables

Table 1: Nielson and Siegel

λ	L_0	S_0	C_0
0.08%	4.58%	−1.85%	−2.25%

Table 2: Calibration list

κ_1	κ_2	σ_1	σ_2	ρ
77.35%	8.20%	2.23%	1.04%	−70.19%

Table 3: Set Calibration 2

κ_1	κ_2	σ_1	σ_2	ρ
52.16%	7.56%	0.58%	1.16%	−98.69%

Table 4: Characteristics of the IRS

type	number	maturity	coupon_period
S^{**}	10 000 000	7 years	6 months

Table 5: Characteristics of the IRS portfolio

type	number	maturity	coupon_period
S_1^{**}	1 000 000	2 years	6 months
S_2^{**}	1 000 000	3 years	1 years
S_3^{**}	1 000 000	10 years	1 years
S_1^{*}	1 000 000	8 years	6 months

Table 6: IRS value change

ε_1	ε_2	**Exactcalib1**	**Exactcalib2**
-5	-5	-446 488.11	723 209.96
-1.5	-1.5	-176 105.01	180 751.85
0.5	0.5	-29 423.64	-147 767.20
2	2	76 982.36	-403 421.92
3.5	3.5	180 374.88	-667 283.58
5	5	280 821.28	-939 598.58
5	-5	1 013 711.90	1 068 057.70
2	-2	381 391.44	410 767.11
0	0	-65 573.81	-64 331.18
-0.5	0.5	-180 636.60	-187 975.41
-2	2	-534 059.15	-571 066.85
-5	5	-1 279 500.21	-1 394 758.01

Table 7: IRS portfolio value change

ε_1	ε_2	**Exactcalib1**	**Exactcalib2**
-5	-5	-50483.9	59015.65
-1.5	-1.5	-14589.1	18517.18
0.5	0.5	5000.355	-5985.92
2	2	19265.68	-25058.3
3.5	3.5	33173.12	-44756.1
5	5	46729.64	-65107.6
-5	5	69103.05	87438.69
-2	2	28830.87	36707.72
0	0	164.49	237.21
0.5	-0.5	-7248.31	-9237.42
2	-2	-30113.2	-38566.9
5	-5	-78877.9	-101598

Table 8: IRS Sensitivities using Set calibration 1

S1	**S2**	**S3**	**S5**	**S6•10^3**	**S7•10^5**	**S12•10^{13}**
78065.42	-183.25	1.82	0.00	0.00	0.00	0.00
-150709.95	1999.40	-14.94	0.00	0.01	0.00	0.00
--	-3196.32	53.27	0.01	-0.05	0.04	0.00
--	--	-60.88	-0.02	0.21	-0.21	0.00
--	--	--	0.03	-0.53	0.70	-0.01
--	--	--	-0.02	0.72	-1.41	0.03
--	--	--	--	-0.41	1.59	-0.12
--	--	--	--	--	-0.77	0.34
--	--	--	--	--	--	-0.72
--	--	--	--	--	--	1.08
--	--	--	--	--	--	-1.09
--	--	--	--	--	--	0.67
--	--	--	--	--	--	-0.19

Table 9: IRS Sensitivities using Set calibration 2

S1	**S2**	**S3**	**S5**	**S6•10^3**	**S7•10^5**	**S12•10^{12}**
205647.24	-5486.29	135.57	0.08	-2.04	5.05	-0.47
-39654.64	2200.41	-81.76	-0.08	2.46	-7.11	1.14
--	-220.33	16.44	0.03	-1.24	4.281	-1.26
--	--	-1.10	-0.01	0.33	-1.43	0.84
--	--	--	0.00	-0.05	0.29	-0.38
--	--	--	0.00	0.00	-0.033	0.12
--	--	--	--	0.00	0.00	-0.03
--	--	--	--	--	0.00	0.00
--	--	--	--	--	--	0.00
--	--	--	--	--	--	0.00
--	--	--	--	--	--	0.00
--	--	--	--	--	--	0.00
--	--	--	--	--	--	0.00

Table 10: IRS Portfolio Sensitivities using Set calibration 1

S1	**S2**	**S3**	**S5**$\bullet 10^2$	**S6**$\bullet 10^4$	**S7**$\bullet 10^5$	**S12**$\bullet 10^{13}$
2576.80	-54.33	0.72	0.01	-0.01	0.00	0.00
-12308.38	121.99	-4.6	-0.12	0.17	-0.02	0.00
--	-231.39	7.13	0.50	-0.91	0.15	-0.01
--	--	-5.37	-1.03	2.56	-0.54	0.06
--	--	--	1.01	-4.02	1.14	-0.31
--	--	--	-0.40	3.26	-1.45	1.11
--	--	--	--	-1.08	1.00	-2.85
--	--	--	--	--	-0.29	5.38
--	--	--	--	--	--	-7.38
--	--	--	--	--	--	7.17
--	--	--	--	--	--	-4.68
--	--	--	--	--	--	1.84
--	--	--	--	--	--	-0.33

Table 11: IRS Portfolio Sensitivities using Set calibration 2

S1	**S2**	**S3**	**S5**$\bullet 10^2$	**S6**$\bullet 10^4$	**S7**$\bullet 10^5$	**S12**$\bullet 10^{12}$
15723.55	-411.26	14.51	2.16	-8.06	2.94	-1.59
-3234.92	162.25	-8.27	-2.04	9.19	-3.93	3.66
--	-16.17	1.58	0.77	-4.36	2.24	-3.86
--	--	-0.00	-0.15	1.11	-0.71	2.46
--	--	--	0.01	-0.16	0.14	-1.06
--	--	--	0.00	0.01	-0.02	0.33
--	--	--	--	0.00	0.00	-0.07
--	--	--	--	--	0.00	0.01
--	--	--	--	--	--	0.00
--	--	--	--	--	--	0.00
--	--	--	--	--	--	0.00
--	--	--	--	--	--	0.00
--	--	--	--	--	--	0.00

Table 12: Error approximation order for IRS using Set Calibration1

ε_1	ε_2	**Er1**	**Er2**	**Er6**$\bullet 10^4$	**Er9**$\bullet 10^9$	**Er12**$\bullet 10^9$
-5	-5	-17691.68	-439.61	-0.11	1.11	1.11
0.5	0.5	-172.09	0.43	0.00	0.33	0.33
2	2	-2732.88	27.45	0.00	-0.25	-0.25
5	5	-16827.53	424.54	0.11	-1.05	-1.05
-5	5	-64591.12	2645.85	7.22	-3.73	-1.86
-2	2	-10585.49	172.43	0.01	1.16	1.16
0	0	0	0	0	0	0
2	-2	-10934.61	-176.69	-0.01	0.35	0.35
5	-5	-70049.57	-2812.60	-7.44	-1.86	0.00

Table 13: Error approximation order for IRS using Set Calibration2

ε_1	ε_2	**Er1**	**Er2**	**Er6**$\bullet 10^3$	**Er9**$\bullet 10^8$	**Er12**$\bullet 10^9$
-5	-5	-42421.87	1405.78	0.16	-0.07	-0.47
0.5	0.5	-439.72	-1.44	0.00	-0.06	-0.58
2	2	-7105.54	-93.11	0.00	0.08	0.76
5	5	-45304.39	-1476.74	-0.17	-0.12	-0.93
-2	2	-15505.48	308.58	0.00	-0.01	-0.06
0	0	0	0	0	0	0
0.5	-0.5	-993.29	-4.91	0.00	0.07	0.67
2.5	-2.5	-25332.58	-623.11	-0.02	0.01	0.12
5	-5	-103917.41	-5079.55	-2.87	-1.28	0.23

Table 14: Error approximation order of the IRS portfolio using Set Calibration 1

ε_1	ε_2	**Er1**	**Er2**	**Er6**$\bullet 10^3$	**Er9**$\bullet 10^8$	**Er12**$\bullet 10^9$
-5	-5	-2086.08	-45.15	0.00	-0.01	-0.04
-1.5	-1.5	-184.89	-1.21	0.00	-0.01	-0.09
2	2	-323.72	2.83	0.00	-0.01	-0.10
5	5	-1997.12	43.81	0.00	-0.02	-0.17
-5	5	-4682.55	349.70	0.69	-0.64	-0.18
-2	2	-782.07	23.09	0.00	0.05	0.52
0	0	0	0	0	0	0
0.5	-0.5	-50.69	-0.37	0.00	-0.01	-0.09
2	-2	-829.26	-24.10	0.00	0.02	0.15
5	-5	-5421.31	-389.06	-0.73	-0.65	-0.14

Table 15: Error approximation order of the IRS portfolio using Set Calibration 2

ε_1	ε_2	**Er1**	**Er2**	**Er6**$\bullet 10^3$	**Er9**$\bullet 10^8$	**Er12**$\bullet 10^9$
-5	-5	-3123.68	153.21	0.10	-0.06	-0.13
-1.5	-1.5	-290.67	4.25	0.00	0.02	0.18
2	2	-534.66	-10.36	0.00	-0.01	-0.10
3.5	3.5	-1661.87	-56.19	-0.01	0.01	0.08
5	5	-3442.70	-165.81	-0.11	-0.07	-0.30
-5	5	-6811.02	476.85	1.50	-2.00	-0.04
-2	2	-1134.49	31.57	0.00	-0.02	-0.19
0	0	0	0	0	0	0
0.5	-0.5	-73.39	-0.51	0.00	-0.02	-0.16
2	-2	-1199.11	-33.05	0.00	0.00	-0.03
5	-5	-7822.94	-535.07	-1.58	-2.07	0.11

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