International Journal of Applied Mathematics

Volume 30 No. 5 2017, 425-435
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v30i5.7

SOME FORMS FOR r^{th} MOVING MAXIMA OF ITERATED LOGARITHM LAW

Bader Almohaimeed

Department of Mathematics
College of Science
Qassim University
Buraydah 51431, P.O. Box 707, SAUDI ARABIA

Abstract: Let $\eta_{r,n}$ be a sequence of independent random variables, which is identically distributed and is defined over common probability space $(\Omega, \mathcal{F}, \mathcal{A})$ for a continuous distribution function F. Let $\eta_{r,n}$ denote the r^{th} upper order statistic between $(X_{n-a_n+1}, X_{n-a_n+2}, ..., X_n)$, for $n \geq 1$ with sequence (a_n) of integers, which is non-decreasing for $0 \leq a_n \leq n$. In this paper, some forms of iterated logarithm law for $\eta_{r,n}$ are obtained.

AMS Subject Classification: 60F15
Key Words: moving maxima, iterated logarithm law, functions varying regularly

1. Introduction

Let (X_n) be a sequence of random variables (r.vs) with independent identically distributed (i.i.d) terms defined over common probability space $(\Omega, \mathcal{F}, \mathcal{A})$, and the common distribution (d.f) F be continuous. Let us represent $r(F)$ as right extremity of F. Observe that if for all real x, $F(x) < 1$, then $r(f) = \infty$. On the same space, define a sequence (U_n) of uniform $(0, 1)$ r.vs. If $(M_{r,n})$ denotes the r^{th} largest among $X_1, X_2, ..., X_n$ and $M^*_{r,n}$ denotes the largest among

Received: October 11, 2017

$U_1, U_2, ..., U_n$, then $(M_{r,n})$ and $M^*_{r,n}$ are called the upper order statistic among $X_1, X_2, ..., X_n$ and the upper order statistic of $U_1, U_2, ..., U_n$, respectively.

Consider a sequence of integers $\{a_n\}$ which is non-decreasing, for $0 < a_n \leq n$ and let $\eta_{r,n}$ denote the largest among $(X_{n-a_n+1}, X_{n-a_n+2}, ..., X_n)$, and $\eta^*_{r,n}$ denote the r^{th} largest among $(U_{n-a_n+1}, U_{n-a_n+2}, ..., U_n)$. Then, $\eta_{r,n}$ is the r^{th} upper order statistic among $(X_1, X_2, ..., X_n)$ and $\eta^*_{r,n}$ the r^{th} upper order statistics of $(U_{n-a_n+1}, U_{n-a_n+2}, ..., U_n)$ and $\eta_{r,n}$ may be called the r^{th} moving maxima.

Through this paper, we assume $\{a_n\}$ is non-decreasing and $a_n/n \sim b_n$, with b_n is non-decreasing as smooth condition. Moreover, we assume that $a_n/\log n \to \infty$ as $n \to \infty$. Also i.o and a.s mean infinitely often and almost surely. For any $\lambda > 0$, $[\lambda]$ stands for the greatest integer less than or equal to λ. With suffix or without, we represent constants N (integer) and C as positive.

Barndorff-Nielson [1] established

$$\limsup_{n \longrightarrow \infty} \frac{n(1 - M^*_{1,n})}{\log \log n} = 1, \quad a.s. \tag{1}$$

The result in (1) is generalized by Rothman-Russo [2] with the conditions on (a_n) for certain classes, to moving maxima $\eta^*_{1,r}$. Using the smoothness conditions on (a_n) stated above, Vasudeva [9] has observed that

$$\limsup_{n \longrightarrow \infty} \frac{n(1 - \eta^*_{1,n})}{\beta_n} = 1, \quad a.s. \tag{2}$$

for $\beta_n = \log \frac{n}{a_n} + log \log n$.

Bahram and Benchikh [3] established that

$$\limsup_{n \longrightarrow \infty} \frac{n(1 - \eta^*_{1,n})}{\beta_n(\alpha)} = 1, \quad a.s., \tag{3}$$

where $\beta_n(\alpha) = \log \frac{n}{a_n} + (1 - \alpha) \log \log a_n + \alpha \log \log n$ for $0 \leq \alpha \leq 1$.

In this paper, we establish Barndoff-Nielson's form of the L.I.L., for $\eta_{r,n}$, using the construction of Vasudeva and Moridani [4].

For our convenience, in extreme value theory C_1 and C_2 are represented as two major classes. We denote L.I.L., for $\{\eta_{r,n}\}$, which is normalized properly for d.f.s F and belongs to C_1 and C_2. The class C_1 is for all d.f.s F with $-\log \bar{F} = x^\gamma L(x)$, with $x \to \infty$ for any constant $\gamma > 0$ and function $L(x)$ is a slowly varying. The distributions along Weibullian right tail (including Normal, Exponential, Gumbel etc.) are contained in this class. Following [5], it is observed that distributions along Weibullian tail contain in domain for attraction in Gumbel law, for $0 < \gamma < 1$. Moreover, we observe that when F is Normal ($\gamma = 2$) or Exponential ($\gamma = 1$), $\{M_n\}$ converges properly normalized to Gumbell r.v [6]. The C_2 class is for all d.f.s along $\bar{F}(x) = x^{-\gamma}L(x)$, with $x \to \infty$, for any constant $\gamma > 0$ and function $L(x)$ is a slowly varying. Galambos [6] observed that the class C_2 of all d.f.s contains in the domain of attraction of Fréchet law. For $F \in C_1$, we define $U(x) = -\log(1 - F(x)), \quad x > 0$. Also denoted is V as the inverse function of $U(x)$. If $U(x) = x^{\gamma}L(x)$, following [7], it is observed that for every functions $a(.)$, $0 \neq a(x) \to 0$, as $x \to \infty$,

$$\frac{V(x(1 + a(x))) - V(x)}{a(x)V(x)} \to \gamma^{-1} \quad \text{as} \quad x \to \infty. \tag{4}$$

This follows that for x large enough, V is continuous and varying regularly along exponent γ^{-1}. For $F \in C_2$, we define $U^*(x) = 1 - F(x),\ x > 0$. Also, we observe that $U^*(x) = x^{-\gamma}L(x)$, for any constant $\gamma > 0$ and function L is slowly varying. Suppose that the inverse of U^* is represented by V^*. Note that, $V^*(y) = y^{-\frac{1}{\gamma}}l(\frac{1}{y})$, $0 < y \leq 1$ with l is varying slowly. Note that the functions $U^*(.)$ and $V^*(.)$ are decreasing. Recently, Vasudeva and Srilakshminarayana [8] established the following theorems.

Theorem 1. *Let $F \in C_1$. Then*

$$\liminf_{n\to\infty} \gamma(\log a_n - \log\beta_n)\left(\frac{\eta_{r,n}}{V(\log a_n - \log\beta_n)} - 1\right) = 0 \qquad a.s.,$$

for $\beta_n = \log\frac{n}{a_n} + \log\log n$, *with* $0 \leq \alpha \leq 1$.

Theorem 2. *Let $F \in C_2$. Then*

$$\liminf_{n\to\infty} \frac{\eta_{r,n}}{V^*\left(\frac{\beta_n}{a_n}\right)} = 1 \quad a.s.,$$

for $\beta_n = \log\frac{n}{a_n} + \alpha\log\log n$, *with* $0 \leq \alpha \leq 1$.

2. Main Results

Our main purpose in this paper is to extend the Vasudeva and Srilakshminarayana's theorem by using $\beta_n(\alpha) = \log\frac{n}{a_n} + (1-\alpha)\log\log a_n + \alpha\log\log n$ for $0 \leq \alpha \leq 1$.

Theorem 3. *Let $F \in C_1$. Then*

$$\liminf_{n\to\infty} \gamma(\log a_n - \log \beta_n(\alpha)) \left(\frac{\eta_{r,n}}{V(\log a_n - \log \beta_n(\alpha)} - 1 \right) = 0 \quad a.s.$$

Remark 2.1. Let us mention some particular cases:

1. For $a_n = n$ and $r = 1$, $\eta_{1,n}$ coincides with the partial maxima, i.e., with $M_{1,n}$. The L.L.I. in Theorem 3, reduces to

$$\liminf_{n\to\infty} \gamma(\log n - \log_3 n) \left(\frac{M_{r,n}}{V(\log n - \log_3 n)} - 1 \right) = 0 \quad a.s.,$$

where $\log_3 = \log\log\log n$.

2. If $\alpha = 1$, we have Theorem 1.

3. If $\alpha = 0$, we also have

$$\liminf_{n\to\infty} (\log a_n - \log \beta_n(0)) \left(\frac{\eta_{r,n}}{V(\log a_n - \log \beta_n(0))} - 1 \right) = 0 \quad a.s.,$$

where $\beta_n(0) = \log \frac{n}{a_n} + \log\log a_n$.

Theorem 4. *Let $F \in C_2$. Then*

$$\liminf_{n\to\infty} \frac{\eta_{r,n}}{V^* \left(\frac{\beta_n(\alpha)}{a_n} \right)} = 1 \quad a.s.$$

Remark 2.2. 1. For $a_n = n$, the above theorem gives:

$$\liminf_{n\to\infty} \frac{M_{r,n}}{V^* \left(\frac{\log\log n}{n} \right)} = 1 \quad a.s.$$

2. If $\alpha = 1$, we have Theorem 2.

3. If $\alpha = 0$, Theorem 4 also implies the following result

$$\liminf_{n\to\infty} \frac{\eta_{r,n}}{V^* \left(\frac{\beta_n(0)}{a_n} \right)} = 1 \quad a.s.$$

We need two lemmas to prove our results. We first present the following Borel-Cantelli lemma, which is presented by Barndorff-Nielson [1].

Lemma 2.1. (See [1]) *Let $\{A_n\}$ be a sequence of events defined over a probability space such that $P(A_n) \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$. Then, $P(A_n i.0.) = 0$.*

The following lemma is crucial from the context of this paper.

Lemma 2.2.
$$\limsup_{n \longrightarrow \infty} \frac{a_n(1-\eta^*_{r,n})}{\beta_n(\alpha)} = 1 \quad a.s.,$$
for $\beta_n(\alpha) = \log \dfrac{n}{a_n} + (1-\alpha)\log_2 a_n + \alpha \log_2 n$ with $0 \leq \alpha \leq 1$.

Proof of Lemma 2.2. Equivalently, we show that for any given $\varepsilon \in (0,1)$,
$$P\left(\frac{a_n(1-\eta^*_{r,n})}{\beta_n(\alpha)} > 1+\varepsilon \quad i.o\right) = 0, \tag{5}$$
and
$$P\left(\frac{a_n(1-\eta^*_{r,n})}{\beta_n(\alpha)} > 1-\varepsilon \quad i.o\right) = 1. \tag{6}$$
We have:
$$P\left(\frac{a_n(1-\eta^*_{r,n})}{\beta_n(\alpha)} > 1+\varepsilon\right) = P\left(\eta^*_{r,n} < 1 - \frac{\beta_n(\alpha)(1+\varepsilon)}{a_n}\right)$$
$$= \quad \sum_{j=0}^{r} \binom{a_n}{j} \left((1+\varepsilon)\frac{\beta_n(\alpha)}{a_n}\right)^j \left(1-(1+\varepsilon)\frac{\beta_n(\alpha)}{a_n}\right)^{a_n-j}.$$
From the fact that $\frac{a_n}{\log n} \to \infty$, note that $\frac{\beta_n(\alpha)}{a_n} \to 0$ and $\left(1-(1+\varepsilon)\frac{\beta_n(\alpha)}{a_n}\right)^{a_n-j} \sim e^{-(1+\varepsilon)\beta_n(\alpha)}$ as $n \to \infty$, $0 \leq j \leq r$. Consequently, for all $n \geq N_1$, and N_1 large enough,
$$\sum_{j=0}^{r} \binom{a_n}{j} \left((1+\varepsilon)\frac{\beta_n(\alpha)}{a_n}\right)^j \left(1-(1+\varepsilon)\frac{\beta_n(\alpha)}{a_n}\right)^{a_n-j}$$
$$\leq 2\beta_n^r(\alpha)e^{-(1+\varepsilon)\beta_n(\alpha)}.$$

In turn, for all $n \geq N_1$, we have

$$
\begin{aligned}
P\left(\frac{a_n(1-\eta^*_{r,n})}{\beta_n(\alpha)} > 1+\varepsilon\right) &\leq 2\beta_n^r(\alpha)e^{-(1+\varepsilon)\beta_n(\alpha)} \\
&= \frac{2\left(\log\left(\frac{n(\log n)^{\alpha}(\log a_n)^{(1-\alpha)}}{a_n}\right)\right)^r}{\left(\frac{n(\log n)^{\alpha}(\log a_n)^{(1-\alpha)}}{a_n}\right)^{1+\varepsilon}}.
\end{aligned}
$$

Since $\frac{\left(\log\left(\frac{n(\log n)^{\alpha}(\log a_n)^{(1-\alpha)}}{a_n}\right)\right)^r}{\left(\frac{n(\log n)^{\alpha}(\log a_n)^{(1-\alpha)}}{a_n}\right)^{\frac{\varepsilon}{2}}} \to 0$ as $n \to \infty$, we may get a $N_2(\geq N_1)$ such that for all $n \geq N_2$,

$$
\begin{aligned}
P\left(\frac{a_n(1-\eta^*_{r,n})}{\beta_n(\alpha)} > 1+\varepsilon\right) &\leq \left(\frac{a_n}{n(\log n)^{\alpha}(\log a_n)^{(1-\alpha)}}\right)^{1+\frac{\varepsilon}{2}} \\
&= \left(\frac{a_n}{n}\left(\left(\frac{\log n}{\log a_n}\right)^{1-\alpha}\frac{1}{\log n}\right)\right)^{1+\frac{\varepsilon}{2}} \\
&\leq \left(\frac{a_n}{n}\left(\left(\frac{\log n}{\log a_n}\right)\frac{1}{\log n}\right)\right)^{1+\frac{\varepsilon}{2}} \\
&= \left(\frac{a_n}{n}\left(\left(\frac{\log n}{\log a_n}\right)\frac{1}{\log n}\right)\right)^{1+\frac{\varepsilon}{2}} \\
&= \left(\frac{a_n}{n\log a_n}\right)^{1+\frac{\varepsilon}{2}}.
\end{aligned} \tag{7}
$$

Define $A_n = \left(\frac{a_n(1-\eta^*_{r,n})}{\beta_n(\alpha)} > 1+\varepsilon\right)$. Remark that $P(A_n) \to 0$ as $n \to \infty$. Observe that

$$
A_n \cap A^c_{n+1} \subseteq \left(\eta^*_{r,n} < 1 - \frac{(1+\varepsilon)\beta_n(\alpha)}{a_n}\right) \cap \left(X_{n+1} > 1 - \frac{(1+\varepsilon)\beta_{n+1}(\alpha)}{a_{n+1}}\right).
$$

Hence, for all $n \geq N_2$,

$$
\begin{aligned}
P(A_n \cap A^c_{n+1}) &= P(A_n)(1+\varepsilon)\frac{\beta_{n+1}(\alpha)}{a_{n+1}} \\
&\leq \left(\frac{a_n}{n\log a_n}\right)^{(1+\frac{\varepsilon}{2})}(1+\varepsilon)\frac{\beta_{n+1}(\alpha)}{a_{n+1}} \\
&\leq \left(\frac{a_n}{n\log a_n}\right)^{(1+\frac{\varepsilon}{2})}(1+\varepsilon)\frac{\beta_{n+1}}{a_{n+1}},
\end{aligned}
$$

since $\frac{a_n}{a_{n+1}} \leq 1$. Let $u_n = \frac{n\log n}{a_n}$. Using $(n+1)\log(n+1) \leq 2n\log(n)$ for large n, we may get a N_3 and a c_1 with $\beta_{n+1} \leq c_1\beta_n$ for all $n \geq N_3$. Consequently, for $n \geq N_3$:

$$\begin{aligned} P(A_n \cap A_{n+1}^c) &\leq c_1 \left(\frac{a_n}{n \log a_n}\right)^{\frac{\varepsilon}{4}} \frac{\beta_n}{n(\log a_n)^{1+\frac{\varepsilon}{4}}} \\ &\leq c_1 \left(\frac{a_n}{n \log a_n}\right)^{\frac{\varepsilon}{4}} \frac{\log u_n}{n(\log a_n)^{1+\frac{\varepsilon}{4}}} \\ &= c_1 \left(\frac{a_n}{n \log n}\right)^{\frac{\varepsilon}{4}} \left(\frac{\log n}{\log a_n}\right)^{\frac{\varepsilon}{4}} \frac{\log u_n}{n(\log a_n)^{1+\frac{\varepsilon}{4}}} \\ &= c_1 \frac{\log u_n}{u_n^{\frac{\varepsilon}{4}}} \left(\frac{\log n}{\log a_n}\right)^{\frac{\varepsilon}{4}} \frac{1}{n(\log a_n)^{1+\frac{\varepsilon}{4}}} \\ &= c_1 \frac{\log u_n}{u_n^{\frac{\varepsilon}{4}}} \left(\frac{\log n}{\log a_n}\right) \frac{1}{n(\log a_n)^{1+\frac{\varepsilon}{4}}}. \end{aligned}$$

Using $\dfrac{\log u_n}{u_n^{\frac{\varepsilon}{4}}} \longrightarrow 0$ as $n \longrightarrow \infty$, one can find a N_4 such that for all $n \geq N_4$,

$$P(A_n \cap A_{n+1}^c) \leq c_3 \left(\frac{\log n}{\log a_n}\right) \frac{1}{n(\log a_n)^{1+\frac{\varepsilon}{4}}}.$$

Let $a_n = [n^p]$, $0 < p < 1$, one can find a N_5 such that for all $n \geq N_5$,

$$P(A_n \cap A_{n+1}^c) \leq c_4 \frac{1}{n(\log n)^{1+\frac{\varepsilon}{4}}}.$$

Consequently, $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$. Recalling $P(A_n) \to 0$ as $n \to \infty$ and applying Lemma 2.1, (5) is established. We now prove (6). From Bahram and Benchikh [3], we have

$$P\left(\frac{a_n(1-\eta_{1,n}^*)}{\beta_n(\alpha)} > 1 - \varepsilon\right) = 0. \tag{8}$$

Since $\eta_{r,n}^* \leq \eta_{1,n}^*$, one can trivially see that

$$\frac{a_n(1-\eta_{r,n})}{\beta_n(\alpha)} > \frac{a_n(1-\eta_{1,n}^*)}{\beta_n(\alpha)}.$$

In turn, (8) implies (6).

3. Proofs of the Theorems

Given that (X_n) is a sequence of i.i.d. r.vs. with a common continuous d.f. F define $U_n = F(X_n)$, $n \geq 1$, and observe that $\{U_n\}$ is a sequence of i.i.d. Uniform $(0,1)$ r.vs. Recall that $\eta_{r,n}$ is the r^{th} maxima of $X_{n-a_n+1}, ..., X_n$ and that $\eta^*_{r,n}$ the r^{th} maxima of $U_{n-a_n+1}, ..., U_n$. Note the relation $\eta^*_{r,n} = F(\eta_{r,n})$.

Proof of Theorem 3. We show that for $0 < \varepsilon < 1/2$,

$$P\left(\gamma(\log a_n - \log \beta_n(\alpha))\left(\frac{\eta_{r,n}}{V(\log a_n - \log \beta_n(\alpha))} - 1\right)\right.$$

$$\left. < \log \frac{1}{1+\frac{\varepsilon}{2}} \quad i.o.\right) = 0 \tag{9}$$

and

$$P\left(\gamma(\log a_n - \log \beta_n(\alpha))\left(\frac{\eta_{r,n}}{V(\log a_n - \log \beta_n(\alpha)} - 1\right)\right.$$

$$\left. < \log \frac{1}{1-\frac{\varepsilon}{2}} \quad i.o.\right) = 1. \tag{10}$$

From Lemma 2.1 we have:

$$P\left(1 - \eta^*_{r,n} > \frac{\beta_n(\alpha)}{a_n}(1+\varepsilon) \quad i.o.\right) = 0 \tag{11}$$

and

$$P\left(1 - \eta^*_{r,n} > \frac{\beta_n(\alpha)}{a_n}(1-\varepsilon) \quad i.o.\right) = 1. \tag{12}$$

Using the same arguments as in Vasudeva and Moridani [4], we have

$$\begin{aligned}
1 - \eta^*_{r,n} &> \frac{\beta_n(\alpha)}{a_n}(1+\varepsilon) \\
&\Leftrightarrow -\log(1 - F(\eta_{r,n})) < -\log\left(\frac{\beta_n(\alpha)}{a_n}(1+\varepsilon)\right) \\
&\Leftrightarrow U(\eta_{r,n}) < -\log\left(\frac{\beta_n(\alpha)}{a_n}(1+\varepsilon)\right)
\end{aligned}$$

$$\Leftrightarrow \eta_{r,n} < V\left(-\log(\frac{\beta_n(\alpha)}{a_n}) + \log(1+\varepsilon)^{-1}\right)$$

$$\Leftrightarrow \eta_{r,n} < V\left((loga_n - \log\beta_n(\alpha))\left(1 + \frac{\log(1+\varepsilon)^{-1}}{loga_n - \log\beta_n(\alpha)}\right)\right)$$

$$\Leftrightarrow \eta_{r,n} - V(\log(a_n/\beta_n(\alpha)))$$

$$< V\left((log(a_n/\beta_n(\alpha))))\left(1 + \frac{\log(1+\varepsilon)^{-1}}{loga_n - \log\beta_n(\alpha)}\right)\right)$$

$$- V(log(a_n/\beta_n(\alpha))).$$

Using equation (4) one can find a $\delta > 0$ such that for all n large,

$$\eta_{r,n} - V(log(a_n - \log\beta_n(\alpha)))$$

$$< \gamma^{-1}(1-\delta)\frac{\log(1+\varepsilon)^{-1}}{loga_n - \log\beta_n(\alpha)}V(loga_n - \log\beta_n(\alpha))$$

$$\Leftrightarrow \left(\frac{\eta_{r,n}}{V(loga_n - \log\beta_n(\alpha))} - 1\right) < \gamma^{-1}(1-\delta)\frac{\log(1+\varepsilon)^{-1}}{loga_n - \log\beta_n(\alpha)}.$$

Choose δ such that $(1-\delta)\log(1-\varepsilon)^{-1} = \log(1+\frac{\varepsilon}{2})^{-1}$ for n large. Then we have

$$1 - \eta^*_{r,n} > \frac{\beta_n(\alpha)}{a_n}(1+\varepsilon)$$

$$\Leftrightarrow \left(\frac{\eta_{r,n}}{V(loga_n - \log\beta_n(\alpha))} - 1\right) < \gamma^{-1}\frac{\log(1+\frac{\varepsilon}{2})^{-1}}{loga_n - \log\beta_n(\alpha)},$$

$$\text{or} \quad \gamma(loga_n - \log\beta_n(\alpha))\left(\frac{\eta_{r,n}}{V(loga_n - \log\beta_n(\alpha))} - 1\right) < \log(1+\frac{\varepsilon}{2})^{-1}.$$

From (11), we hence have (9). Proceeding on similar lines one can show (10) from (12). The details are omitted.

Proof of Theorem 4. From Lemma 2.2, we have

$$P\left(1 - \eta^*_{r,n} > \frac{\beta_n(\alpha)}{a_n}(1+\varepsilon) \quad i.o.\right) = 0 \tag{13}$$

and

$$P\left(1 - \eta^*_{r,n} > \frac{\beta_n(\alpha)}{a_n}(1-\varepsilon) \quad i.o.\right) = 1. \tag{14}$$

Using the relations

$$\eta^*_{r,n} = F(\eta_{r,n}) \quad \text{and} \quad U^*(x) = 1 - F(x) = x^{-\gamma}L(x),$$

where L is slowly varying, from (13) we get

$$P\left(U^*(\eta_{r,n}) > \frac{\beta_n(\alpha)}{a_n}(1+\varepsilon) i.o.)\right) = 0. \tag{15}$$

Note that

$$U^*(\eta_{r,n}) > \frac{\beta_n(\alpha)}{a_n}(1+\varepsilon) \Leftrightarrow V^*(U^*(\eta_{r,n})) < V^*\left(\frac{\beta_n(\alpha)}{a_n}(1+\varepsilon)\right)$$

$$\Leftrightarrow \eta_{r,n} < a_n^{\frac{1}{\gamma}}(\beta_n(\alpha)(1+\varepsilon))^{-\frac{1}{\gamma}} l\left(\frac{a_n}{(\beta_n(\alpha)(1+\varepsilon)}\right)$$

$$\Leftrightarrow \eta_{r,n} < \left(\frac{\beta_n(\alpha)}{a_n}\right)^{-\frac{1}{\gamma}} l\left(\frac{1}{1+\varepsilon}\frac{1}{\frac{\beta_n(\alpha)}{a_n}}\right)(1+\varepsilon)^{-\frac{1}{\gamma}}$$

$$\Leftrightarrow \eta_{r,n} < V^*\left(\frac{\beta_n(\alpha)}{a_n}\right)(1+\varepsilon)^{-\frac{1}{\gamma}}.$$

Hence we get

$$P\left(\frac{\eta_{r,n}}{V^*\left(\frac{\beta_n(\alpha)}{a_n}\right)} < \frac{1}{(1+\varepsilon)^{\frac{1}{\gamma}}} \quad i.o.\right) = 0.$$

Similarly from (14) we get

$$P\left(\frac{\eta_{r,n}}{V^*\left(\frac{\beta_n(\alpha)}{a_n}\right)} > \frac{1}{(1-\varepsilon)^{\frac{1}{\gamma}}} \quad i.o.\right) = 1.$$

Hence the theorem is proved.

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