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# A UNIFORM ABSOLUTE CONTINUITY OF INTEGRAL RESULT IN $L^{p(x)}$

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**Abstract:** In this paper we prove a uniform absolute continuity of integral result in variable exponent Lebesgue space. The idea of our proof is similar to that for the classical Lebesgue space.

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**Key Words:** variable exponent Lebesgue spaces, uniform absolute continuity of integral, modular

## 1. Introduction

The field of variable exponent Lebesgue and Sobolev spaces is in active development at present, and has found many important applications (see for example, the books [1] and [2], and the references there). One of the reasons for the huge development of the theory of classical Lebesgue and Sobolev spaces $L^p$ and $W^{k,p}$ is the description of many phenomena arising in applied sciences. For instance, many materials can be modeled with sufficient accuracy using the function spaces $L^p$ and $W^{k,p}$, where $p$ is a fixed constant. For some nonhomogeneous materials, for instance electrorheological fluids, this approach is not adequate, but rather the exponent $p$ should be allowed to vary. This leads us to the study of variable exponent Lebesgue and Sobolev spaces $L^{p(x)}$ and $W^{k,p(x)}$, respectively, where $p$ is a real-valued function.

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Spaces of variable exponent can be traced back to Orlicz [5], but the current investigation goes back to a paper Kováčik and Rákosník [3]. The basic properties of these spaces can be found in the paper [3]; many of these properties were independently established by Fan and Zhao [4].

## 2. Variable Exponent Lebesgue Spaces

By $\Omega$ we denote a non empty subset of $\mathbb{R}^n$. Let $p:\Omega\to[1,\infty)$ be a measurable function, called a variable exponent on $\Omega$ and denote $p^+ = \text{ess sup}\, p(x)$ and $p^- = \text{ess inf}\, p(x)$. We define the variable exponent Lebesgue $L^{p(x)}(\Omega)$ space to consist of all measurable functions $u:\Omega\to\mathbb{R}$ for which the modular

$$\varrho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx$$

is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda>0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

Equipped with this norm, $L^{p(x)}$ is a Banach space. When $p(x) = p$ is a constant then $L^{p(x)}(\Omega)$ coincides with the classical Lebesgue space $L^p(\Omega)$. One important property of $L^{p(x)}(\Omega)$ is that

$$\int_\Omega |u_j(x)|^{p(x)}\,dx \to 0$$

if and only if $\|u_j(x)\|_{L^{p(x)}(\Omega)} \to 0$, so that the norm and modular topologies are the same. In a classical Lebesgue space the relation between the norm and modular $\varrho_p(u)$ is obtained directly: $\|u\|_{L^p(\Omega)} = (\varrho_p(u))^{\frac{1}{p}}$. However, in the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ the relation is given by the following inequalities:

If $0 < \|u\|_{L^{p(x)}(\Omega)} \le 1$, then

$$\left(\varrho_{p(.)}(u)\right)^{\frac{1}{p^-}} \le \|u\|_{L^{p(x)}(\Omega)} \le \left(\varrho_{p(.)}(u)\right)^{\frac{1}{p^+}}.$$

If $\|u\|_{L^{p(x)}(\Omega)} > 1$, then

$$\left(\varrho_{p(.)}(u)\right)^{\frac{1}{p^+}} \le \|u\|_{L^{p(x)}(\Omega)} \le \left(\varrho_{p(.)}(u)\right)^{\frac{1}{p^-}}.$$

The Hölder inequality, i.e.

$$\|u(x)\,v(x)\|_{L^1(\Omega)} \leq\ 2\,\|u(x)\|_{L^{p(x)}(\Omega)}\ \|v(x)\|_{L^{p'(x)}(\Omega)}$$

holds. If $0 <$meas$(\Omega) < \infty$ and $p(x) \leq q(x)$ in $\Omega$, then there exists an imbedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ whose norm does not exceed meas$(\Omega)+1$. $L^{p(x)}(\Omega)$ are reflexive if and only if $1 < p^- < p^+ < \infty$, smooth functions are dense if $p^+ < \infty$. Some basic properties of the classic Lebesgue spaces are not transferred to the variable exponent case. For example, the variable exponent Lebesgue space is no longer translation invariant. As a consequence, Young's theorem and the so called mean continuity property fail in general. Many other basic results were proven in [3].

The proof of the following theorem follows the pattern in the classical Lebesgue spaces.

**Theorem 1.** *Let* $q(x) \in (0,\infty)$, $\{u_\alpha : \alpha \in A\} \subset L^{p(x)}(\Omega)$ *such that* $\sup_{\alpha\in A}\int\limits_\Omega |u_\alpha(x)|^{p(x)}\,dx = c < \infty$. *Let* $0 < p(x) < q(x)$ *and* $q(x) - p(x) \geq \beta > 0$. *Then,*

*i)*

$$\lim_{\theta\to\infty}\sup_{\alpha\in A}\int\limits_{\left\{x\in\Omega:|u_\alpha|^{p(x)}>\theta\right\}} |u_\alpha(x)|^{p(x)}\,dx = 0.$$

*ii) For every* $\epsilon > 0$ *there exist* $\delta > 0$ *such that for every* $E \subset \Omega$ *with meas*$(E) < \delta$ *have*

$$\int\limits_E |u_\alpha(x)|^{p(x)}\,dx < \epsilon \quad \text{for all } \alpha \in A.$$

*Proof.* i) If $0 < \omega < \sigma$ then $\sigma^{p(x)} \leq \omega^{p(x)-q(x)}\sigma^{q(x)}$ holds. By using this inequality in the following integral, we obtain

$$\int\limits_{\left\{x\in\Omega:|u_\alpha|^{p(x)}>\omega^{p^+}\right\}} |u_\alpha(x)|^{p(x)}\,dx \leq \int\limits_{\left\{x\in\Omega:|u_\alpha|^{p(x)}>\omega^{p^+}\right\}} \omega^{p(x)-q(x)}\,|u_\alpha(x)|^{q(x)}\,dx$$

$$= \int\limits_{\left\{x\in\Omega:|u_\alpha|^{p(x)}>\omega^{p^+}\right\}} \frac{|u_\alpha(x)|^{q(x)}}{\omega^{q(x)-p(x)}}\,dx$$

$$\leq \int_{\Omega} \frac{|u_\alpha(x)|^{q(x)}}{\omega^\beta} dx$$
$$= \frac{1}{\omega^\beta} \int_{\Omega} |u_\alpha(x)|^{q(x)} \leq \frac{c}{\omega^\beta}.$$

By taking supremum we have

$$\sup_{\alpha \in A} \int_{\left\{x\in\Omega: |u_\alpha|^{p(x)} > \omega^{p^+}\right\}} |u_\alpha(x)|^{p(x)}\, dx \leq \frac{c}{\omega^\beta}.$$

Taking limit of both sides leads to

$$\lim_{\omega\to\infty} \sup_{\alpha \in A} \int_{\left\{x\in\Omega: |u_\alpha|^{p(x)} > \omega^{p^+}\right\}} |u_\alpha(x)|^{p(x)}\, dx \leq \lim_{\omega\to\infty} \frac{c}{\omega^\beta} = 0.$$

If we take $\theta = \omega^{p^+}$, we have

$$\lim_{\theta\to\infty} \sup_{\alpha \in A} \int_{\left\{x\in\Omega: |u_\alpha|^{p(x)} > \theta\right\}} |u_\alpha(x)|^{p(x)}\, dx = 0.$$

ii)
From i), for given any $\epsilon > 0$ there exist $\theta > 0$ such that

$$\lim_{\theta\to\infty} \sup_{\alpha \in A} \int_{\left\{x\in\Omega: |u_\alpha|^{p(x)} > \theta\right\}} |u_\alpha(x)|^{p(x)}\, dx < \frac{\epsilon}{2} \quad \text{for all } \alpha \in A.$$

Hence, for every $E \subset \Omega$ we have

$$\int_E |u_\alpha(x)|^{p(x)}\, dx = \int_{E\cap\left\{x\in\Omega: |u_\alpha|^{p(x)} > \theta\right\}} |u_\alpha(x)|^{p(x)}\, dx$$
$$+ \int_{E\cap\left\{x\in\Omega: |u_\alpha|^{p(x)} \leq \theta\right\}} |u_\alpha(x)|^{p(x)}\, dx < \frac{\epsilon}{2} + \theta\left[\text{meas}(E)\right].$$

If we take $\delta = \frac{\epsilon}{2\theta}$, then for every $E \subset \Omega$ with meas$(E) < \delta$ we have

$$\int_E |u_\alpha(x)|^{p(x)}\, dx < \frac{\epsilon}{2} + \theta \frac{\epsilon}{2\theta} = \epsilon. \qquad \square$$

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