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PERMANENTAL REPRESENTATION OF PERFECT MATCHINGS IN BIPARTITE GRAPHS ASSOCIATED WITH PADOVAN NUMBERS

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Abstract: In this paper, we consider the relationship between Padovan numbers and perfect matchings of a certain type of bipartite graphs. Then we give a Maple procedure in order to calculate the number of perfect matchings of this family.

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1. Introduction

The famous integer sequences (e.g. Fibonacci, Padovan) provide invaluable opportunities for exploration, and contribute handsomely to the beauty of mathematics, especially number theory [1]–[2]. Among these sequences, Padovan numbers have achieved a kind of celebrity status. The Padovan sequence $\{P(n)\}$ is defined by the recurrence relation, for $n > 2$, as

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$$P(n) = P(n-2) + P(n-3)$$

with $P(0) = P(1) = P(2) = 1$, see [3]. The number $P(n)$ is called nth Padovan number. The Padovan numbers are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, ...$$

for $n = 0, 1, 2, ...$ This sequence is named as $A000931$ in [4].

A bipartite graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex in V_1 and a vertex in V_2. The investigation of the properties of bipartite graphs was begun by König. His work was motivated by an attempt to give a new approach to the investigation of matrices on determinants of matrices. As a practical matter, bipartite graphs form a model of the interaction between two different types of objects. For example; social network analysis, railway optimization problem, marriage problem, etc., [5].

A perfect matching (or 1-factor) of a graph is a matching in which each vertex has exactly one edge incident on it. Namely, every vertex in the graph has degree 1. The enumeration or actual construction of perfect matching of a bipartite graph has many applications, for example, in maximal flow problems and in assignment and scheduling problems arising in operational research [6]. The number of perfect matchings of bipartite graphs also plays a significant role in organic chemistry, [7].

Let $A(G)$ be adjacency matrix of the bipartite graph G and $\mu(G)$ denote the number of perfect matchings of G. Then, one can find the following fact in [6]: $\mu(G) = \sqrt{\mathrm{per}\,(A(G))}$.

Let G be a bipartite graph whose vertex set V is partitioned into two subsets V_1 and V_2 such that $|V_1| = |V_2| = n$. We construct the bipartite adjacent matrix $B(G) = (b_{ij})$ of G as following: $b_{ij} = 1$ if and only if G contains an edge from $v_i \in V_1$ to $v_j \in V_2$, and otherwise $b_{ij} = 0$. Then, the number of perfect matchings of bipartite graph G is equal to the permanent of its bipartite adjacency matrix, [6].

The permanent of an $n \times n$ matrix $A = (a_{ij})$ is defined by

$$\mathrm{per}\,(A) = \sum_{\sigma \epsilon S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group S_n. The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive. One can find the basic properties and more applications of permanents in [8]–[11].

Let $A = [a_{ij}]$ be an $m \times n$ real matrix with row vectors $\alpha_1, \alpha_2, ..., \alpha_m$. We say that A is contractible on column (resp. row) k, if column (resp. row) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A^T_{ij:k}\right]^T$ is called the contraction of A on row k relative to columns i and j. We say that A can be contracted to a matrix B if either $B = A$ or there exist matrices $A_0, A_1, ..., A_t$ $(t \geq 1)$ such that $A_0 = A$, $A_t = B$, and A_r is a contraction of A_{r-1} for $r = 1, ..., t$, [8].

Brualdi and Gibson [8] proved the following result about the permanent of a matrix.

Lemma 1. *Let A be a nonnegative integral matrix of order n for $n > 1$ and let B be a contraction of A. Then*

$$per A = per B. \tag{1}$$

The permanents have many applications in physics, chemistry and electrical engineering. Some of the most important applications of permanents are via graph theory. A more difficult problem with many applications is the enumeration of perfect matchings of a graph [6]. Therefore, counting the number of perfect matchings in bipartite graphs has been very popular problem.

The relationships between perfect matchings (1-factors) of bipartite graphs and the famous integer sequences and their generalizations have been extensively discussed by many researchers. For example, Lee et al. [12], consider a bipartite graph $G(A_n = (a_{i,j}))$ with bipartite adjacency matrix is the $n \times n$ tridiagonal matrix of the form

$$A_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 1 & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

Then they obtain the number of perfect matchings of $G(A_n)$ is the $(n+1)$th Fibonacci number $F(n+1)$. They also consider a bipartite graph $G\left(\mathcal{F}_{(n,k)}\right)$

with bipartite adjacency matrix $\mathcal{F}_{(n,k)} = (f_{i,j})$ such that $f_{i,j} = 1$ if $-1 \leq j - i \leq k - 1$ and $f_{i,j} = 0$ otherwise, for $k \leq n + 1$. Then the number of perfect matchings of $G\left(\mathcal{F}_{(n,k)}\right)$ is $g^k\left(n + k - 1\right)$, where $g^k\left(n\right)$ is the nth k-Fibonacci number.

In [13], Lee considers a bipartite graph $G\left(B_n = (b_{i,j})\right)$ with bipartite adjacency matrix is the $n \times n$ matrix of the form

$$B_n = \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

Then for $n \geq 3$, they obtain the number of perfect matchings of $G\left(B_n\right)$ is the $(n-1)$st Lucas number $L\left(n-1\right)$. He also considers a bipartite graph $G\left(\mathcal{L}_{(n,k)}\right)$ with bipartite adjacency matrix $\mathcal{L}_{(n,k)} = \mathcal{F}_{(n,k)} + E_{1,k+1} - \sum_{j=2}^{k} E_{1,j}$ for $n \geq 3$, where $E_{i,j}$ denotes the $n \times n$ matrix with 1 at the (i,j)-entry and zeros elsewhere. Then the number of perfect matchings of $G\left(\mathcal{L}_{(n,k)}\right)$ is $l^k\left(n-1\right)$, where $l^k\left(n\right)$ is the nth k-Lucas number.

In [14], Shiu et al. firstly define the (k,α)-sequences as: For $k \geq 2$, $n \geq 1$ and $\alpha = (a_1, a_2, \ldots, a_m) \in R^m$, where R is a ring. The k-sequence $\left\{s_\alpha^k\left(n\right)\right\}$ is

$$\begin{aligned} s_\alpha^k\left(n\right) &= a_1 f^k\left(n + k - 2\right) + \ldots + a_m f^k\left(n + k - m - 1\right) \\ &= \sum_{i=1}^{m} a_i f^k\left(n - 1 + k - i\right). \end{aligned}$$

The number $s_\alpha^k\left(n\right)$ is called nth (k,α)-number. Then they give the following result:

For a fixed $m \geq 1$, suppose $n, k \geq 2$ and $n \geq m$. Let $G\left(\mathcal{B}_{(n,k)}\left(\alpha\right)\right)$ a bipartite graph with bipartite adjacency matrix has the form

$$\mathcal{B}_{(n,k)}\left(\alpha\right) = \begin{pmatrix} a_1 & a_2 & \ldots & a_m & 0 & \ldots & 0 \\ 1 & & & & & & \\ 0 & & & \mathcal{F}_{(n-1,k)} & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{pmatrix}.$$

Then the number of perfect matching of $G\left(\mathcal{B}_{(n,k)}(\alpha)\right)$ is nth (k,α)-number $s_{\alpha}^{k}(n)$.

In [15], Kılıç et al. consider a bipartite graph $G(\mathcal{R}_n)$ with bipartite adjacency matrix $\mathcal{R}_n = (r_{i,j})$ such that $r_{i,j} = 1$ if $-1 \leq j-i \leq 1$ or $i = 1$ and $r_{i,j} = 0$ otherwise. Then the number of perfect matchings of $G(\mathcal{R}_n)$ is $\sum_{i=0}^{n} F(i) = F(n+2) - 1,$ where $F(n)$ is the nth Fibonacci number. They also consider a bipartite graph $G(W_n)$ with bipartite adjacency matrix $W_n = \mathcal{R}_n + \mathcal{S}_n,$ where $\mathcal{S}_n$ denotes the $n \times n$ matrix with -1 at the $(1,2)$-entry, 1 at the $(2,4)$-entry and zeros elsewhere. Then for $n \geq 4,$ the number of perfect matchings of $G(W_n)$ is $\sum_{i=0}^{n-2} L(i) = L(n) - 1,$ where $L(n)$ is the nth Lucas number.

One can find more applications on the relationship between the number of perfect matchings of bipartite graphs and the well-known integer sequences, [16]–[22].

In this paper we consider a class of bipartite graph. Then we show that the numbers of perfect matchings of this graph generate the Padovan numbers by the contraction method. Finally, we give a *Maple* procedure in order to calculate the numbers of perfect matchings of above-mentioned bipartite graph.

2. Main Results

In this section, we determine a class of bipartite graphs whose number of perfect matchings is nth Padovan number $P(n)$.

A matrix is said to be a $(0,1)$-matrix if each of its entries is either 0 or 1.

Let $H_n = (h_{ij})$ be $n \times n$ $(0,1)$-matrix as the following

$$H_n = \begin{pmatrix} 1 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & & & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}. \qquad (2)$$

Theorem 2. *Let $G(H_n)$ be the bipartite graph with bipartite adjacency matrix H_n given by (2). Then the number of perfect matchings of $G(H_n)$ is* *nth Padovan number* $P(n)$.

Proof. Let H_n^k be the kth contraction of H_n, $1 \leq k \leq n-2$. According to the definition of the matrix H_n, the matrix H_n can be contracted on column 1 so that

$$H_n^1 = \begin{pmatrix} 1 & 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & & & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}_{(n-1)\times(n-1)},$$

the matrix H_n^1 can be contracted on column 1,

$$H_n^2 = \begin{pmatrix} 2 & 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & & & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}_{(n-2)\times(n-2)}.$$

Furthermore, the matrix H_n^2 can be contracted on column 1 and $P(3) = P(4) = 2$, $P(5) = 3$, so that

$$H_n^3 = \begin{pmatrix} 2 & 3 & 2 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & & & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}_{(n-3)\times(n-3)}$$

$$= \begin{pmatrix} P(4) & P(5) & P(3) & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & & & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}_{(n-3)\times(n-3)} .$$

Continuing this process, we have

$$H_n^k = \begin{pmatrix} P(k+1) & P(k+2) & P(k) & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & & & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}_{(n-k)\times(n-k)}$$

for $3 \leq k \leq n-4$. Hence,

$$H_n^{n-3} = \begin{pmatrix} P(n-2) & P(n-1) & P(n-3) \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_{3\times 3}$$

which, by contraction of H_n^{n-3} on column 1, gives

$$H_n^{n-2} = \begin{pmatrix} P(n-1) & P(n) \\ 1 & 0 \end{pmatrix}_{2\times 2} .$$

By applying equation (1), we obtain $\mathrm{per} H_n = \mathrm{per} H_n^{n-2} = P(n)$ which is the desired result. □

2.1. Maple Procedure

The following Maple procedure calculates the number of perfect matchings of bipartite graph $G(H_5)$ given by (2).

```
>restart:
with(LinearAlgebra):
permanent:=proc(n)
local i,j,r,f,A;
f:=(i,j)->piecewise(i=1 and j=1,1,abs(j-i)=1,1,j-i=2,1,0);
A:=Matrix(n,n,f):
for r from 0 to n-2 do
print(r,A):
for j from 2 to n-r do
A[1,j]:=A[2,1]*A[1,j]+A[1,1]*A[2,j]:
od:
A:=DeleteRow(DeleteColumn(Matrix(n-r,n-r,A),1),2):
od:
print(r,eval(A)):
end proc:with(LinearAlgebra):
permanent(n);
```

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