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# ALGORITHMS AND IDENTITIES FOR BIVARIATE $(h_1, h_2)$-BLOSSOMING

Ilija Jegdić

Department of Mathematics
Texas Southern University
Houston, Texas 77004, USA

**Abstract:** We extend the definition of $h$-blossoming for polynomials in one variable to the polynomials in two variables, and we use this bivariate $(h_1, h_2)$-blossoming to study various properties, identities, and algorithms associated with $(h_1, h_2)$-Bézier surfaces. We construct a recursive $(h_1, h_2)$-midpoint subdivision algorithm for the $(h_1, h_2)$-Bézier surfaces and we prove its geometric rate of convergence.

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**Key Words:** $h$-blossoming, Bernstein basis functions, Bézier surfaces

## 1. Introduction and Summary

The classical Bernstein polynomials, named after Sergei Natanovich Bernstein, were introduced in 1912, while Bézier curves and surfaces were studied by Paul de Casteljau and Pierre Bézier in late 1950's and early 1960's. One important property of Bézier curves and surfaces is that they could be computed using recursive evaluation algorithms based on certain structural properties of Bernstein basis functions. The classical Bernstein basis functions, Bernstein polynomials, and Bézier curves and surfaces are used in various areas of numerical analysis, computational geometry, computer aided geometric design, approximation theory, and other fields.

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The quantum $q$-analogues of Bernstein basis functions were introduced and studied by Phillips et al. in [6]-[10], while the $h$-analogous were studied by Stancu in [16, 17], Goldman in [1, 2], and Goldman and Barry in [3]. The theory of quantum $q$- and $h$-Bézier curves, in the context of quantum $q$- and $h$-blossoming for polynomials in one variable, was introduced by Simeonov et al. in [14, 15]. The importance and usefulness of quantum $q$- and $h$-blossoming is the quantum blossoming representation of quantum $q$- and $h$-Bézier curves, surfaces, and splines, which gives efficient algorithms for recursive evaluation, degree evaluation, subdivision, and other properties (for example, see [11]-[13] by Simeonov and Goldman). Some of these properties, algorithms and identities were also derived using the standard mathematical induction and other elementary techniques in [4] by Jegdić, Larson, and Simeonov. Recently, Jegdić, Simeonov, and Zafiris used the tensor product and generalized concept of $q$-blossoming for polynomials in one variable introduced in [15] to define $q$-blossoming for polynomials in two variables leading to the study of $q$-Bézier surfaces in [5].

The main goal of this paper is to extend the definition of $h$-blossoming for polynomials in one variable to $(h_1, h_2)$-blossoming for polynomials in two variables, and to use it to generalize identities and algorithms for $h$-Bernstein polynomials in one variable from [14] to the case of polynomials of two variables and to study $(h_1, h_2)$-Bézier surfaces.

The paper is organized as follows. In Section 2 we define the bivariate $(h_1, h_2)$-Bernstein basis functions and $(h_1, h_2)$-Bézier surfaces. We obtain an analogue of the de Casteljau evaluation algorithm based on the recurrence relations for $(h_1, h_2)$-Bernstein basis functions. In Section 3 we define the $(h_1, h_2)$-blossoming and in Theorem 5 we prove that there exists a unique $(h_1, h_2)$-blossom for any polynomial of two variables. Section 4 deals with recursive evaluation algorithms (Theorems 6 and 7) and in Theorem 8 we prove that every polynomial surface is an $(h_1, h_2)$-Bézier surface. We obtain several results analogous to the univariate $h$-blossoming such as: that $(h_1, h_2)$-Bernstein basis functions form a basis for polynomials, uniqueness of $(h_1, h_2)$-Bézier control points, dual functional property of the $(h_1, h_2)$-blossom, Marsden's identity in bivariate $(h_1, h_2)$-case, and representation of constant and linear functions. We conclude the paper with Section 5 in which we present a midpoint subdivision algorithm for $(h_1, h_2)$-Bézier surfaces and we prove its convergence (Theorems 16 and 17). We illustrate this algorithm on several examples of $(h_1, h_2)$-Bézier surfaces using *Mathematica*.

## 2. Definition of $(h_1, h_2)$-Bernstein Basis Functions and $(h_1, h_2)$-Bézier Surfaces

In this section we introduce the definition of $(h_1, h_2)$-Bernstein basis functions in two variables and the definition of $(h_1, h_2)$-Bézier surfaces. Using the recurrence relations for bivariate $(h_1, h_2)$-Bernstein basis functions we derive the $(h_1, h_2)$-de Casteljau evaluation algorithm.

We recall the definition of the $h$-Bernstein basis functions over an interval $[a, b]$

$$B_k^n(t; [a, b]; h) := \binom{n}{k} \frac{\prod_{i=0}^{k-1}(t - a + ih) \prod_{i=0}^{n-k-1}(b - t + ih)}{\prod_{i=0}^{n-1}(b - a + ih)},$$

where $k = 0, \ldots, n$ and the values of the parameter $h$ for which $b = a - ih$ for some $i = 0, \ldots, n - 1$ are excluded.

**Definition 1.** The bivariate $(h_1, h_2)$-Bernstein basis functions of degree $m$ in $t$ and $n$ in $s$ over $[a, b] \times [c, d]$ are given by

$$B_{k,l}^{m,n}(t, s; [a, b] \times [c, d]; h_1, h_2) := B_k^m(t; [a, b]; h_1)\, B_l^n(s; [c, d]; h_2),$$

where $k = 0, \ldots, m$ and $l = 0, \ldots, n$, and the values of $h_1$ and $h_2$ for which $b = a - ih_1$ (for some $i = 0, \ldots, m-1$) and $d = c - jh_2$ (for some $j = 0, \ldots, n-1$) are excluded.

Using the fact that $h$-Bernstein basis functions satisfy the translation and scale invariance properties, we obtain these properties in the bivariate case:

$$\begin{aligned} &B_{k,l}^{m,n}(t + \alpha, s + \beta; [a + \alpha, b + \alpha] \times [c + \beta, d + \beta]; h_1, h_2) \\ &\quad = B_{k,l}^{m,n}(t, s; [a, b] \times [c, d]; h_1, h_2), \text{ for every } \alpha, \beta \in \mathbb{R}, \qquad (1) \\ &B_{k,l}^{m,n}(\alpha t, \beta s; [\alpha a, \alpha b] \times [\beta c, \beta d]; \alpha h_1, \beta h_2) \\ &\quad = B_{k,l}^{m,n}(t, s; [a, b] \times [c, d]; h_1, h_2), \text{ for every } \alpha, \beta \in \mathbb{R} \setminus \{0\}. \qquad (2) \end{aligned}$$

From now on, to shorten the notation in the formulae, we write $\mathcal{R}$ and $\vec{h}$ instead of $[a, b] \times [c, d]$ and $(h_1, h_2)$, respectively.

Using the recurrence relations for $h$-Bernstein basis functions, we obtain the following recurrence relations for the bivariate case:

$$B_{0,0}^{0,0}(t, s; \mathcal{R}; \vec{h}) = 1,$$

and if $k = 1, \ldots, m-1$ and $l = 1, \ldots, n-1$, we have

$$\begin{aligned}
&B_{k,l}^{m,n}(t,s;\mathcal{R};\vec{h}) \qquad (3)\\
&\quad = \frac{t-a+(k-1)h_1}{b-a+(m-1)h_1}\frac{s-c+(l-1)h_2}{d-c+(n-1)h_2}B_{k-1,l-1}^{m-1,n-1}(t,s;\mathcal{R};\vec{h})\\
&\quad + \frac{b-t+(m-k-1)h_1}{b-a+(m-1)h_1}\frac{s-c+(l-1)h_2}{d-c+(n-1)h_2}B_{k,l-1}^{m-1,n-1}(t,s;\mathcal{R};\vec{h})\\
&\quad + \frac{t-a+(k-1)h_1}{b-a+(m-1)h_1}\frac{d-s+(n-l-1)h_2}{d-c+(n-1)h_2}B_{k-1,l}^{m-1,n-1}(t,s;\mathcal{R};\vec{h})\\
&\quad + \frac{b-t+(m-k-1)h_1}{b-a+(m-1)h_1}\frac{d-s+(n-l-1)h_2}{d-c+(n-1)h_2}B_{k,l}^{m-1,n-1}(t,s;\mathcal{R};\vec{h}).
\end{aligned}$$

**Definition 2.** The $(h_1, h_2)$-Bézier surface of degree $m$ in $t$ and $n$ in $s$ over a rectangle $\mathcal{R}$ with control points $P_{i,j}$, where $i = 0, \ldots, m$ and $j = 0, \ldots, n$, is defined by

$$P(t,s) := \sum_{i=0}^{m}\sum_{j=0}^{n} P_{i,j}B_{i,j}^{m,n}(t,s;\mathcal{R};\vec{h}), \quad (t,s) \in \mathcal{R}.$$

Using the recurrence relation (3) we obtain the $(h_1, h_2)$-de Casteljau evaluation algorithm:

- define $P_{k,l}^{0,0}(t,s) = P_{k,l}$, for $k = 0, \ldots, m$ and $l = 0, \ldots, n$,
- if $u = 1, \ldots, m$ and $v = 1, \ldots, n$, define recursively

$$\begin{aligned}
&P_{k,l}^{u,v}(t,s)\\
&\quad = \frac{b-t+(m-k-u)h_1}{b-a+(m-u)h_1}\frac{d-s+(n-l-v)h_2}{d-c+(n-v)h_2}P_{k,l}^{u-1,v-1}(t,s)\\
&\quad + \frac{t-a+kh_1}{b-a+(m-u)h_1}\frac{d-s+(n-l-v)h_2}{d-c+(n-v)h_2}P_{k+1,l}^{u-1,v-1}(t,s)\\
&\quad + \frac{b-t+(m-k-u)h_1}{b-a+(m-u)h_1}\frac{s-c+lh_2}{d-c+(n-v)h_2}P_{k,l+1}^{u-1,v-1}(t,s)\\
&\quad + \frac{t-a+kh_1}{b-a+(m-u)h_1}\frac{s-c+lh_2}{d-c+(n-v)h_2}P_{k+1,l+1}^{u-1,v-1}(t,s),
\end{aligned}$$

for $k = 0, \ldots, m-u$ and $l = 0, \ldots, n-v$.

The induction on $m$ and $n$ shows that $P_{0,0}^{m,n}(t,s) = P(t,s)$.

## 3. Definition and Properties of $(h_1, h_2)$-Blossoming

In this section we define an $(h_1, h_2)$-blossom, for a given polynomial of two variables, and we prove its existence and uniqueness.

**Definition 3.** Let $P(t, s)$ be a polynomial of degree $m$ in $t$ and $n$ in $s$. The $(h_1, h_2)$-blossom of $P(t, s)$ is a polynomial

$$p(u_1, \ldots, u_m; v_1, \ldots, v_n; \vec{h})$$

which satisfies the following $(h_1, h_2)$-blossoming axioms:

• *symmetry*: for every two permutations $\sigma_1$ and $\sigma_2$ of the sets $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively,

$$p(u_1, \ldots, u_m; v_1, \ldots, v_n; \vec{h}) = p(u_{\sigma_1(1)}, \ldots, u_{\sigma_1(m)}; v_{\sigma_2(1)}, \ldots, v_{\sigma_2(n)}; \vec{h}),$$

• *multi-affine*:

$$\begin{aligned} &p(u_1, \ldots, (1-\alpha)u_k + \alpha z_k, \ldots, u_m; v_1, \ldots, v_n; \vec{h}) \\ &= (1-\alpha) p(u_1, \ldots, u_k, \ldots, u_m; v_1, \ldots, v_n; \vec{h}) \\ &\quad + \alpha p(u_1, \ldots, z_k, \ldots, u_m; v_1, \ldots, v_n; \vec{h}) \end{aligned}$$

and

$$\begin{aligned} &p(u_1, \ldots, u_m; v_1, \ldots, (1-\beta)v_k + \beta w_k, \ldots v_n; \vec{h}) \\ &= (1-\beta) p(u_1, \ldots, u_m; v_1, \ldots, v_k, \ldots, v_n; \vec{h}) \\ &\quad + \beta p(u_1, \ldots, u_m; v_1, \ldots, w_k, \ldots, v_n; \vec{h}), \end{aligned}$$

• $(h_1, h_2)$-*diagonal*:

$$p(t, t-h_1, \ldots, t-(m-1)h_1; s, s-h_2 \ldots, s-(n-1)h_2; \vec{h}) = P(t, s).$$

**Example 4.** *Consider $P(t, s)$ as a cubic polynomial in $t$ and quadratic in $s$.*

• *If $P(t, s) = 1$, then $p(u_1, u_2, u_3; v_1, v_2; \vec{h}) = 1$.*
• *If $P(t, s) = t$, then $p(u_1, u_2, u_3; v_1, v_2; \vec{h}) = (u_1 + u_2 + u_3)/3 + h_1$.*
• *If $P(t, s) = s$, then $p(u_1, u_2, u_3; v_1, v_2; \vec{h}) = (v_1 + v_2)/2 + h_2/2$.*
• *If $P(t, s) = t^2$, then*

$$p(u_1, u_2, u_3; v_1, v_2; \vec{h}) = \frac{u_1 u_2 + u_2 u_3 + u_1 u_3}{3} + \frac{2}{3}(u_1 + u_2 + u_3)h_1 + \frac{4}{3}h_1^2.$$

- *If* $P(t,s) = ts$, *then*

$$p(u_1, u_2, u_3; v_1, v_2; \vec{h}) = \left(\frac{u_1 + u_2 + u_3}{3} + h_1\right)\left(\frac{v_1 + v_2}{2} + \frac{h_2}{2}\right).$$

Let

$$\Phi_{m,n}(t, s; \vec{h}) := \prod_{i=1}^{m}(t - (i-1)h_1)\prod_{j=1}^{n}(s - (j-1)h_2).$$

We use partial derivatives of $\Phi_{m,n}$ to define

$$\Phi_{m,n;k,l}(t, s; \vec{h}) := \frac{1}{(m-k)!(n-l)!}\frac{\partial^{m+n-(k+l)}}{\partial^{m-k}t\partial^{n-l}s}\Phi_{m,n}(t, s; \vec{h}),$$

where $k = 0, \ldots, m$ and $l = 0, \ldots, n$. Note that $\Phi_{m,n;k,l}(t, s; \vec{h})$ is a polynomial of degree $k$ in $t$ and $l$ in $s$, and that $\{\Phi_{m,n;k,l}(t, s; \vec{h})\}$, $k = 0, \ldots, m$, $l = 0, \ldots, n$, is a basis for polynomials $P(t,s)$ of degree $m$ in $t$ and $n$ in $s$. Let $\varphi_{m;0}(u_1, \ldots, u_m) := 1$ and

$$\varphi_{m;k}(u_1, \ldots, u_m) := \sum_{1 \le i_1 < \ldots < i_k \le m} u_{i_1} \ldots u_{i_k}, \quad k = 1, \ldots, m$$

denote the elementary symmetric functions in $m$ variables, and let

$$\varphi_{m,n;k,l}(u_1, \ldots, u_m; v_1, \ldots, v_n) := \varphi_{m;k}(u_1, \ldots, u_m)\,\varphi_{n;l}(v_1, \ldots, v_n).$$

**Theorem 5.** *For every polynomial* $P(t,s)$ *of degree* $m$ *in* $t$ *and* $n$ *in* $s$, *there exists a unique* $(h_1, h_2)$*-blossom.*

*Proof.* Let $P(t,s)$ be a polynomial of degree $m$ in $t$ and $n$ in $s$. Then we write

$$P(t, s) = \sum_{k=0}^{n}\sum_{l=0}^{n} c_{k,l}\Phi_{m,n;k,l}(t, s; \vec{h})$$

and the polynomial

$$\begin{aligned} &p(u_1, \ldots, u_m; v_1, \ldots, v_n; \vec{h}) \\ &\quad := \sum_{k=0}^{m}\sum_{l=0}^{n} c_{k,l}\,\varphi_{m,n;k,l}(u_1, \ldots, u_m; v_1, \ldots, v_n) \end{aligned}$$

is an $(h_1, h_2)$-blossom of $P(t, s)$. To prove uniqueness, we assume that there are two $(h_1, h_2)$-blossoms $\mathbf{p} := p(u_1, \ldots, u_m; v_1, \ldots, v_n; \vec{h})$ and $\mathbf{r} := r(u_1, \ldots, u_m; v_1, \ldots, v_n; \vec{h})$. Since every symmetric multi-affine polynomial has a unique representation in terms of elementary symmetric functions, we have

$$\mathbf{p} = \sum_{k=0}^{m} \sum_{l=0}^{n} a_{k,l} \varphi_{m,n;k,l}(u_1, \ldots, u_m; v_1, \ldots, v_n),$$
$$\mathbf{r} = \sum_{k=0}^{m} \sum_{l=0}^{n} b_{k,l} \varphi_{m,n;k,l}(u_1, \ldots, u_m; v_1, \ldots, v_n),$$

for some constants $a_{k,l}$ and $b_{k,l}$, $k = 0, \ldots, m$ and $l = 0, \ldots, n$. Using the $(h_1, h_2)$-diagonal property we have

$$\begin{aligned} P(t, s) &= p(t, \ldots, t-(m-1)h_1; s, \ldots, s-(n-1)h_2; \vec{h}) \\ &= \sum_{k=0}^{m} \sum_{l=0}^{n} a_{k,l} \Phi_{m,n;k,l}(t, s; \vec{h}), \\ P(t, s) &= r(t, \ldots, t-(m-1)h_1; s, \ldots, s-(n-1)h_2; \vec{h}) \\ &= \sum_{k=0}^{m} \sum_{l=0}^{n} b_{k,l} \Phi_{m,n;k,l}(t, s; \vec{h}). \end{aligned}$$

Hence $a_{k,l} = b_{k,l}$ for all $k = 0, \ldots, m$ and $l = 0, \ldots, n$. □

## 4. $(h_1, h_2)$-Evaluation Algorithms

We construct $(h_1, h_2)$-recursive evaluation algorithms and we prove that every polynomial in two variables is an $(h_1, h_2)$-Bézier surface. We also prove the dual functional property of the $(h_1, h_2)$-blossom and we present an analogue of the Marsden Identity.

**Theorem 6.** *Let $P(t, s)$ be a polynomial of degree $m$ in $t$ and $n$ in $s$ with $(h_1, h_2)$-blossom $p(u_1, \ldots, u_m; v_1, \ldots, v_n; \vec{h})$. Define*

$$\begin{aligned} Q_{i,j}^{0,0} :=& p(a - ih_1, \ldots, a-(m-1)h_1, b, b-h_1, \ldots, b-(i-1)h_1; \\ & c - jh_2, \ldots, c-(n-1)h_2, d, d-h_2, \ldots, d-(j-1)h_2; \vec{h}), \end{aligned}$$

$i = 0, \dots, m$, $j = 0, \dots, n$, *and for* $k = 0, \dots, m-1$, $l = 0, \dots, n-1$, *define recursively*

$$\begin{aligned} Q_{i,j}^{k+1,l+1}&(u_1, \dots, u_{k+1}; v_1, \dots, v_{l+1}; \vec{h}) \\ &= (1-\alpha_{k,i})(1-\beta_{l,j}) Q_{i,j}^{k,l}(u_1, \dots, u_k; v_1, \dots, v_l; \vec{h}) \\ &+ \alpha_{k,i}(1-\beta_{l,j}) Q_{i+1,j}^{k,l}(u_1, \dots, u_k; v_1, \dots, v_l; \vec{h}) \\ &+ (1-\alpha_{k,i})\beta_{l,j} Q_{i,j+1}^{k,l}(u_1, \dots, u_k; v_1, \dots, v_l; \vec{h}) \\ &+ \alpha_{k,i}\beta_{l,j} Q_{i+1,j+1}^{k,l}(u_1, \dots, u_k; v_1, \dots, v_l; \vec{h}), \end{aligned}$$

$i = 0, \dots, m-k-1$ *and* $j = 0, \dots, n-l-1$. *Here,*

$$\alpha_{k,i} = \frac{u_{k+1} - a + (i+k)h_1}{b - a + kh_1} \quad \text{and} \quad \beta_{l,j} = \frac{v_{l+1} - c - (j+l)h_2}{d - c + lh_2}.$$

*Then for each* $k = 0, \dots, m$ *and* $l = 0, \dots, n$, *we have*

$$\begin{aligned} &Q_{i,j}^{k,l}(u_1, \dots, u_k; v_1, \dots, v_l; \vec{h}) = \\ &p(a-(k+i)h_1, \dots, a-(m-1)h_1, b, \dots, b-(i-1)h_1, u_1, \dots, u_k; \\ &c-(l+j)h_2, \dots, c-(n-1)h_2, d, \dots, d-(j-1)h_2, v_1, \dots, v_l; \vec{h}), \end{aligned}$$

$i = 0, \dots, m-k$, $j = 0, \dots, n-k$. *In particular,*

$$Q_{0,0}^{m,n}(u_1, \dots, u_m; v_1, \dots, v_n; \vec{h}) = p(u_1, \dots, u_m; v_1, \dots, v_n; \vec{h}).$$

*Proof.* The result follows by induction on $k$ and $l$. □

**Theorem 7.** *Let* $P(t,s)$ *be a polynomial of degree* $m$ *in* $t$ *and* $n$ *in* $s$ *with* $(h_1, h_2)$*-blossom* $p(u_1, \dots, u_m; v_1, \dots, v_n; \vec{h})$. *There exist* $n!m!$ *affine invariant, recursive evaluation algorithms for* $P(t,s)$ *defined recursively as follows. Let* $\sigma_1, \sigma_2$ *be permutations of* $\{1, \dots, m\}$ *and* $\{1, \dots, n\}$, *respectively. Define*

$$\begin{aligned} P_{i,j}^{0,0} =& p(a - ih_1, \dots, a-(m-1)h_1, b, b-h_1, \dots, b-(i-1)h_1; \\ & c - jh_2, \dots, c-(n-1)h_2, d, d-h_2, \dots, d-(j-1)h_2; \vec{h}), \end{aligned}$$

$i = 0, \dots, m$, $j = 0, \dots, n$, *and for* $k = 0, \dots, m-1$ *and* $l = 0, \dots, n-1$, *define recursively*

$$\begin{aligned} P_{i,j}^{k+1,l+1}(t,s) := (1-\gamma_{k,i})(1-\delta_{l,j}) P_{i,j}^{k,l}(t,s) + \gamma_{k,i}(1-\delta_{l,j}) P_{i+1,j}^{k,l}(t,s) \\ + (1-\gamma_{k,i})\delta_{l,j} P_{i,j+1}^{k,l}(t,s) + \gamma_{k,i}\delta_{l,j} P_{i+1,j+1}^{k,l}(t,s), \end{aligned}$$

*$i = 0, \dots, m-k-1$ and $j = 0, \dots, n-l-1$, where*

$$\gamma_{k,i} = \frac{t-a-(\sigma_1(k+1)-1-i-k)h_1}{b-a+kh_1}$$

*and*

$$\delta_{l,j} = \frac{s-c-(\sigma_2(l+1)-1-j-l)h_2}{d-c+lh_2}.$$

*Then for $k = 0, \dots, m$ and $l = 0, \dots, n$, we have*

$$\begin{aligned} P_{i,j}^{k,l}(t,s) =& p(a-(k+i)h_1, \dots, a-(m-1)h_1, b, \dots, b-(i-1)h_1, \\ & t-(\sigma_1(1)-1)h_1, \dots, t-(\sigma_1(k)-1)h_1; \\ & c-(l+j)h_2, \dots, c-(n-1)h_2, d, \dots, d-(j-1)h_2, \\ & s-(\sigma_2(1)-1)h_2, \dots, s-(\sigma_2(l)-1)h_2; \vec{h}), \end{aligned} \tag{4}$$

*$i = 0, \dots, m-k$ and $j = 0, \dots, n-l$. In particular,*

$$\begin{aligned} P_{0,0}^{m,n}(t,s) =& p(t-(\sigma_1(1)-1)h_1, \dots, t-(\sigma_1(m)-1)h_1; \\ & s-(\sigma_2(1)-1)h_2, \dots, s-(\sigma_2(n)-1)h_2; \vec{h}) \\ =& P(t,s). \end{aligned}$$

*Proof.* This result follows from the previous theorem by substituting $u_i = t-(\sigma_1(i)-1)h_1$ and $v_j = s-(\sigma_2(j)-1)h_2$, $i = 0, \dots, m$ and $j = 1, \dots, n$. □

**Theorem 8.** *Let $P(t,s)$ be a polynomial of degree $m$ in $t$ and $n$ in $s$ with $(h_1,h_2)$-blossom $p(u_1, \dots, u_m; v_1, \dots, v_n; \vec{h})$. Then*

$$\begin{aligned} P(t,s) =& \sum_{i=0}^{m} \sum_{j=0}^{n} p(a-ih_1, \dots, a-(m-1)h_1, b, \dots, b-(i-1)h_1; \\ & c-jh_2, \dots, c-(n-1)h_2, d, \dots, d-(j-1)h_2; \vec{h}) \\ & \times B_{i,j}^{m,n}(t,s;\mathcal{R};\vec{h}). \end{aligned} \tag{5}$$

*Proof.* We proceed by induction on $m$ and $n$. Note that (5) is true if $m = n = 0$, since $B_{0,0}^{0,0}(t,s;\mathcal{R};\vec{h}) = 1$. Assume that (5) is true for all bivariate polynomials of degree at most $m-1$ in $t$ and at most $n-1$ in $s$. Let $P(t,s)$ be a polynomial of degree $m$ in $t$ and $n$ in $s$. By the previous theorem we have

$$P(t,s) = P_{0,0}^{m,n}(t,s) = (1-\gamma_{m-1,0})(1-\delta_{n-1,0})P_{0,0}^{m-1,n-1}(t,s)$$

$$
\begin{aligned}
&+ \gamma_{m-1,0}(1-\delta_{n-1,0})P_{1,0}^{m-1,n-1}(t,s)\\
&+ (1-\gamma_{m-1,0})\delta_{n-1,0}P_{0,1}^{m-1,n-1}(t,s)\\
&+ \gamma_{m-1,0}\delta_{n-1,0}P_{1,1}^{m-1,n-1}(t,s).
\end{aligned}
$$

Furthermore, using (4) with the identity permutations yields

$$
\begin{aligned}
P_{0,0}^{m-1,n-1}(t,s) =& p(a-(m-1)h_1, t, t-h_1, \dots, t-(m-2)h_1;\\
& c-(n-1)h_2, s, s-h_2, \dots, s-(n-2)h_2; \vec{h}),\\
P_{1,0}^{m-1,n-1}(t,s) =& p(b, t, t-h_1, \dots, t-(m-2)h_1;\\
& c-(n-1)h_2, s, s-h_2, \dots, s-(n-2)h_2; \vec{h}),\\
P_{0,1}^{m-1,n-1}(t,s) =& p(a-(m-1)h_1, t, t-h_1, \dots, t-(m-2)h_1;\\
& d, s, s-h_2, \dots, s-(n-2)h_2; \vec{h}),\\
P_{1,1}^{m-1,n-1}(t,s) =& p(b, t, t-h_1, \dots, t-(m-2)h_1;\\
& d, s, s-h_2, \dots, s-(n-2)h_2; \vec{h}).
\end{aligned}
$$

Notice that the $(h_1, h_2)$-blossoms of $P_{i,j}^{m-1,n-1}(t,s)$, $i,j = 0,1$, are

$$
\begin{aligned}
& p_{0,0}(u_1, \dots, u_{m-1}; v_1, \dots, v_{n-1}; \vec{h})\\
&\quad = p(a-(m-1)h_1, u_1, \dots, u_{m-1}; c-(n-1)h_2, v_1, \dots, v_{n-1}; \vec{h}),\\
& p_{1,0}(u_1, \dots, u_{m-1}; v_1, \dots, v_{n-1}; \vec{h})\\
&\quad = p(b, u_1, \dots, u_{m-1}; c-(n-1)h_2, v_1, \dots, v_{n-1}; \vec{h}),\\
& p_{0,1}(u_1, \dots, u_{m-1}; v_1, \dots, v_{n-1}; \vec{h})\\
&\quad = p(a-(m-1)h_1, u_1, \dots, u_{m-1}; d, v_1, \dots, v_{n-1}; \vec{h}),\\
& p_{1,1}(u_1, \dots, u_{m-1}; v_1, \dots, v_{n-1}; \vec{h})\\
&\quad = p(b, u_1, \dots, u_{m-1}; d, v_1, \dots, v_{n-1}; \vec{h}).
\end{aligned}
$$

Therefore, by applying the induction hypothesis to $P_{i,j}^{m-1,n-1}(t,s)$ on $[a-ih_1, b-ih_1] \times [c-jh_2, d-jh_2]$, $i,j = 0,1$, we get

$$
\begin{aligned}
& P_{0,0}^{m-1,n-1}(x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} p(a-(m-1)h_1, a-ih_1, \dots, a-(m-2)h_1,\\
& b, b-h_1, \dots, b-(i-1)h_1; c-(n-1)h_2, c-jh_2, \dots, c-(n-2)h_2,\\
& d, d-h_2, \dots, d-(j-1)h_2; h_1, h_2) \times B_{i,j}^{m-1,n-1}(t,s;\mathcal{R};\vec{h}),
\end{aligned}
$$

$$P_{1,0}^{m-1,n-1}(t,s) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} p(b, a-(i+1)h_1, \dots, a-(m-1)h_1,$$
$$b-h_1, \dots, b-ih_1; c-(n-1)h_2, c-jh_2, \dots, c-(n-2)h_2,$$
$$d, \dots, d-(j-1)h_2; h_1, h_2) \times B_{i,j}^{m-1,n-1}(t,s;[a-h_1, b-h_1]\times[c,d];\vec{h}),$$

$$P_{0,1}^{m-1,n-1}(t,s) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} p(a-(m-1)h_1, a-ih_1, \dots, a-(m-2)h_1,$$
$$b, b-h_1, \dots, b-(i-1)h_1; d, c-(j+1)h_2, \dots, c-(n-1)h_2,$$
$$d-h_2, \dots, d-jh_2; h_1, h_2) \times B_{i,j}^{m-1,n-1}(t,s;[a,b]\times[c-h_2, d-h_2];\vec{h}),$$

and

$$P_{1,1}^{m-1,n-1}(t,s) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} p(b, a-(i+1)h_1, \dots, a-(m-1)h_1,$$
$$b-h_1, \dots, b-ih_1; d, c-(j+1)h_2, \dots, c-(n-1)h_2,$$
$$d-h_2, \dots, d-jh_2; h_1, h_2)$$
$$\times B_{i,j}^{m-1,n-1}(t,s;[a-h_1, b-h_1]\times[c-h_2, d-h_2];\vec{h}).$$

By (4), we obtain

$$P_{0,0}^{m-1,n-1}(t,s) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P_{i,j}^{0,0} B_{i,j}^{m-1,n-1}(t,s;\mathcal{R};\vec{h}).$$

By (4) and (1), we have

$$\begin{aligned} &P_{1,0}^{m-1,n-1}(t,s) \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P_{i+1,j}^{0,0} B_{i,j}^{m-1,n-1}(t,s;[a-h_1, b-h_1]\times[c,d];\vec{h}) \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P_{i+1,j}^{0,0} B_{i,j}^{m-1,n-1}(t+h_1, s;\mathcal{R};\vec{h}). \end{aligned}$$

Similarly,

$$P_{0,1}^{m-1,n-1}(t,s) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P_{i,j+1}^{0,0} B_{i,j}^{m-1,n-1}(t, s+h_2;\mathcal{R};\vec{h})$$

and

$$P_{1,1}^{m-1,n-1}(t,s) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P_{i+1,j+1}^{0,0} B_{i,j}^{m-1,n-1}(t+h_1, s+h_2; \mathcal{R}; \vec{h}).$$

Hence,

$$\begin{aligned}
P(t,s) &= (1-\gamma_{m-1,0})(1-\delta_{n-1,0}) \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P_{i,j}^{0,0} B_{i,j}^{m-1,n-1}(t,s;\mathcal{R};\vec{h}) \\
&+ \gamma_{m-1,0}(1-\delta_{n-1,0}) \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P_{i+1,j}^{0,0} B_{i,j}^{m-1,n-1}(t+h_1, y;\mathcal{R};\vec{h}) \\
&+ (1-\gamma_{m-1,0})\delta_{n-1,0} \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P_{i,j+1}^{0,0} B_{i,j}^{m-1,n-1}(t, s+h_2;\mathcal{R};\vec{h}) \\
&+ \gamma_{m-1,0}\delta_{n-1,0} \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P_{i+1,j+1}^{0,0} B_{i,j}^{m-1,n-1}(t+h_1, s+h_2;\mathcal{R};\vec{h}).
\end{aligned}$$

We rewrite this expression as

$$\begin{aligned}
P(t,s) &= (1-\gamma_{m-1,0})(1-\delta_{n-1,0}) \\
&\times \left( \sum_{i=0}^{m-1} P_{i,0}^{0,0} B_{i,0}^{m-1,n-1}(t,s) + \sum_{j=1}^{n-1} P_{0,j}^{0,0} B_{0,j}^{m-1,n-1}(t,s) \right) \\
&+ \gamma_{m-1,0}(1-\delta_{n-1,0}) \\
&\times \left( \sum_{i=0}^{m-1} P_{i+1,0}^{0,0} B_{i,0}^{m-1,n-1}(t+h_1,s) + \sum_{j=1}^{n-1} P_{m,j}^{0,0} B_{m-1,j}^{m-1,n-1}(t+h_1,s) \right) \\
&+ (1-\gamma_{m-1,0})\delta_{n-1,0} \\
&\times \left( \sum_{j=0}^{n-1} P_{0,j+1}^{0,0} B_{0,j}^{m-1,n-1}(t,s+h_2) + \sum_{i=1}^{m-1} P_{i,n}^{0,0} B_{i,n-1}^{m-1,n-1}(t,s+h_2) \right) \\
&+ \gamma_{m-1,0}\delta_{n-1,0} \left( \sum_{i=1}^{m} P_{i,n}^{0,0} B_{i-1,n-1}^{m-1,n-1}(t+h_1,s+h_2) \right. \\
&\qquad \left. + \sum_{j=1}^{n-1} P_{m,j}^{0,0} B_{m-1,j-1}^{m-1,n-1}(t+h_1,s+h_2) \right)
\end{aligned}$$

$$+ (1-\gamma_{m-1,0})(1-\delta_{n-1,0}) \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} P_{i,j}^{0,0} B_{i,j}^{m-1,n-1}(t,s)$$
$$+ \gamma_{m-1,0}(1-\delta_{n-1,0}) \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} P_{i,j}^{0,0} B_{i-1,j}^{m-1,n-1}(t+h_1,s)$$
$$+ (1-\gamma_{m-1,0})\delta_{n-1,0} \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} P_{i,j}^{0,0} B_{i,j-1}^{m-1,n-1}(t,s+h_2)$$
$$+ \gamma_{m-1,0}\delta_{n-1,0} \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} P_{i,j}^{0,0} B_{i-1,j-1}^{m-1,n-1}(t+h_1,s+h_2), \qquad (6)$$

where the domain for all of the above $(h_1, h_2)$-Bernstein polynomials is the rectangle $\mathcal{R} = [a,b] \times [c,d]$. Note that

$$\gamma_{m-1,0} = \frac{t-a}{b-a+(m-1)h_1}, \qquad 1-\gamma_{m-1,0} = \frac{b-t+(m-1)h_1}{b-a+(m-1)h_1}$$

and

$$\delta_{n-1,0} = \frac{s-c}{d-c+(n-1)h_2}, \qquad 1-\delta_{n-1,0} = \frac{d-s+(n-1)h_2}{d-c+(n-1)h_2}.$$

First we remark that in (6) the coefficient of $P_{0,0}^{0,0}$ is

$$(1-\gamma_{m-1,0})(1-\delta_{n-1,0})B_{0,0}^{m-1,n-1}(t,s) = B_{0,0}^{m,n}(t,s).$$

Similarly, the coefficients of $P_{0,n}^{0,0}$, $P_{m,0}^{0,0}$ and $P_{m,n}^{0,0}$ are $B_{0,n}^{m,n}(t,s)$, $B_{m,0}^{m,n}(t,s)$ and $B_{m,n}^{m,n}(t,s)$, respectively.

Next we consider the expression containing $\{P_{0,j}^{0,0}\}_{j=1}^{n-1}$, in (6):

$$(1-\gamma_{m-1,0})(1-\delta_{n-1,0}) \sum_{j=1}^{n-1} P_{0,j}^{0,0} B_{0,j}^{m-1,n-1}(t,s)$$
$$+ (1-\gamma_{m-1,0})\delta_{n-1,0} \sum_{j=0}^{n-2} P_{0,j+1}^{0,0} B_{0,j}^{m-1,n-1}(t,s+h_2)$$
$$= \sum_{j=1}^{n-1} P_{0,j}^{0,0}(1-\gamma_{m-1,0})B_0^{m-1}(t)$$
$$\times \left\{(1-\delta_{n-1,0})B_j^{n-1}(s) + \delta_{n-1,0}B_{j-1}^{n-1}(s+h_2)\right\}.$$

By a straightforward calculation we show that $(1-\gamma_{m-1,0})B_0^{m-1}(t) = B_0^m(t)$ and $(1-\delta_{n-1,0})B_j^{n-1}(s) + \delta_{n-1,0}B_{j-1}^{n-1}(s+h_2) = B_j^n(s)$. Hence the expression containing $\{P_{0,j}^{0,0}\}_{j=1}^{n-1}$ in (6) is

$$\sum_{j=1}^{n-1} P_{0,j}^{0,0} B_{0,j}^{m,n}(t,s).$$

Similarly, the expressions containing $\{P_{i,0}^{0,0}\}_{i=1}^{m-1}$, $\{P_{i,n}^{0,0}\}_{i=1}^{m-1}$, and $\{P_{m,j}^{0,0}\}_{j=1}^{n-1}$ are

$$\sum_{i=1}^{m-1} P_{i,0}^{0,0} B_{i,0}^{m,n}(t,s), \quad \sum_{i=1}^{m-1} P_{i,n}^{0,0} B_{i,n}^{m,n}(t,s), \quad \text{and} \quad \sum_{j=1}^{n-1} P_{n,j}^{0,0} B_{n,j}^{m,n}(t,s),$$

respectively. Finally, we consider the remaining four lines in (6):

$$\begin{aligned}
&\sum_{i=1}^{m-1}\sum_{j=1}^{n-1} P_{i,j}^{0,0} \left\{(1-\gamma_{m-1,0})B_i^{m-1}(t) + \gamma_{m-1,0}B_{i-1}^{m-1}(t+h_1)\right\} \\
&\qquad\qquad \times \Big\{(1-\delta_{n-1,0})B_j^{n-1}(s) + \delta_{n-1,0}B_{j-1}^{n-1}(s+h_2)\Big\} \\
&= \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} P_{i,j}^{0,0} B_i^m(t) B_j^n(s) = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} P_{i,j}^{0,0} B_{i,j}^{m,n}(t,s).
\end{aligned}$$

Therefore,

$$P(t,s) = \sum_{i=0}^{m}\sum_{j=0}^{n} P_{i,j}^{0,0} B_{i,j}^{m,n}(t,s;\mathcal{R};\vec{h}).$$

□

**Corollary 9.** *The $(h_1,h_2)$-Bernstein basis functions of degree $m$ in $t$ and $n$ in $s$ on a rectangle $\mathcal{R}$ form a basis for the polynomials of degree $m$ in $t$ and $n$ in $s$.*

**Corollary 10.** *The $(h_1,h_2)$-Bézier control points of an $(h_1,h_2)$-Bézier surface on a rectangle $\mathcal{R}$ are unique.*

**Theorem 11.** *Let $P(t,s)$ be an $(h_1,h_2)$-Bézier surface of degree $m$ in $t$ and $n$ in $s$ on a rectangle $\mathcal{R}$ with $(h_1,h_2)$-blossom $p(u_1,\ldots,u_m;v_1,\ldots,v_n;\vec{h})$. Then the $(h_1,h_2)$-Bézier control points of $P(t,s)$ are given by*

$$P_{i,j} = p(a-ih_1,\ldots,a-(m-1)h_1, b, b-h_1,\ldots,b-(i-1)h_1;$$

$$c - jh_2, \dots, c - (n-1)h_2, d, d - h_2, \dots, d - (j-1)h_2; \vec{h}),$$

*where $i = 0, \dots, m$ and $j = 0, \dots, n$.*

*Proof.* This result follows from Theorem 8 and Corollary 10. □

**Theorem 12.** *Let $P(t,s)$ be an $(h_1,h_2)$-Bézier surface of degree $m$ in $t$ and $n$ in $s$ on a rectangle $\mathcal{R}$ with control points $\{P_{i,j}\}$, $i = 0, \dots, m$ and $j = 0, \dots, n$. Let $\{P_{i,j}^{k,l}\}$, $k = 0, \dots, m$, $l = 0, \dots, n$ and $i = 0, \dots, m-k$, $j = 0, \dots, n-l$, be the nodes in the $(h_1,h_2)$-evaluation algorithm for $P(t,s)$ for the identity permutations. Then*

$$P_{i,j}^{k,l}(t,s) = \sum_{r=0}^{k} \sum_{q=0}^{l} P_{i+r,j+q} B_{r,q}^{k,l}(t + ih_1, s + jh_2; \mathcal{R}; \vec{h}).$$

*Proof.* From Theorems 6 and 7 we have that the $(h_1,h_2)$-blossom of $P_{i,j}^{k,l}(t,s)$ is $Q_{i,j}^{k,l}(u_1, \dots, u_k; v_1, \dots, v_l; \vec{h})$. Furthermore, we have

$$\begin{aligned} P_{i,j}^{k,l}(t,s) = &\sum_{r=0}^{k} \sum_{q=0}^{l} Q_{i,j}^{k,l}(A - rh_1, \dots, A - (k-1)h_1, \\ &B, B - h_1, \dots, B - (r-1)h_1; C - qh_2, \dots, C - (l-1)h_2, \\ &D, D - h_2, \dots, D - (q-1)h_2; \vec{h}) \times B_{r,q}^{k,l}(t,s; [A,B] \times [C,D]; \vec{h}) \\ = &\sum_{r=0}^{k} \sum_{q=0}^{l} P_{i+r,j+q} B_{r,q}^{k,l}(t + ih_1, s + jh_2; \mathcal{R}; \vec{h}), \end{aligned}$$

using the property (1) and the intervals $[A,B] = [a - ih_1, b - ih_1]$ and $[C,D] = [c - jh_2, d - jh_2]$. □

Using Propositions 5.1, 5.2 and 5.9 from [14], we state their analogues in two variables.

**Proposition 13.** *(Marsden's Identity)*

$$\begin{aligned} &\frac{\prod_{i=1}^{m-1} \prod_{j=1}^{n-1} (x - t + ih_1)(y - s + jh_2)}{\prod_{i=1}^{m-1} \prod_{j=1}^{n-1} (b - a + ih_1)(d - c + jh_2)} = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{(-1)^{i+j}}{\binom{m}{i}\binom{n}{j}} \\ &\times B_{m-i,n-j}^{m,n}(x, y; [a - (m-1)h_1, b] \times [c - (n-1)h_2, d]; -h_1, -h_2) \\ &\times B_{i,j}^{m,n}(t, s; \mathcal{R}; \vec{h}). \end{aligned}$$

**Proposition 14.** *(Representation of Constant and Linear Functions)*

$$1 = \sum_{k=0}^{n} \sum_{l=0}^{n} B_{k,l}^{n}(x, y; \mathcal{R}; \vec{h}),$$

$$t = \sum_{i=0}^{m} \sum_{j=0}^{n} \left( \frac{m-i}{m} a + \frac{i}{m} b \right) B_{i,j}^{m,n}(t, s; \mathcal{R}; \vec{h}),$$

$$s = \sum_{i=0}^{m} \sum_{j=0}^{n} \left( \frac{n-j}{n} c + \frac{j}{n} d \right) B_{i,j}^{m,n}(t, s; \mathcal{R}; \vec{h}).$$

**Proposition 15.** *The functions* $\Phi_{m,n;k,l}(t, s; \vec{h})$ *satisfy*

$$\Phi_{m,n;k,l}(t, s; \vec{h}) =$$
$$\sum_{i=0}^{m} \sum_{j=0}^{n} \varphi_{m,n;k,l}(a - ih_1, \ldots, a - (m-1)h_1, b, \ldots, b - (i-1)h_1;$$
$$c - jh_2, \ldots, c - (n-1)h_2, d, \ldots, d - (h-1)h_2; \vec{h})$$
$$\times B_{i,j}^{m,n}(t, s; \mathcal{R}; \vec{h}).$$

## 5. A Subdivision Algorithm for $(h_1, h_2)$-Bézier Surfaces

In this section we present a subdivision algorithm for $(h_1, h_2)$-Bézier surfaces and we illustrate it on several examples.

**Theorem 16.** *Let* $\{P_{i,j}\}$*,* $i = 0, \ldots, m$*,* $j = 0, \ldots, n$*, be the control points of an* $(h_1, h_2)$*-Bézier surface* $P(t, s)$ *of degree* $m$ *in* $t$ *and* $n$ *in* $s$ *on a rectangle* $\mathcal{R} := [a, b] \times [c, d]$ *with* $(h_1, h_2)$*-blossom* $p(u_1, \ldots, u_m; v_1, \ldots, v_n; \vec{h})$*. Let* $x \in (a, b)$ *and* $y \in (c, d)$*.*

- **LOWER-LEFT** $(h_1, h_2)$**-subdivision**

  *A control polygon for the surface* $P(t, s)$ *over the subrectangle* $[a, x] \times [c, y]$ *is generated by selecting* $\sigma_1(k) = k$*,* $k = 1, \ldots, m$ *and* $\sigma_2(l) = l$*,* $l = 1, \ldots, n$ *in Theorem 7. Then*

$$P(t, s) = \sum_{k=0}^{m} \sum_{l=0}^{n} P_{k,l}^{LL} B_{k,l}^{m,n}(t, s; [a, x] \times [c, y]; \vec{h}),$$

*where*

$$\begin{aligned}P^{LL}_{k,l} =& p(a-kh_1,\dots,a-(m-1)h_1,x,\dots,x-(k-1)h_1;\\ & c-lh_2,\dots,c-(n-1)h_2,y,\dots,y-(l-1)h_2;\vec{h}),\end{aligned}$$

$k=0,\dots,m$ *and* $l=0,\dots,n$. *Moreover,*

$$P^{LL}_{k,l} = \sum_{j=0}^{k}\sum_{r=0}^{l} P_{j,r} B^{k,l}_{j,r}(x,y;\mathcal{R};\vec{h}).$$

- **UPPER-LEFT** $(h_1,h_2)$**-subdivision**

  *A control polygon for the surface* $P(t,s)$ *over the subrectangle* $[a,x]\times[y,d]$ *is generated by selecting* $\sigma_1(k)=k$, $k=1,\dots,m$ *and* $\sigma_2(l)=n+1-l$, $l=1,\dots,n$ *in Theorem 7. Then*

$$P(t,s) = \sum_{k=0}^{m}\sum_{l=0}^{n} P^{UL}_{k,l} B^{m,n}_{k,l}(t,s;[a,x]\times[y,d];\vec{h}),$$

*where*

$$\begin{aligned}P^{UL}_{k,l} =& p(a-kh_1,\dots,a-(m-1)h_1,x,\dots,x-(k-1)h_1;\\ & y-lh_2,\dots,y-(n-1)h_2,d,\dots,d-(l-1)h_2;\vec{h}),\end{aligned}$$

$k=0,\dots,m$ *and* $l=0,\dots,n$. *Moreover,*

$$P^{UL}_{k,l} = \sum_{j=0}^{k}\sum_{r=l}^{n} P_{j,r} B^{k,n-l}_{j,r-l}(x,y;\mathcal{R};\vec{h}).$$

- **LOWER-RIGHT** $(h_1,h_2)$**-subdivision**

  *A control polygon for the surface* $P(t,s)$ *over* $[x,b]\times[c,y]$ *is generated by selecting* $\sigma_1(k)=m+1-k$, $k=1,\dots,m$ *and* $\sigma_2(l)=l$, $l=1,\dots,n$ *in Theorem 7. Then*

$$P(t,s) = \sum_{k=0}^{m}\sum_{l=0}^{n} P^{LR}_{k,l} B^{m,n}_{k,l}(t,s;[x,b]\times[c,y];\vec{h}),$$

*where*

$$P^{LR}_{k,l} = p(x-kh_1,\dots,x-(m-1)h_1,b,\dots,b-(k-1)h_1;$$

$$c - lh_2, \ldots, c - (n-1)h_2, y, \ldots, y - (l-1)h_2; \vec{h}),$$

*$k = 0, \ldots, m$ and $l = 0, \ldots, n$. Moreover,*

$$P_{k,l}^{LR} = \sum_{j=k}^{m} \sum_{r=0}^{l} P_{j,r} B_{j-k,r}^{m-k,l}(x, y; \mathcal{R}; \vec{h}).$$

- **UPPER-RIGHT $(h_1, h_2)$-subdivision**

  *A control polygon for the surface $P(t,s)$ over $[x,b] \times [y,d]$ is generated by selecting $\sigma_1(k) = m+1-k$, $k = 1, \ldots, m$ and $\sigma_2(l) = n+1-l$, $l = 1, \ldots, n$ in Theorem 7. Then*

$$P(t,s) = \sum_{k=0}^{m} \sum_{l=0}^{n} P_{k,l}^{UR} B_{k,l}^{m,n}(t, s; [x,b] \times [y,d]; \vec{h}),$$

*where*

$$\begin{aligned} P_{k,l}^{UR} =& p(x - kh_1, \ldots, x - (n-1)h_1, b, \ldots, b - (k-1)h_1; \\ & y - lh_2, \ldots, y - (n-1)h_2, d, \ldots, d - (l-1)h_2; \vec{h}), \end{aligned}$$

*$k = 0, \ldots, m$ and $l = 0, \ldots, n$. Moreover,*

$$P_{k,l}^{UR} = \sum_{j=k}^{m} \sum_{r=l}^{n} P_{j,r} B_{j-k,r-l}^{m-k,n-l}(x, y; \mathcal{R}; \vec{h}).$$

*Proof.* The results follow from Theorem 7. □

**Theorem 17.** *Let $P(t,s)$ be an $(h_1, h_2)$-Bézier surface defined on the rectangle $\mathcal{R} := [a,b] \times [c,d]$. The control polygons generated by $(h_1, h_2)$-midpoint subdivision algorithms converge uniformly to $P(t,s)$ at the rate of $2^{-N}$, where $N$ is the number of iterations.*

*Proof.* Let $x \in (a,b)$ and $y \in (c,d)$ be arbitrary. In the first iteration we subdivide the surface $P(t,s)$ into four segments over subrectangles $[a,x] \times [c,y]$, $[a,x] \times [y,d]$, $[x,b] \times [c,y]$ and $[x,b] \times [y,d]$ and we use Theorem 16 to compute the control points for each segment. In the $N$th iteration, $N \geq 2$, we subdivide each of the four surfaces generated at the $(N-1)$th iteration.

Next, we estimate the areas of the corresponding 3d control polygons and we prove that, in the case when $x = (a+b)/2$ and $y = (c+d)/2$, they converge to the original $(h_1, h_2)$-Bézier surface uniformly at the rate $2^{-N}$. First, assume

$$P(t,s) = \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} A_{\mu,\nu}\Phi_{m,n;\mu,\nu}(t,s;\vec{h}).$$

Then the $(h_1, h_2)$-blossom of $P(t,s)$, $p(u_1,\dots,u_m;v_1,\dots,v_n;\vec{h})$, is

$$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} A_{\mu,\nu}\varphi_{m,n;\mu,\nu}(u_1,\dots,u_m;v_1,\dots,v_n).$$

To estimate the area of the 3d surface polygon $\mathcal{P}_{k,l}$, resulting from the lower-left subdivision, with vertices $P^{LL}_{k,l}$, $P^{LL}_{k+1,l}$, $P^{LL}_{k+1,l+1}$ and $P^{LL}_{k,l+1}$, we estimate the areas of triangles $\Delta_1$ (with vertices $P^{LL}_{k,l}$, $P^{LL}_{k+1,l}$ and $P^{LL}_{k,l+1}$) and $\Delta_2$ (with vertices $P^{LL}_{k+1,l}$, $P^{LL}_{k+1,l+1}$ and $P^{LL}_{k,l+1}$). We start by estimating the length of the segment $[P^{LL}_{k+1,l}, P^{LL}_{k,l}]$. By Theorem 16 we have

$$\begin{aligned}
&P^{LL}_{k+1,l} - P^{LL}_{k,l}\\
&= p(a-(k+1)h_1,\dots,a-(m-1)h_1,x,x-h_1,\dots,x-(k-1)h_1;\\
&\qquad c-lh_2,\dots,c-(n-1)h_2,y,y-h_2,\dots,y-(l-1)h_2;\vec{h})\\
&- p(a-kh_1,\dots,a-(m-1)h_1,x,x-h_1,\dots,x-(k-1)h_1;\\
&\qquad c-lh_2,\dots,c-(n-1)h_2,y,y-h_2,\dots,y-(l-1)h_2;\vec{h})\\
&= \sum_{\mu=1}^{m}\sum_{\nu=0}^{n} A_{\mu\nu}(x-a)\varphi_{m-1;\mu-1}(a-(k+1)h_1,\dots,a-(m-1)h_1,\\
&\qquad x,x-h_1,\dots,x-(k-1)h_1;h_1)\\
&\times \varphi_{n;\nu}(c-lh_2,\dots,c-(n-1)h_2,y,y-h_2,\dots,y-(l-1)h_2;h_2),
\end{aligned}$$

and, therefore,

$$\begin{aligned}
&|P^{LL}_{k+1,l} - P^{LL}_{k,l}|\\
&\le |x-a|\sum_{\mu=1}^{n}\sum_{\nu=0}^{n} |A_{\mu\nu}|\,(\max\{|a|,|b|\}+(m-1)|h_1|)^{\mu-1}\binom{m-1}{\mu-1}\\
&\qquad\qquad\times(\max\{|c|,|d|\}+(n-1)|h_2|)^{\nu}\binom{n}{\nu}
\end{aligned}$$

$$\leq |x-a|M_1 \sum_{\mu=1}^{m}\sum_{\nu=0}^{n}\binom{m-1}{\mu-1}\binom{n}{\nu} \leq |x-a|M_1 \sum_{\mu=0}^{m}\binom{m}{\mu}\sum_{\nu=0}^{n}\binom{n}{\nu}$$
$$= |x-a|M_1 2^{m+n} = M_2|x-a|,$$

where

$$M_1 = \max_{0\leq\mu\leq m, 0\leq\nu\leq n} |A_{\mu,\nu}| \max_{0\leq\mu\leq m} (\max\{|a|, |b| + (m-1)|h_1|\})^{\mu}$$
$$\times \max_{0\leq\nu\leq n} (\max\{|c|, |d| + (n-1)|h_2|\})^{\nu}.$$

Similarly, $|P^{LL}_{k,l+1} - P^{LL}_{k,l}| \leq M_2|y-c|$, and, therefore, $|\Delta_1| \leq M_3|x-a||y-c|$, with $M_3 = M_2^2$, implying $|\mathcal{P}_{k,l}| \leq M_4|x-a||y-c|$, where $M_4 = 2M_3$. Similarly, the areas of the control 3d subpolygons $\mathcal{P}_{k,l}$ arising from the upper-left, lower-right and upper-right $(h_1,h_2)$-subdivisions can be estimated by $|\mathcal{P}_{k,l}| \leq M_4|x-a||d-y|$, $|\mathcal{P}_{k,l}| \leq M_4|b-x||y-c|$ and $|\mathcal{P}_{k,l}| \leq M_4|b-x||d-y|$, respectively.

Let $\tilde{P}(t,s)$ be a segment of the original $(h_1,h_2)$-Bézier surface generated after $N$ iterations and let $\mathcal{P}(t,s)$ be the corresponding control polygon. Then $\tilde{P}(t,s)$ is the restriction of $P(t,s)$ over the subrectangle $R = [t_0,t_1]\times[s_0,s_1] \subset \mathcal{R}$ of area $(b-a)(d-c)/4^N$ and $P(t,s)$ and $\mathcal{P}(t,s)$ coincide at the corner points. Hence,

$$|\tilde{P}(t,s) - \mathcal{P}(t,s)| = |P(t,s) - \mathcal{P}(t,s)|$$
$$\leq |P(t,s) - P(t_0,s_0)| + |P(t_0,s_0) - \mathcal{P}(t,s)|$$
$$\leq \max_{(\tau,\sigma)\in R}\left|\frac{\partial P}{\partial t}(\tau,\sigma)\right| |t-t_0| + \max_{(\tau,\sigma)\in R}\left|\frac{\partial P}{\partial s}(\tau,\sigma)\right| |s-s_0|$$
$$+ M_4|t-t_0||s-s_0| \leq \frac{M_5(b-a)(d-c)}{2^N},$$

for every $(t,s)\in R$, where

$$M_5 = \frac{1}{d-c}\max_{(\tau,\sigma)\in R}\left|\frac{\partial P}{\partial t}(\tau,\sigma)\right| + \frac{1}{b-a}\max_{(\tau,\sigma)\in R}\left|\frac{\partial P}{\partial s}(\tau,\sigma)\right| + \frac{M_4}{2^N}.$$

□

**Example 18.** *We consider three $(h_1,h_2)$-Bézier surfaces on the rectangle $[0,1]\times[0,1]$. We perform the recursive $(h_1,h_2)$-midpoint subdivision algorithm on each surface and we plot both the surface and the control points obtained in the 4th iteration of the subdivision algorithm.*

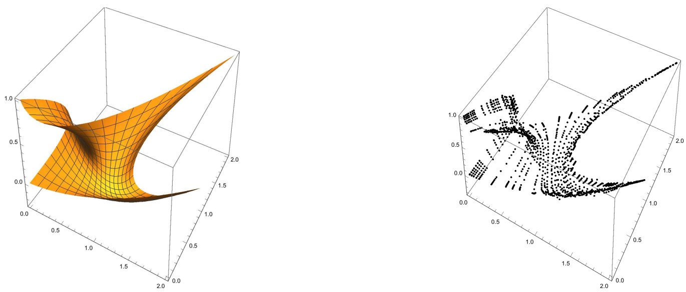

Figure 1: An $(h_1, h_2)$-Bézier surface cubic in $t$ and quadratic in $s$.

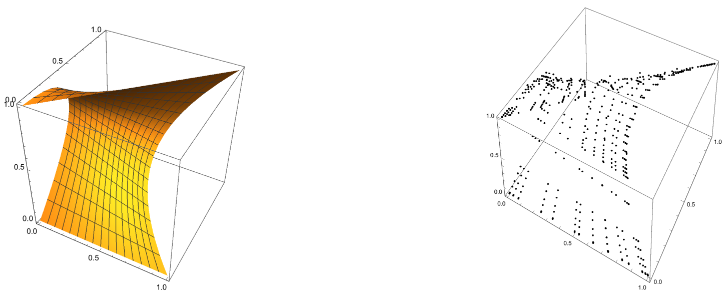

Figure 2: An $(h_1, h_2)$-Bézier surface linear in $t$ and quadratic in $s$.

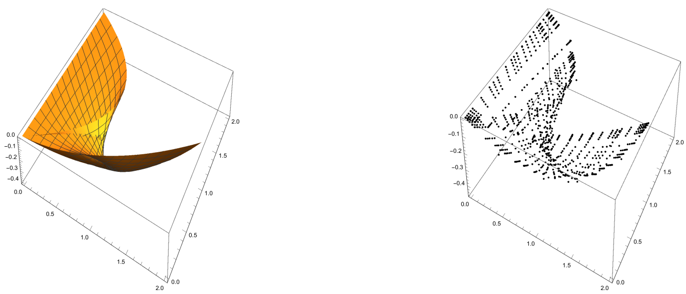

Figure 3: An $(h_1, h_2)$-Bézier surface quadratic in both $t$ and $s$.

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