International Journal of Applied Mathematics

Volume 30 No. 4 2017, 309-319

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v30i4.4

SOME RESULTS ON THE q, k AND p, q-GENERALIZED GAMMA FUNCTIONS

İnci Ege[1 §], Emrah Yıldırım[2]

[1,2]Department of Mathematics
Adnan Menderes University
09010, Aydın, TURKEY

Abstract: In this paper, the authors present some properties and inequalities for the p, q-generalized psi-function. Also they obtain double inequalities bounding ratios of q, k and p, q-generalized Gamma functions.

Some of the results in this paper are presented at International Conference on Applied and Mathematical Modeling ICAAMM 2017.

AMS Subject Classification: 33B15, 26A48, 26D15
Key Words: Gamma function, psi- (digamma-) function, q, k-generalized Gamma function, p, q-generalized Gamma function, inequality

1. Introduction

The classical Euler's Gamma function is one of the most important special functions with applications in many fields such as analysis, mathematical physics, statistics and probability theory. In [4], Diaz and Truel introduced the q, k-generalized Gamma function, and also in [10], Krasniqi and Merovci defined the p, q-generalized Gamma function. This work is devoted to establish some properties and also inequalities concerning ratios of these generalized functions.

The paper is organized as follows: In next Section 2, we present some notations and preliminaries that will be helpful in the sequel. In Section 3 we give some properties and inequalities for the functions $\Gamma_{p,q}(x)$ and $\psi_{p,q}(x)$ for $x > 0$. Also, we present double inequalities involving a ratio of the functions $\Gamma_{q,k}(x)$ and $\Gamma_{p,q}(x)$.

Received: June 9, 2017

§Correspondence author

2. Notations and Preliminaries

In this section, we present some definitions to make this paper self-containing. The reader can find details, e.g. in [3, 4, 5, 8, 11].

The well-known Euler's Gamma function is defined by the following integral for $x > 0$,

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt,$$

and it has also an equivalent limit expression as

$$\lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)(x+2)\dots(x+n)},$$

see [1, 2, 12]. The psi- or digamma-function, $\psi(x)$, is defined as the logarithmic derivative of the Gamma function. That is,

$$\psi(x) = \frac{d}{dx} \ln\Gamma(x) = \frac{\Gamma^{'}(x)}{\Gamma(x)}$$

for $x > 0$. The series representation is

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n\geq 1} \frac{x}{n(n+x)},$$

where γ denotes Euler's constant.

Diaz and Teruel [4] defined the q, k-generalized Gamma function $\Gamma_{q,k}(x)$ for $k > 0$, $q \in (0, 1)$ and $x > 0$ by the formula

$$\Gamma_{q,k}(x) = \frac{(1-q^k)_{q,k}^{\frac{x}{k}-1}}{(1-q)^{\frac{x}{k}-1}} = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^x)_{q,k}^{\infty}\,(1-q)^{\frac{x}{k}-1}},$$

where
$(x+y)_{q,k}^n = \prod_{i=0}^{n-1}(x+q^{ik}y)$, $\quad (1+x)_{q,k}^\infty = \prod_{i=0}^{\infty}(1+q^{ik}x)$,
$(1+x)_{q,k}^t = \frac{(1+x)_{q,k}^\infty}{(1+q^{kt}x)_{q,k}^\infty}$ for $x, y, t \in \mathbb{R}$ and $n \in \mathbb{N}$ and $\Gamma_{q,k}(x) \to \Gamma(x)$ as $q \to 1$ and $k \to 1$.

Also, Krasniqi and Merovci [10] defined the p, q extension of the Gamma function for $p \in \mathbb{N}$, $q \in (0, 1)$ and $x > 0$ as

$$\Gamma_{p,q}(x) = \frac{[p]_q!\, [p]_q^x}{[x]_q [x+1]q[x+2]_q \dots [x+p]_q},$$

where $[p]_q = \frac{1-q^p}{1-q}$ and $\Gamma_{p,q}(x) \to \Gamma(x)$ as $p \to \infty$ and $q \to 1$.

The functions $\Gamma_{q,k}(x)$ and $\Gamma_{p,q}(x)$ satisfy the following identities: $\Gamma_{q,k}(x+k) = [x]_q \Gamma_{q,k}(x)$, $\Gamma_{q,k}(k) = 1$ and $\Gamma_{p,q}(x+1) = [x]_q \Gamma_{p,q}(x)$, $\Gamma_{p,q}(1) = 1$.

Similarly to the definition of $\psi(x)$, the q, k and p, q-generalized of psi- (or digamma-) functions are defined respectively as:

$$\psi_{q,k}(x) = \frac{d}{dx} \ln \Gamma_{q,k}(x) = \frac{\Gamma'_{q,k}(x)}{\Gamma_{q,k}(x)}, \psi_{q,k}(x) = \frac{d}{dx} \ln \Gamma_{p,q}(x) = \frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)}$$

for $x > 0$, and they satisfy the series representations

$$\psi_{q,k}(x) = -\frac{1}{k} \ln(1-q) + (\ln q) \sum_{n=0}^{\infty} \frac{q^{nk+x}}{1-q^{nk+x}}, \tag{1}$$

$$\psi_{p,q}(x) = \ln[p]_q + (\ln q) \sum_{n=0}^{p} \frac{q^{n+x}}{1-q^{n+x}}, \tag{2}$$

where $\psi_{q,k}(x) \to \psi(x)$ as $q \to 1$ and $k \to 1, \psi_{p,q}(x) \to \psi(x)$ as $p \to \infty$ and $q \to 1$, [7, 10].

The function f is called log-convex if for all $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and for all $x, y > 0$ the following inequality holds:

$$\log f(\alpha x + \beta y) \leq \alpha \log f(x) + \beta \log f(y).$$

Note that the functions $\Gamma_{q,k}$ and $\Gamma_{p,q}$ are log-convex, [9, 10].

In the paper [6], the authors proved the inequality

$$\frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i x)^{\mu_i}}{\prod_{i=1}^{n} \Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i)^{\lambda}} \leq \frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i + \alpha_i x)^{\mu_i}}{\prod_{i=1}^{n} \Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)^{\lambda}} \leq \frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i)^{\mu_i}}{\prod_{i=1}^{n} \Gamma_{q,k}(\beta)^{\lambda}} \tag{3}$$

by using the method based on some monotonocity properties of q, k-extension of the Gamma function.

In this paper, one of our aim is to establish a generalization of equation (3) by using techniques similar to those of [6].

3. Main Results

We now present the results of this paper. Let us begin with the following theorem.

Theorem 1. *For $x > 0,\ p, n \in \mathbb{N}$ and $0 < q < 1$, the following inequality is valid:*

$$\frac{\Gamma_{p,q}(nx)}{\Gamma_{p,q}(x)} < [p]_q^{nx-1}. \tag{4}$$

Proof. Using the definition of $\Gamma_{p,q}$ for x and nx, we get

$$\begin{aligned}\frac{\Gamma_{p,q}(nx)}{\Gamma_{p,q}(x)} &= \frac{[p]_q^{nx}}{[p]_q^{x}}.\frac{[x]_q[x+1]_q\dots[x+p]_q}{[nx]_q[nx+1]_q\dots[nx+p]_q}\\ &< [p]_q^{nx-1},\end{aligned}$$

and thus the result follows. □

Corollary 2. *The inequality*

$$\Gamma_{p,q}(x+y) \leq [p]_q^{x+y-1}\sqrt{\Gamma_{p,q}(x)\Gamma_{p,q}(y)}$$

holds for $x, y > 0,\ p, n \in \mathbb{N}$ and $0 < q < 1$.

Proof. Since $\Gamma_{p,q}$ is log-convex, we can write

$$\Gamma_{p,q}(\frac{x+y}{2}) \leq \sqrt{\Gamma_{p,q}(x)\Gamma_{p,q}(y)}. \tag{5}$$

Then

$$\Gamma_{p,q}(x+y) \leq \sqrt{\Gamma_{p,q}(2x)\Gamma_{p,q}(2y)}.$$

From equation (4) in the last theorem we get for $n = 2$ that

$$\Gamma_{p,q}(2x) \leq \Gamma_{p,q}(x)[p]_q^{2x-1},\ \ \Gamma_{p,q}(2y) \leq \Gamma_{p,q}(y)[p]_q^{2y-1}.$$

Hence we get the result. □

The p, q-extension of the psi-function is similarly defined as

$$\psi_{p,q}(x) = \frac{d}{dx}\ln\Gamma_{p,q}(x) = \frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)}.$$

It satisfies the series representation:

$$\psi_{p,q}(x)=\ln[p]_q+(\ln q)\sum_{n=0}^{p}\frac{q^{n+x}}{1-q^{n+x}}, \tag{6}$$

where $\psi_{p,q}(x)\to\psi(x)$ as $p\to\infty$ and $q\to 1,$ [6].

Lemma 3. *For $x>0,\ p\in\mathbb{N}$ and $0<q<1$, the function $\psi_{p,q}(x)$ satisfies the equation:*

$$\psi_{p,q}(x+1)=-\ln q\frac{q^x}{1-q^x}+\psi_{p,q}(x). \tag{7}$$

Proof. Since

$$\Gamma_{p,q}(x+1)=[x]_q\Gamma_{p,q}(x), \tag{8}$$

by differentiating with respect to x both parts of equation (8), it follows:

$$\frac{d}{dx}\Gamma_{p,q}(x+1)=-\ln q\frac{q^x}{1-q}\Gamma_{p,q}(x)+[x]_q\frac{d}{dx}\Gamma_{p,q}(x). \tag{9}$$

By dividing both parts of (9) by $\Gamma_{p,q}(x)$, taking in mind the definition of $\psi_{p,q}(x)$ and equation (8), we obtain the desired equation. □

Remark 4. *By induction and using*

$$\Gamma_{p,q}(x+1)=[x]_q\Gamma_{p,q}(x),$$

we get

$$\Gamma_{p,q}(x+n)=[x]_{n,q}\Gamma_{p,q}(x)$$

for $x>0,\ p\in\mathbb{N},\ 0<q<1$ and $n\in\mathbb{N}$ where

$$[x]_{n,q}=[x]_q[x+1]_q[x+2]_q\dots[x+(n-1)]_q.$$

Theorem 5. *The function $\psi_{p,q}(x)$ satisfies the recurrence formula*

$$\psi_{p,q}(x+n)=\psi_{p,q}(x)-\ln q\sum_{j=0}^{n-1}\frac{q^{x+j}}{1-q^{x+j}}$$

for $x>0,\ p\in\mathbb{N}$ and $0<q<1$.

Proof. The equality will be proved by induction. For $n = 1$ it holds, because of equation (7). We suppose that our assumption holds for n and we will prove that it holds also for $n + 1$.

Since we have

$$\begin{aligned} \psi_{p,q}(x + (n + 1)) &= \psi_{p,q}((x + n) + 1) \\ &= \psi_{p,q}(x + n) - \ln q \frac{q^{x+n}}{1 - q^{x+n}} \\ &= \psi_{p,q}(x) - \ln q \sum_{j=0}^{n-1} \frac{q^{x+j}}{1 - q^{x+j}} - \ln q \frac{q^{x+n}}{1 - q^{x+n}} \\ &= \psi_{p,q}(x) - \ln q \sum_{j=0}^{n} \frac{q^{x+j}}{1 - q^{x+j}}, \end{aligned}$$

then our assumption is true for every $n \in \mathbb{N}$. Hence the result follows. □

Theorem 6. *The following inequalities are valid for $x > 0$, $p \in \mathbb{N}$ and $0 < q < 1$:*

$$\frac{q^x}{1 - q^x} \ln q + \ln[x]_q < \psi_{p,q}(x) < \ln[x]_q. \tag{10}$$

Proof. Let $f(x) = \ln \Gamma_{p,q}(x)$. We apply the mean value theorem to this function in the interval $(x, x + 1)$.

Then, there is $x_0 \in (x, x + 1)$ such that the equality

$$\ln \Gamma_{p,q}(x + 1) - \ln \Gamma_{p,q}(x) = \psi_{p,q}(x_0)$$

holds, and using

$$\Gamma_{p,q}(x + 1) = [x]_q \Gamma_{p,q}(x)$$

we get

$$\psi_{p,q}(x_0) = \ln[x]_q.$$

Since

$$\psi'_{p,q}(x) = \ln^2 q \sum_{k=0}^{p} \frac{q^{x+k}}{(1 - q^{x+k})^2} > 0,$$

we have $\psi_{p,q}(x)$ is increasing on $(0, \infty)$. Then we obtain

$$\psi_{p,q}(x) < \psi_{p,q}(x_0) < \psi_{p,q}(x + 1).$$

Since we got

$$\psi_{p,q}(x+1) = -\ln q \frac{q^x}{1-q^x} + \psi_{p,q}(x),$$

we have

$$\psi_{p,q}(x) < \ln[x]_q < -\ln q \frac{q^x}{1-q^x} + \psi_{p,q}(x),$$

and the result follows. $\square$

Corollary 7. *For $p \in \mathbb{N}$ and $0 < q < 1$ we have*

$$\frac{q}{1-q} \ln q < \psi_{p,q}(1) < 0$$

and for $x \in (0, 1]$ we have

$$\psi_{p,q}(x) < 0.$$

Lemma 8. *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function on any open interval and $\alpha, \beta, \gamma_i, \mu, \lambda, b$ be real numbers such that*

$$b + \alpha x \leq \beta + \sum_{i=1}^{n} \gamma_i x\,,\ \gamma_i \lambda \geq \alpha\mu > 0.$$

If

$$f(b + \alpha x) > 0 \quad \text{or} \quad f(\beta + \sum_{i=1}^{n} \gamma_i x) > 0,$$

then

$$\alpha\mu f(b + \alpha x) - \lambda\gamma_i f(\beta + \sum_{i=1}^{n} \gamma_i x) \leq 0 \tag{11}$$

is valid.

Proof. Let $f(b+\alpha x) > 0$. Since f is increasing, $f(b+\alpha x) \leq f(\beta+\sum_{i=1}^{n} \gamma_i x)$. Then $f(\beta + \sum_{i=1}^{n} \gamma_i x) > 0$.
Writing

$$\alpha\mu f(b + \alpha x) \leq \alpha\mu f(\beta + \sum_{i=1}^{n} \gamma_i x) \leq \lambda\gamma_i f(\beta + \sum_{i=1}^{n} \gamma_i x),$$

leads us to equation (11). This time, let $f(\beta+\sum_{i=1}^{n}\gamma_i x)>0$. Then $f(b+\alpha x)>0$ or $f(b+\alpha x)\leq 0$.

If $f(b+\alpha x)>0$, then the proof is completed. And if $f(b+\alpha x)\leq 0$; since $\gamma_i\lambda\geq\alpha\mu>0$ we have

$$\gamma_i\lambda f(b+\alpha x)\leq\alpha\mu f(\beta+\sum_{i=1}^{n}\gamma_i x)\leq\gamma_i\lambda f(\beta+\sum_{i=1}^{n}\gamma_i x).$$

Hence equation (11) holds. □

One can prove the following lemma immediately:

Lemma 9. *Let $f:\mathbb{R}\to\mathbb{R}$ be an increasing function on any open interval and $\alpha,\beta,\gamma_i,\mu,\lambda,b$ be real numbers such that*

$$b+\alpha x\leq\beta+\sum_{i=1}^{n}\gamma_i x\,,\ \alpha\mu\geq\gamma_i\lambda>0.$$

Then if

$$f(b+\alpha x)<0\ \ \text{or}\ \ f(\beta+\sum_{i=1}^{n}\gamma_i x)<0,$$

the inequality (11) still holds.

Preparation for Applications:

Since $\psi_{q,k}(x)$ and $\psi_{p,q}(x)$ are increasing functions on the open interval $(0,\infty)$, we can write $\psi_{q,k}(x)$ or $\psi_{p,q}(x)$ in equation (11) instead of f.

Applications to the q,k Generalized Gamma Function:

We apply Lemmas 8 and 9 to the function $\Gamma_{q,k}$. Note that one can get similar results for the generalized p,q-Gamma function $\Gamma_{p,q}$.

Theorem 10. *Let $\alpha_i,\beta,\gamma_i,\mu_i,\lambda,b_i$ be positive real numbers such that*

$$b_i+\alpha_i x\leq\beta+\sum_{i=1}^{n}\gamma_i x\,,\ \gamma_i\lambda\geq\alpha_i\mu_i>0.$$

If

$$\psi_{q,k}(b_i+\alpha_i x)>0\ \ \text{or}\ \ \psi_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i x)>0,$$

then

$$g(x)=\frac{\prod_{i=1}^{n}\Gamma_{q,k}(b_i+\alpha_i x)^{\mu_i}}{\prod_{i=1}^{n}\Gamma_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i x)^{\lambda}}$$

is decreasing function for $x\geq 0$.

Proof. Let $H(x)=\ln g(x)$. Then,

$$\begin{aligned} H(x) &= \ln\frac{\prod_{i=1}^{n}\Gamma_{q,k}(b_i+\alpha_i x)^{\mu_i}}{\prod_{i=1}^{n}\Gamma_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i x)^{\lambda}} \\ &= \mu_i\ln\prod_{i=1}^{n}\Gamma_{q,k}(b_i+\alpha_i x)-\lambda\ln\prod_{i=1}^{n}\Gamma_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i x). \end{aligned}$$

We have

$$\begin{aligned} H^{'}(x) &= \sum_{i=1}^{n}\mu_i\alpha_i\frac{\Gamma_{q,k}(b_i+\alpha_i x)}{\Gamma^{'}_{q,k}(b_i+\alpha_i x)}-\lambda\sum_{i=1}^{n}\frac{\Gamma_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i x)}{\Gamma^{'}_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i x)} \\ &= \sum_{i=1}^{n}\Big[\mu_i\alpha_i\psi_{q,k}(b_i+\alpha_i x)-\lambda\gamma_i\psi_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i x)\Big]\leq 0. \end{aligned}$$

This implies that H is decreasing on $x\in[0,\infty)$. As a result, g is decreasing on $x\in[0,\infty)$. □

Corollary 11. *Let* $\alpha_i,\beta,\gamma_i,\mu_i,\lambda,b_i$ *be positive real numbers such that*

$$b_i+\alpha_i x\leq\beta+\sum_{i=1}^{n}\gamma_i x\,,\ \ \gamma_i\lambda\geq\alpha_i\mu_i>0$$

and let

$$\psi_{q,k}(b_i+\alpha_i x)>0\ \ \text{or}\ \ \psi_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i x)>0.$$

Then for $x \in [0,1]$ we have

$$\frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i+\alpha_i x)^{\mu_i}}{\prod_{i=1}^{n} \Gamma_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i)^{\lambda}} \leq \frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i+\alpha_i x)^{\mu_i}}{\prod_{i=1}^{n} \Gamma_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i x)^{\lambda}} \leq \frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i)^{\mu_i}}{\prod_{i=1}^{n} \Gamma_{q,k}(\beta)^{\lambda}}, \tag{12}$$

and for $x \in [1,\infty)$ we have

$$\frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i+\alpha_i x)^{\mu_i}}{\prod_{i=1}^{n} \Gamma_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i x)^{\lambda}} \leq \frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i+\alpha_i)^{\mu_i}}{\prod_{i=1}^{n} \Gamma_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i)^{\lambda}}. \tag{13}$$

Proof. Since $g(x) = \frac{\prod_{i=1}^{n} \Gamma_{q,k}(b_i+\alpha_i x)^{\mu_i}}{\prod_{i=1}^{n} \Gamma_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i x)^{\lambda}}$ is decreasing function, for $x \in [0,1]$ we have

$$g(1) \leq g(x) \leq g(0),$$

and for $x \in [1,\infty)$

$$g(x) \leq g(1);$$

yielding the results. □

Remark 12. *Let $\alpha, \beta, \gamma_i, \mu, \lambda, b$ be real numbers such that*

$$b+\alpha x \leq \beta + \sum_{i=1}^{n} \gamma_i x\,, \;\; \alpha\mu \geq \gamma_i \lambda > 0.$$

Then if

$$\psi_{q,k}(b+\alpha x) < 0 \;\; \text{or} \;\; \psi_{q,k}(\beta+\sum_{i=1}^{n}\gamma_i x) < 0,$$

inequalities (12) and (13) are hold.

Remark 13. *If we set $\gamma_i = \alpha_i$ in Theorem 10 and Corollary 11, we obtain inequalities (3.3) and (3.4) from [6].*

References

[1] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopedia Math. Appl. **71**, Cambridge University Press, Cambridge (2000).

[2] T.M. Apostol, *Introduction to Analytic Number Theory*, Springer (1976).

[3] R. Diaz and E. Pariguan, On hypergeometric functions and Pochhammer k-symbol, *Divulgaciones Matematicas*, **15**, No 2 (2007), 179-192.

[4] R. Diaz and C. Teruel, q, k-generalized gamma and beta functions, *Journal of Nonlinear Mathematical Physics*, **12**, No 1 (2005), 118-134.

[5] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge (1990).

[6] K. Nantomah, E. Prempeh and S.B. Twun, On (p, q) and (q, k)-extensions of a double-inequality bounding a ratio of gamma functions, *Mathematical Sciences and Applications E-Notes*, **4**, No 2 (2016), 23-28.

[7] K. Nantomah, E. Prempeh and S.B. Twun, The $(q, k)-$extension of some Gamma function inequalities, *Konuralp Journal of Mathematics*, **4**, No 1 (2016), 148-154.

[8] V.G. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer-Verlag, New York (2002).

[9] C.G. Kokologiannaki, Some properties of $\Gamma_{q,k}(t)$ and related functions, *International Journal of Contemporary Mathematical Sciences*, **11**, No 1 (2016), 1-8.

[10] V. Krasniqi and F. Merovci, Some completely monotonic properties for the (p, q)-gamma function, *Mathematica Balkanica, New Series*, **26** No. 1-2 (2012), 133-145.

[11] A.D. Sole, V.G. Kac, On integral representations of q-gamma and q-beta functions, *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni*, **16**, No 1 (2005), 11-29.

[12] N.M. Temme *Special Functions, an Introduction to Classical Functions of Mathematical Physiscs*, John Wiley and Sons, New York (1996).