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COMMUTATIVE NEUTRIX CONVOLUTION PRODUCT INVOLVING GAUSSIAN ERROR FUNCTION

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Abstract: In this paper, using neutrix calculus, several commutative neutrix convolution products are evaluated, involving the Gaussian error function $\mathrm{erf}(x)$ and its associated functions $\mathrm{erf}(x_+)$ and $\mathrm{erf}(x_-)$.

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1. Gaussian Error Function

The error function (also known as Gaussian error function) $\mathrm{erf}(x)$ [20] is defined for $x \in \mathbf{R}$ by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du = \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(2i+1)} x^{2i+1}.$$

The error function is odd, convex on $(-\infty, 0]$, concave on $[0, \infty)$, and strictly increasing on $\mathbf{R}$. Some other properties of this function the reader can see in [1, 2].

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The Gaussian error function plays an important role in applied mathematics. That is why Dirschmid and Fischer extended the classical Gaussian error function $\operatorname{erf}(x)$ to a family of infinite extended Gaussian error functions $\operatorname{erf}_i(x)$ (for $i \geq 1$) which can be easily programmed by current computational tools. The generalized Gaussian error function for $i \in \mathbf{N}$ are defined by

$$\operatorname{erf}_i(x) = \frac{2}{\sqrt{\pi}} \int_0^x u^i e^{-u^2}\, du,$$

see [4].

It can be easily noted that

$$\lim_{x \to 0} \operatorname{erf}_i(x) = 0, \quad \lim_{x \to \infty} \operatorname{erf}_i(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty u^i e^{-u^2}\, du = \frac{1}{\sqrt{\pi}} \Gamma\big(\frac{i+1}{2}\big),$$

$$\lim_{x \to -\infty} \operatorname{erf}_i(x) = \frac{2}{\sqrt{\pi}} \int_0^{-\infty} u^i e^{-u^2}\, du = \frac{(-1)^{i+1}}{\sqrt{\pi}} \Gamma\big(\frac{i+1}{2}\big),$$

see [15].

The locally summable functions $\operatorname{erf}(x_+)$ and $\operatorname{erf}(x_-)$ are defined by

$$\operatorname{erf}(x_+) = H(x) \operatorname{erf}(x), \quad \operatorname{erf}(x_-) = H(-x) \operatorname{erf}(x),$$

where H denotes the Heaviside function.

The functions $\operatorname{erf}(|x|^{1/2})$, $\operatorname{erf}(x_+^{1/2})$ and $\operatorname{erf}(x_-^{1/2})$ are similarly defined by

$$\begin{aligned} \operatorname{erf}(|x|^{1/2}) &= \frac{2}{\sqrt{\pi}} \int_0^{|x|^{1/2}} \exp(-u^2)\, du, \\ \operatorname{erf}(x_+^{1/2}) &= H(x) \operatorname{erf}(|x|^{1/2}), \quad \operatorname{erf}(x_-^{1/2}) = H(-x) \operatorname{erf}(|x|^{1/2}). \end{aligned}$$

Similarly, we define the locally summable functions $\operatorname{erf}_i(x_+)$ and $\operatorname{erf}_i(x_-)$ by

$$\operatorname{erf}_i(x_+) = H(x) \operatorname{erf}_i(x), \quad \operatorname{erf}_i(x_-) = H(-x) \operatorname{erf}_i(x).$$

In [8] the functions $\operatorname{erf}_{2i,+}(|x|^{1/2})$ and $\operatorname{erf}_{2i,-}(|x|^{1/2})$ were defined by

$$\operatorname{erf}_{2i,+}(|x|^{1/2}) = H(x) \operatorname{erf}_{2i}(|x|^{1/2}), \quad \operatorname{erf}_{2i,-}(|x|^{1/2}) = H(-x) \operatorname{erf}_{2i}(|x|^{1/2}),$$

for $i = 0, 1, 2, \ldots$.

2. Convolution Product

The classical definition of the convolution of two locally summable functions f and g is as follows:

Definition 1. Let f and g be functions. Then the *convolution* $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$$

for all points x for which the integral exists.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and

$$f * g = g * f$$

and if $(f * g)'$ and $f * g'$ (or $f' * g$) exists, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{2.1}$$

Definition 1 can be extended to define the convolution $f * g$ of two distributions f and g in $'$, the space of infinitely differentiable functions with compact support, see Gel'fand and Shilov [14].

Definition 2. Let f and g be distributions in $\mathcal{D}'$. Then the *convolution* $f * g$ is defined by the equation

$$\langle (f * g)(x), \varphi \rangle = \langle f(y), \langle g(x), \varphi(x+y) \rangle \rangle$$

for arbitrary φ in , provided f and g satisfy either of the conditions

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side.

Note that if f and g are locally summable functions satisfying either of the above conditions and the classical convolution $f * g$ exists, then it is in agreement with Definition 1.

The convolution product of distributions may be defined in a more general way without any restriction on the supports. The most well-known is given by Jones, see [18]. However, there still exist many pairs of distributions such that the convolution products do not exist in the sense of these definitions.

The following results concerning the Gaussian error function and convolution product were proved in [13]:

$$\begin{aligned}
x_+^r * \operatorname{erf}_+(x) &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i \operatorname{erf}_i(x) x_+^{r-i+1}, \\
x_+^r * [x\exp_+(-x^2)] &= \frac{\sqrt{\pi}}{2(r+1)}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{-i}\operatorname{erf}_i(x) x_+^{r-i+2} \\
&+ \frac{\sqrt{\pi}}{2(r+1)}\sum_{i=1}^{r+1}\binom{r+1}{i}(-1)^{-i-1}\operatorname{erf}_i(x) x_+^{r-i+1},
\end{aligned}$$

for $r = 0, 1, 2, \dots$, where $\exp_+(-x^2) = H(x)\exp(-x^2)$.

Also, for proving our main results, we will need following results proved in [8]:

$$\begin{aligned}
x_+^r * \operatorname{erf}(x_+^{1/2}) &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i \operatorname{erf}_{2i}(x_+^{1/2}) x_+^{r-i+1}, \\
x_-^r * \operatorname{erf}(x_-^{1/2}) &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i \operatorname{erf}_{2i}(|x|^{1/2}) x_-^{r-i+1}, \\
x_+^r * [x_+^{-1/2}\exp(-|x|)] &= \sqrt{\pi}\sum_{i=0}^{r}\binom{r}{i}(-1)^i \operatorname{erf}_{2i}(x_+^{1/2}) x_+^{r-i}, \qquad (2.2)
\end{aligned}$$

for $r = 0, 1, 2, \dots$.

3. Commutative Neutrix Convolution Product

In [11] the *commutative neutrix convolution product* is defined so that it exists for a considerably large class of pairs of distributions. In that definition, unit-sequences of function in $\mathcal{D}$ are used which allows one to approximate a given distribution by a sequence of distributions of bounded support.

The method of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part. Using the concepts of the neutrix and the neutrix limit due to van der Corput [3], Fisher gave the general principle for the discarding of unwanted infinite quantities from asymptotic expansions and this has been exploited in context of distributions, particularly in connection with convolution product and distributional multiplication, see [5, 6, 7, 17, 19, 9].

In order to introduce Fisher's definition of neutrix convolution product, we first of all let τ be a function in $\mathcal{D}$ satisfying the following properties:

To recall the definition of the commutative neutrix convolution product we first let τ be a function in $\mathcal{D}$, see [18], satisfying the conditions:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leq \tau(x) \leq 1$,
(iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function τ_n is then defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \ldots$.

We have the following definition of the commutative neutrix convolution product.

Definition 3. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ and $g_n = g\tau_n$ for $n = 1, 2, \ldots$. Then the commutative neutrix convolution product $f \boxed{*} g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided that the limit h exists in the sense that

$$\mathop{\mathrm{N-lim}}_{\nu \to \infty} \langle f_n * g_n, \varphi \rangle = \langle h, \varphi \rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix (see van der Corput [3]), having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n, \quad (\lambda \neq 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Note that in this definition the convolution product $f_n * g_n$ is in the sense of Definition 1, the distribution f_n and g_n having bounded support since the support of τ_n is contained in the interval $[-n - n^{-n}, n + n^{-n}]$. This neutrix convolution product is also commutative. Some results concerning commutative neutrix convolution product are given in [16].

It is obvious that any results proved with the original definition hold with the new definition. The following theorem, proved in [11], therefore hold, the first showing that the commutative neutrix convolution product is a generalization of the convolution product.

Theorem 4. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the commutative neutrix convolution product $f \boxed{*} g$ then exists and*

$$f \boxed{*} g = f * g.$$

Note however that $(f \boxed{*} g)'$ is not necessarily equal to $f' \boxed{*} g$ but we do have the following theorem proved in [12].

Theorem 5. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the commutative neutrix convolution product $f \boxed{*} g$ exists. If $\mathrm{N}-\lim\limits_{\nu\to\infty}\langle (f\tau_n') * g_n, \varphi\rangle$ exists and equals $\langle h, \varphi\rangle$ for all φ in , then $f' \boxed{*} g$ exists and*

$$(f \boxed{*} g)' = f' \boxed{*} g + h\,. \tag{3.1}$$

4. Main Results

Before proving our results on the convolution, we need the following lemma, which is easily proved by induction, see [8]:

Lemma 1.

$$\mathrm{erf}_{2i}(x) = \frac{2}{\sqrt{\pi}}\int_0^x u^{2i}e^{-u^2}\,du$$

$$= -\sum_{j=0}^{i-1}\frac{(2i)!(i-j)!}{\sqrt{\pi}2^{2j}i!(2i-2j)!}x^{2i-2j-1}\exp(-x^2) + \frac{(2i)!}{2^{2i}i!}\mathrm{erf}(x), \tag{4.1}$$

$$\mathrm{erf}_{2i+1}(x) = \frac{2}{\sqrt{\pi}}\int_0^x u^{2i+1}e^{-u^2}\,du$$

$$= -\sum_{j=0}^{i}\frac{i!}{\sqrt{\pi}(i-j)!}x^{2i-2j}\exp(-x^2) + \frac{i!}{\sqrt{\pi}},$$

for $i = 0, 1, 2, \ldots,$ where the sum in (4.1) is empty when $i = 0$.

In order to prove our next results we need to extend our set of negligible functions given in Definition 3 to include also finite linear sums of the function

$$n^r \operatorname{erf}[(x+n)^{1/2})], \qquad r = 1, 2, \dots .$$

We now prove our main results.

Theorem 6. *The commutative neutrix convolution product* $x^r \boxed{*} \operatorname{erf}(x_+^{1/2})$ *exists and*

$$x^r \boxed{*} \operatorname{erf}(x_+^{1/2}) = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{(-1)^i (2i)!}{2^{2i} i!} x^{r-i+1}, \tag{4.2}$$

for $r = 0, 1, 2, \dots$.

Proof. We put $(x^r)_n = x^r \tau_n(x)$ and $\left[\operatorname{erf}(x_+^{1/2})\right]_n = \operatorname{erf}(x_+^{1/2})\tau_n$ for $n = 1, 2, \dots$. Since these functions have compact support, the classical convolution product $(x^r)_n * \left[\operatorname{erf}(x_+^{1/2})\right]_n$ exists by Definition 1, and

$$\begin{aligned}
(x^r)_n * \left[\operatorname{erf}(x_+^{1/2})\right]_n &= \int_{-\infty}^{\infty} (x-t)^r \operatorname{erf}(t^{1/2}) \tau_n(x-t) \tau_n(t)\, dt \\
&= \frac{2}{\sqrt{\pi}} \int_0^{x+n} (x-t)^r \operatorname{erf}(t^{1/2}) \tau_n(x-t)\, dt \\
&\quad + \frac{2}{\sqrt{\pi}} \int_{x+n}^{x+n+n^{-n}} (x-t)^r \tau_n(x-t) \operatorname{erf}(t^{1/2}) \tau_n(t)\, dt \\
&= I_1 + I_2.
\end{aligned}$$

For I_1 we have:

$$\begin{aligned} I_1 &= \frac{2}{\sqrt{\pi}} \int_0^{x+n} (x-t)^r \int_0^{t^{1/2}} \exp(-u^2)\, du\, dt \\ &= \frac{2}{\sqrt{\pi}} \int_0^{(x+n)^{1/2}} \exp(-u^2) \int_{u^2}^{x+n} (x-t)^r\, dt\, du \\ &= \frac{2}{\sqrt{\pi}(r+1)} \int_0^{(x+n)^{1/2}} (x-u^2)^{r+1} \exp(-u^2)\, du \\ &\qquad - \frac{2(-n)^{r+1}}{\sqrt{\pi}(r+1)} \int_0^{(x+n)^{1/2}} \exp(-u^2)\, du \\ &= \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i \operatorname{erf}_{2i}[(x+n)^{1/2})] x^{r-i+1} \\ &\qquad - \frac{(-n)^{r+1}}{r+1} \operatorname{erf}[(x+n)^{1/2}]. \end{aligned}$$

It follows easily from Lemma 1 and on noting that

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-u^2)\, du = 1,$$

we have

$$\mathrm{N}\!-\!\lim_{n\to\infty} \operatorname{erf}_{2i}[(x+n)^{1/2}] = \frac{(2i)!}{2^{2i} i!},$$

and so

$$\mathrm{N}\!-\!\lim_{n\to\infty} I_1 = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{(-1)^i (2i)!}{2^{2i} i!} x^{r-i+1}$$

for $i = 0, 1, 2, \ldots$.

Further, it is easily seen that

$$\lim_{n\to\infty} I_2 = \lim_{n\to\infty} \frac{2}{\sqrt{\pi}} \int_{x+n}^{x+n+n^{-n}} (x-t)^r \operatorname{erf}(t^{1/2}) \tau_n(x-t) \tau_n(t)\, dt = 0\,.$$

Equation (4.2) follows, thus proving the theorem. □

Replacing x by $-x$ in equation (4.2), we get

Corollary 7. *The commutative neutrix convolution product* $x^r \boxed{*} \operatorname{erf}(x_-^{1/2})$ *exists and*

$$x^r \boxed{*} \operatorname{erf}(x_-^{1/2}) = -\frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}\frac{(2i)!}{2^{2i}i!}x^{r-i+1} \tag{4.3}$$

for $r = 0, 1, 2, \ldots$.

Corollary 8. *The commutative neutrix convolution product* $x^r \boxed{*} \operatorname{erf}(|x|^{1/2})$ *exists and*

$$x^r \boxed{*} \operatorname{erf}(|x|^{1/2}) = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}\frac{[(-1)^i - 1](2i)!}{2^{2i}i!}x^{r-i+1} \tag{4.4}$$

for $r = 0, 1, 2, \ldots$.

Proof. Note that

$$x^r \boxed{*} \operatorname{erf}(|x|^{1/2}) = x^r \boxed{*} \operatorname{erf}(x_+^{1/2}) + x^r \boxed{*} \operatorname{erf}(x_-^{1/2}).$$

Then equation (4.4) follows from equations (4.2) and (4.3). □

Theorem 9. *The commutative neutrix convolution product* $x_+^r \boxed{*} \operatorname{erf}(|x|^{1/2})$ *exists and*

$$\begin{aligned} x_+^r \boxed{*} \operatorname{erf}(|x|^{1/2}) &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i[x_+^{r-i+1} - (-1)^r x_-^{r-i+1}] \\ \times \quad & \operatorname{erf}_{2i}(|x|^{1/2}) - \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}\frac{(2i)!}{2^{2i}i!}x^{r-i+1} \end{aligned} \tag{4.5}$$

for $r = 0, 1, 2, \ldots$.

Proof. Using equation (4.2), we have

$$\begin{aligned} x_+^r \boxed{*} \operatorname{erf}(|x|^{1/2}) &= x_+^r * \operatorname{erf}(x_+^{1/2}) + x_+^r \boxed{*} \operatorname{erf}(x_-^{1/2}) \\ &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i x_+^{r-i+1}\operatorname{erf}_{2i}(|x|^{1/2}) \\ &\quad + x_+^r \boxed{*} \operatorname{erf}(x_-^{1/2}). \end{aligned} \tag{4.6}$$

Using equations (2.2) and (4.3), we have

$$\begin{aligned} x^r \boxed{*}\, \mathrm{erf}(x_-^{1/2}) &= x_+^r \boxed{*}\, \mathrm{erf}(x_-^{1/2}) + (-1)^r x_-^r * \mathrm{erf}(x_-^{1/2}) \\ &= x_+^r \boxed{*}\, \mathrm{erf}(x_-^{1/2}) \\ &\quad + \frac{(-1)^r}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i x_-^{r-i+1}\, \mathrm{erf}_{2i}(|x|^{1/2}) \\ &= -\frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{(2i)!}{2^{2i} i!} x^{r-i+1}. \end{aligned} \tag{4.7}$$

Equation (4.5) now follows from equations (4.6) and (4.7), proving the theorem. □

Replacing x by $-x$, we get

Corollary 10. *The commutative neutrix convolution product* $x_-^r \boxed{*}\, \mathrm{erf}(|x|^{1/2})$ *exists and*

$$x_-^r \boxed{*}\, \mathrm{erf}(|x|^{1/2}) = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^i [x_-^{r-i+1} - (-1)^r x_+^{r-i+1}]$$

$$\times \quad \mathrm{erf}_{2i}(|x|^{1/2}) + \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i} \frac{(2i)!}{2^{2i} i!} x^{r-i+1} \tag{4.8}$$

for $r = 0, 1, 2, \ldots$.

Theorem 11. *The commutative neutrix convolution product* $x^r \boxed{*}\, [x_+^{-1/2} \exp(-|x|)]$ *exists and*

$$x^r \boxed{*}\, [x_+^{-1/2} \exp(-|x|)] = \frac{\sqrt{\pi}}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \frac{(-1)^i (r-i+1)(2i)!}{2^{2i} i!} x^{r-i}, \tag{4.9}$$

for $r = 0, 1, 2, \ldots$.

Proof. Differentiating equation (4.2) using equation (3.1), we get

$$\begin{aligned} [x^r \boxed{*}\, \mathrm{erf}(x_+^{1/2})]' &= \frac{1}{\sqrt{\pi}} x^r \boxed{*}\, [x_+^{-1/2} \exp(-|x|)] \\ &= \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \frac{(-1)^i (r-i+1)(2i)!}{2^{2i} i!} x^{r-i}, \end{aligned}$$

and equation (4.9) follows for $r = 0, 1, 2, \ldots$. □

Replacing x by $-x$ in equation (4.9), we get

Corollary 12. *The commutative neutrix convolution product* $x^r \boxed{*} [x_-^{-1/2} \exp(-|x|)]$ *exists and*

$$x^r \boxed{*} [x_-^{-1/2} \exp(-|x|)] = \frac{\sqrt{\pi}}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \frac{(r-i+1)(2i)!}{2^{2i} i!} x^{r-i}, \tag{4.10}$$

for $r = 0, 1, 2, \ldots$.

Adding equations (4.9) and (4.10), we get

Corollary 13. *The commutative neutrix convolution product* $x^r \boxed{*} [|x|^{-1/2} \exp(-|x|)]$ *exists and*

$$x^r \boxed{*} [|x|^{-1/2} \exp(-|x|)] = \frac{\sqrt{\pi}}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{[1 + (-1)^i](r-i+1)(2i)!}{2^{2i} i!} x^{r-i}$$

for $r = 0, 1, 2, \ldots$.

Taking equation (4.10) from equation (4.9), we get

Corollary 14. *The commutative neutrix convolution product* $x^r \boxed{*} [\operatorname{sgn} x .|x|^{-1/2} \exp(-|x|)]$ *exists and*

$$x^r \boxed{*} [\operatorname{sgn} x .|x|^{-1/2} \exp(-|x|)] = \frac{\sqrt{\pi}}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{[(-1)^i - 1](r-i+1)(2i)!}{2^{2i} i!} x^{r-i}$$

for $r = 0, 1, 2, \ldots$.

For further related results, see [10, 13].

References

[1] H. Alzer, A Kuczma-type functional inequality for error and complementary error functions, *Aequat. Math.*, **89** (2015), 927-935.

[2] H. Alzer, M.N. Kwong, A subaddivity property of the error function, *Proc. Amer. Math. Soc.*, **142**, No 8 (2014), 2697-2704.

[3] J.G. van der Corput, Introduction to the neutrix calculus, *J. Analyse Math.*, **7** (1959-1960), 291-398.

[4] H. Dirschmid, F.D. Fischer, Generalized Gausssian error functions and their applications, *Acta Mech.*, **226** (2015), 2887-2897.

[5] B. Fisher, A noncommutative neutrix product of distributions, *Math. Nachr.*, **108** (1982), 117-127.

[6] B. Fisher, Neutrices and the convolution of distributions, *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.*, **17** (1987), 119-135.

[7] B. Fisher, B. Jolevska-Tuneska, A. Takaci, On convolutions and neutrix convolutions involving the incomplete gamma function, *Integral Transforms Spec. Funct.*, **15**, No 5 (2004), 405-414.

[8] B. Fisher, F. Al-Sirehy, E. Ozcag, Results involving gaussian error function $\operatorname{erf}\left(|x|^{1/2}\right)$ and the neutrix convolution, *Advances in Math., Sci. J.*, **5**, No 2 (2016), 231-240.

[9] B. Fisher, F. Al-Sirehy, Results on the dilagorithm integral, *International Journal of Applied Mathematics*, **28**, No 6 (2015), 727-738; DOI: 10.12732/ijam.v28i6.7.

[10] B. Fisher, F. Al-Sirehy, M. Telci, Convolutions involving the error function, *International Journal of Applied Mathematics*, **13**, No 4 (2003), 317-326.

[11] B. Fisher, C.K. Li, A commutative neutrix convolution product of distributions, *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.*, **23** (1993), 13-27.

[12] B. Fisher, E. Özçağ, Ü Gülen, The exponential integral and the commutative neutrix convolution product, *J. Analysis*, **7** (1999), 7-20.

[13] B. Fisher, M. Telci, F. Al-Sirehy, The error function and the neutrix convolution, *International Journal of Applied Mathematics*, **8** (2002), 295-309.

[14] I.M. Gel'fand, G.E. Shilov, *Generalized Functions*, Vol. I, Academic Press (1964).

[15] W. Gröbner, N. Hofreter, *Integrafel II: Bestimmte Integrale, Band 2*, Springer, New York (1973).

[16] B. Jolevska-Tuneska, A. Takaci, Results on the commutative neutrix convolution product of distributions, *Hacettepe J. of Mathematics and Statistics*, **37**, No 2 (2008), 135-141.

[17] B. Jolevska-Tuneska, A. Takaci, B. Fisher, On some neutrix convolution product of distributions, *Integral Trans. Spec. Funct.*, **14**, No 3 (2003), 243-250.

[18] D.S. Jones, The convolution of generalized functions, *Quart. J. Math.*, **24** (1973), 145-163.

[19] E. Özçağ, H. Kansu, A result of neutrix convolution of distributions, *Indian J. Pure Appl. Math.*, **28**, No 10 (1997), 1399-1411.

[20] I.N. Sneddon, *Special Functions of Mathematical Physics and Chemistry*, Oliver and Boyd (1961).