International Journal of Applied Mathematics

Volume 30 No. 2 2017, 107-115

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v30i2.3

STRENGTH OF TENSOR PRODUCT OF CERTAIN STRONG FUZZY GRAPHS

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Abstract: In this paper we find the strength of tensor product of two strong fuzzy graphs with their underlying crisp graphs P_2 and P_n, P_2 and S_n, P_2 and C_n and also K_n and K_m.

AMS Subject Classification: 05C72
Key Words: tensor product of fuzzy graphs, fuzzy star graphs, fuzzy cycle, fuzzy complete graphs

1. Introduction

In 1965, Lofti A. Zadeh [9] introduced the notion of a fuzzy subset. Azriel Rosenfeld [6] defined fuzzy graph based on the definitions of fuzzy sets and fuzzy relations and developed the theory of fuzzy graphs in 1975. John N. Mordeson and Premchand S. Nair [4] introduced different types of operations on fuzzy graphs. M.B. Sheeba [7] introduced the concept of strength of fuzzy graphs. She determined the strength of fuzzy graphs in two different ways. One by introducing weight matrix of a fuzzy graph and the other by introducing the concept of extra strong path between its vertices. In this paper we use the concept of extra strong path to find the strength of various fuzzy graphs.

Received: January 15, 2017

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Throughout this paper only undirected fuzzy graphs are considered.

2. Preliminaries

A fuzzy graph $G = (V, \mu, \sigma)$ [4] is a nonempty set V together with a pair of functions $\mu : V \longrightarrow [0, 1]$ and $\sigma : V \times V \longrightarrow [0, 1]$ such that for all $u, v \in V$, $\sigma(u, v) = \sigma(uv) \leq \mu(u) \wedge \mu(v)$. We call μ the fuzzy vertex set of G and σ the fuzzy edge set of G. Here after we denote the fuzzy graph $G(\mu, \sigma)$ simply by G and the underlying crisp graph of G by $G^*(V, E)$ with V as vertex set and $E = \{(u, v) \in V \times V : \sigma(u, v) > 0\}$ as the edge set or simply by G^*. If for $(u, v) \in E$ we say that u and v are adjacent in G^*. In that case we also say that u and v are adjacent in G. A fuzzy graph G is complete (see [4]) if $\sigma(uv) = \mu(u) \wedge \mu(v)$ for all $u, v \in V$. A fuzzy graph G is a strong fuzzy graph ([4]) if $\sigma(uv) = \mu(u) \wedge \mu(v)$, $\forall u, v \in E$.

A strong fuzzy complete bipartite graph is a strong fuzzy graph with its underlying crisp graph is a complete bipartite graph, [5]. A path P of length $n-1$ in a fuzzy graph G ([4]) is a sequence of distinct vertices $v_1, v_2, v_3, \ldots, v_n$, such that $\sigma(v_i, v_{i+1}) > 0$, $i = 1, 2, 3, \ldots, n-1$. We call P a fuzzy cycle if $v_1 = v_n$ and $n \geq 3$. The strength of a path is defined as the weight of the weakest edge of the path, [4]. A path P is said to connect the vertices u and v of G strongly if its strength is maximum among all paths between u and v. Such paths are called strong paths, [9]. Any strong path between two distinct vertices u and v in G with minimum length is called an extra strong path between them, [7]. There may exists more than one extra strong paths between two vertices in a fuzzy graph G. But, by the definition of an extra strong path each such path between two vertices has the same length. The maximum length of extra strong paths between every pair of distinct vertices in G is called the strength of the graph G, [7].

Theorem 1. [7] *For a fuzzy graph G, if G^* is the path $P = v_1 v_2 \ldots v_n$ on n vertices then the strength of the graph G is its length $(n-1)$.*

Theorem 2. [7] *The strength of a strong fuzzy complete graph is one.*

Hereafter, for a fuzzy graph G we use $\mathscr{S}(G)$ to denote its strength. The following theorems determine the strength of a fuzzy cycle.

Theorem 3. [8] *In a fuzzy cycle G of length n, suppose there are l weakest* *edges where $l \leq [\frac{n+1}{2}]$. If these weakest edges altogether form a subpath then $\mathscr{S}(G)$ is $n-l$.*

Theorem 4. [8] *Let G be a fuzzy cycle with crisp graph G^* a cycle of length n, having l weakest edges which altogether form a subpath. If $l > [\frac{n+1}{2}]$, then $\mathscr{S}(G)$ is $[\frac{n}{2}]$.*

Theorem 5. [8] *Let G be a fuzzy cycle with crisp graph G^* a cycle of length n, having l weakest edges which do not altogether form a subpath. If $l > [\frac{n}{2}] - 1$ then the strength of the graph is $[\frac{n}{2}]$ and if $l = [\frac{n}{2}] - 1$ then $\mathscr{S}(G)$ is $[\frac{n+1}{2}]$.*

Theorem 6. [8] *In a fuzzy cycle of length n suppose there are $l < [\frac{n}{2}] - 1$ weakest edges which do not altogether form a subpath. Let s denote the maximum length of a subpath which does not contain any weakest edge. If $s \leq [\frac{n}{2}]$ then the strength of the graph is $[\frac{n}{2}]$ and if $s > [\frac{n}{2}]$ then the strength of the graph is s.*

3. Main Results

Definition 7. Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two fuzzy graphs with underlying crisp graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ respectively. Then the tensor product G, denoted by $G_1 \otimes G_2$, of G_1 and G_2 is $G(V, \mu_1 \otimes \mu_2, \sigma_1 \otimes \sigma_2)$ with the underlying crisp graph $G(V, E_1 \otimes E_2)$ is the tensor product of $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ where $V = V_1 \times V_2$ and $E_1 \otimes E_2 = \{(u_1, u_2)(v_1, v_2) : u_1 v_1 \in E_1, u_2 v_2 \in E_2\}$, $(\mu_1 \otimes \mu_2)(u_1, u_2) = \mu_1(u_1) \wedge \mu_2(u_2), (u_1, u_2) \in V$, and
$(\sigma_1 \otimes \sigma_2)((u_1, u_2)(v_1, v_2)) = \sigma_1(u_1, v_1) \wedge \sigma_2(u_2, v_2), (u_1, u_2) \in E_1$,
$(v_1, v_2) \in E_2$.

Theorem 8. *Let G_1 and G_2 be two fuzzy graphs with underlying crisp graphs P_2 and P_n respectively. Then the strength $\mathscr{S}(G_1 \otimes G_2)$ of the tensor product of G_1 and G_2 is $n-1$.*

Proof. If $n = 1$ then $G_1 \otimes G_2$ is a null fuzzy graph. Therefore $\mathscr{S}(G_1 \otimes G_2) = 0 = n-1$. If $n > 1$, then it is the disjoint union of two fuzzy paths on n vertices (see Figure 1). So by Theorem 1 $\mathscr{S}(G) = n-1$. □

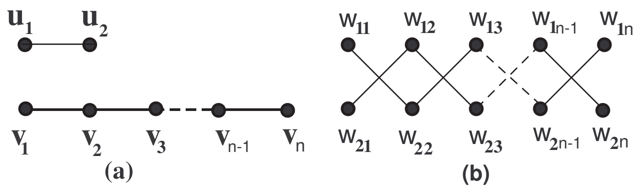

Figure 1: Tensor product of two fuzzy paths

If we replace the fuzzy graph G_2 of Theorem 8 by an another fuzzy graph, having star graph as the underlying crisp graph on n vertices and keeping G_1 as it is then their tensor product G is a null fuzzy graph, if $n = 1$. It is a disjoint union of two fuzzy paths if $n = 2$ and if $n > 2$ it is a disjoint union of two fuzzy star graphs on n vertices. Therefore in the first case, that is if $n = 1$ then $\mathscr{S}(G) = 0$ and in the second case that is if $n = 2$, $\mathscr{S}(G) = 1$ and when $n = 3$, $\mathscr{S}(G) = 2$ by Theorem 8. We can summarise these results as follows.

Theorem 9. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two fuzzy graphs with underlying crisp graph P_2 and the star graph S_n, respectively. Then the strength of the tensor product G is*

$$\mathscr{S}(G) = \begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 2 & \text{if } n \geq 3 \end{cases}.$$

Theorem 10. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with the underlying crisp graphs the path P_2 with vertex set $V_1 = \{u_1, u_2\}$ and the cycle C_n with vertex set $V_2 = \{v_1, v_2, \ldots, v_n\}$. Let $\mu_\circ = \mu_1(u_1) \wedge \mu_1(u_2) \wedge \mu_2(v_1) \wedge \mu_2(v_2) \ldots \wedge \mu_2(v_n)$. Then the strength of the tensor product of $G_1 \otimes G_2(V, \mu, \sigma)$ with vertex set $V = \{w_{ij} : i = 1, 2; j = 1, 2, \ldots, n\}$ is*

$$\mathscr{S}(G) = \begin{cases} [\frac{n}{2}] & \text{if } |V(G_2)| \text{ is even and} \\ & \text{there exist } \ w \in V(G_1), \text{such that } \mu_1(w) = \mu_\circ. \\ \mathscr{S}(G_2) & \text{if } |V(G_2)| \text{ is even and} \\ & \text{there exist no } w \in V(G_1), \text{such that } \mu_2(w) = \mu_\circ \\ n & \text{if } |V(G_2)| \text{ is odd }. \end{cases}$$

Proof.

Case 1. $|V(G_2)|$ is even.

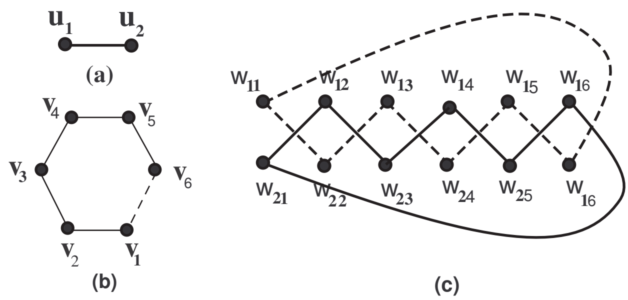

Figure 2

Then $G = G_1 \otimes G_2$ is a disjoint union of two fuzzy cycles H_1 with vertex set

$$\{w_{11}, w_{22}, w_{13}, w_{24}, \ldots, w_{1n-1}, w_{2n}\},$$

and H_2 with vertex set

$$\{w_{12}, w_{23}, w_{14}, w_{25}, \ldots, w_{2n-1}, w_{1n}, w_{21}\}$$

(see Figure 2).

Subcase 1. There exist $w \in V_1$ such that $\mu_1(w) = \mu_\circ$.

In this case all the edges of G have the same weight. So, the strength of G = strength of H_1 = strength of H_2 = $[\frac{n}{2}]$.

Subcase 2. There exist no $w \in V_1$ such that $\mu_1(w) = \mu_\circ$.

In this there exists a $w \in V_2$ such that $\mu_2(w) = \mu_\circ$. Without loss of generality assume that $w = v_1$. Then w_{11} and w_{21} are two weakest vertices of G. In fact each weakest vertex of G_2 determines exactly one weakest vertex in H_1 as well as in H_2. So the number of weakest vertices of H_1 and that of H_2 are equal and equal to that of G_2. Note only that if G_2 has m consecutive weakest vertices then both H_1 and H_2 have the same number of consecutive weakest vertices. From this we can conclude that the strength of G is equal to that of G_2.

Case 2. $V(G_2)$ is odd.

In this case $G = G_1 \otimes G_2$ is a strong fuzzy cycle with vertex set $\{w_{11}, w_{22}, w_{13}, w_{24}, \ldots, w_{2n-1}, w_{1n}, w_{21}, w_{12}, w_{23}, \ldots, w_{1n-1}, w_{2n}\}$ (see Figure 3).

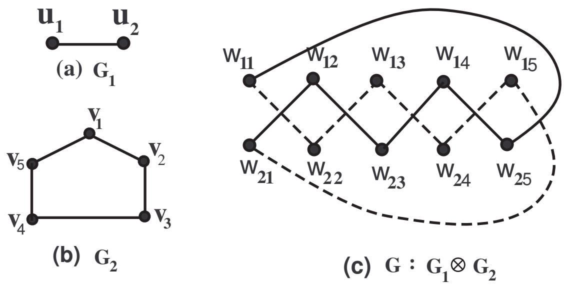

Figure 3

Subcase 1. Suppose there exists $w \in V_1$ such that $\mu_1(w) = \mu_\circ$.

Then all the edges of G have the same weight. Therefore $\mathscr{S}(G) = [\frac{2n}{2}] = n$.

Subcase 2. There exist no $w \in V_1$ such that $\mu_1(w) = \mu_\circ$.

By our assumption there exists a $w \in V_2$ such that $\mu_2(w) = \mu_\circ$. Assume that $w = v_1$. Then w_{11} and w_{21} are weakest vertices of the partial fuzzy subgraph

$$P =< \{w_{11}, w_{22}, w_{13}, w_{24}, \ldots, w_{2n-1}, w_{1n} >\}$$

and

$$Q =< \{w_{21} w_{12} w_{23} \ldots w_{1n-1} w_{2n} >\}$$

of G. Also corresponding to each weakest path of length m in G_2 there exist weakest paths of the same length in P and in Q. Let u and v be any two vertices of G. Then the path joining u and v having length $\geq n$ passes through at least one weakest edge of G. So the length of the extra strong $u - v$ path in G is $\leq n$. If $u = w_{11}$ and $v = w_{21}$ then the length of the extra strong $u - v$ path is exactly n. Hence follows the proof. □

Theorem 11. *Let $G_1(V_1, \mu_1, \sigma_1)$ and $G_2(V_2, \mu_2, \sigma_2)$ be two strong fuzzy graphs with underlying crisp graphs K_m and K_n, respectively. Let $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ be the set of all vertices of K_m and K_n respectively. Then the strength of the tensor product $G_1 \otimes G_2(V, \mu, \sigma)$ of G_1 and G_2 is*

$$\mathscr{S}(G_1 \otimes G_2) = \begin{cases} 0, & \text{for} \quad n = 1, m \geq 1 \quad \text{or } n \geq 1, m = 1 \\ 1, & \text{for} \quad n = m = 2, \\ 2, & \text{for} \quad n > 2 \quad \text{and} \quad m > 2, \\ 3, & n = 2, m > 2 \text{ or } n > 2, m = 2. \end{cases}$$

Proof. Let u and v be two non-adjacent vertices of $G = G_1 \otimes G_2$, say $u = w_{ij}$ and $v = w_{kl}$. Then u_i is not adjacent to u_k in G_1 or v_j is not adjacent to v_l in G_2.

Case 1. $n = 1, m \geq 1$ or $m = 1, n \geq 1$.

In this case $G = G_1 \otimes G_2$ is a null fuzzy graph on m (or n) vertices. Therefore $\mathscr{S}(G)$ is 0.

Case 2. $n = m = 2$.

In this case the tensor product is the disjoint union of two fuzzy paths with P_2 as the underlying crisp graphs. So strength of G is 1 by Theorem 1.

Case 3. $n > 2$ and $m > 2$.

Since G_1 and G_2 are complete fuzzy graphs, there exist atleast one vertex in $G_1 \otimes G_2$ which is adjacent to both u and v in $G_1 \otimes G_2$.

Whether $i = k$ or not, since n and $m > 2$, we can find a $u_r \in V(G_1)$ different from u_i and u_k such that $\mu_1(u_r) = \vee\{\mu_1(u_p) : 1 \leq p \neq i, k \leq n\}$ and a $v_s \in V(G_2)$ such that $\mu_2(v_s) = \vee\{\mu_2(v_q) : 1 \leq q \neq l, j \leq m\}$, so that w_{rs} is adjacent to both u and v in G. By the choice of w_{rs} the $u - v$ path $uw_{rs}v$ is an extra strong path joining u and v in G of length 2.

Case 4. $n > 2$ and $m = 2$ (or $m > 2$ and $n = 2$).

First of all suppose that $n = 2$ and $m > 2$. The case $m > 2$ and $n = 2$ can be dealt as in the same way. We have the following cases:

i $u = w_{1j}, v = w_{1l}, 1 \leq j \neq l \leq m$,

ii $u = w_{2j}, v = w_{2l}, 1 \leq j \neq l \leq m$,

iii $u = w_{1j}$ and $v = w_{2j}$ for some j.

In the first two cases we can proceed as in the proof of Case 3 and prove that the length of the extra strong path joining u and v is 2.

When $u = w_{1j}$ and $v = w_{2j}$, there is no vertex in G which is adjacent to both u and v. Since w_{1j} is adjacent to w_{2k}, for $k \neq j$ and w_{2j} is adjacent to w_{1l}, for $l \neq j$, the extra strong path joining u and v is $uw_{2r}w_{1s}u$ where $(v_r), r \neq j$ is chosen so that $\mu_2(v_r) \geq \vee\{\mu_2(v_p); r \neq j\}$ and $v_s, s \neq j, r$, is chosen such that $\mu_2(v_s) \geq \vee\{\mu_2(v_q);\ q \neq j, r\}$. Hence the length of the extra strong path joining u and v is 3. □

4. Acknowledgments

The first author is grateful to University of Calicut for the fellowship under which this research was conducted.

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