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PRICING OPTION UNDER STOCHASTIC VOLATILITY DOUBLE JUMP MODEL (SVJJ)

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Abstract: Through this paper, we introduce Fourier transform as an alternative approach to pricing option when the underlying asset follows Stochastic Volatility double Jump model (SVJJ). In fact the weakness of the traditional approaches does not depend on closed formula of probability density function which is explicitly unknown under this model. The advantage of Fourier transform technique is that for a wide class of stock price the only thing necessary to evaluate European call is a so called characteristic function since there is one-to-one relation-ship between a p.d.f & ch.f and both of which uniquely determine a probability distribution. For accuracy and validation we implement pricing formulas FFT, Monte Carlo simulation and we compare both of them to the benchmark model BS.

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1. Introduction

Although the fame of Black & Scholes formula [4], [12], this log-normal pure

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diffusion model miscalculate option. First of all, model leaves out the effect of jump size uncertainty of return volatility and the arrival of abnormal information [13]. Moreover, normality distribution contradict empirical evidence. In fact, market exhibit a non-zero skewness and higher kurtosis.

Luckily, the stochastic volatility double jump model corrects this flaw [11]. The literature point out that this model [7], [15] and [1] is the most accurate one to describe precisely the logarithm underlying asset price distribution with stochastic volatility under jumps [14]. However, the problem arises when the probability density function is not explicitly known in term of closed formula.

In this paper, we will discuss the problem of valuing a European call option though Fourier transform (see [5], [15]). This approach is able to give in wide class of underlying asset a distribution of the model by substitutes as so called a characteristic function in some predefined integral. We present the outline of this paper in Section 2 where we briefly describe SVJJ model. Then, in Section 3 and Section 4 we present the characteristic function of the model and show the accurate closed form expression of Fourier inversion transform option pricing. Finally, we conclude with numerical results.

2. Stochastic Volatility Double Jump

2.1. Model Descriptions

Throughout this section, we consider a complete probability space $(\Omega, \mathcal{F}, P)$ with an information filtration $\{\mathcal{F}_t; 0 \leq t \leq T\}$, satisfying the usual conditions. We denote by $\{e^{-rt}S_t; 0 \leq t \leq T\}$ a discounted asset price process where T is a finite time horizon. In an arbitrage-free market, prices of any asset can be calculated as an expected terminal payoffs under discounted by a risk-free interest rate r such that $E^Q[e^{-rT}S_T|\mathcal{F}_t] = e^{-rt}S_t$, which is a martingale condition. We define the stochastic volatility double Jump model (SVJJ) as

$$dS_t = (r - \lambda^s\kappa)S_t dt + \sqrt{V_t}S_t dB_t^s + J^s(Y)S_{t-}dN_t^s, \tag{1}$$

$$dV_t = \epsilon(\bar{v} - V_t)dt + \sigma\sqrt{V_t}dB_t^v + J^v(Z)dN_t^v. \tag{2}$$

Here r is the riskless interest rate which is assumed to be constant, $\epsilon \geq 0$, $\bar{v} \geq 0$ and $\sigma > 0$ are called the speed of mean reversion, the mean level of variance and the volatility of the volatility, respectively. Furthermore, the Brownian motions B^s and B^v are assumed to be correlated with correlation coefficient ρ. N_t^s and N_t^v are independent Poisson processes with constant

intensities λ^s and λ^v respectively. Moreover, we assume that the jump processes N_t^s and N_t^v are independent with standard Brownian motion B^s and B^v.

In equation (1), $J^s(Y)$ (rep. $J^v(Z)$) is the Poisson jump-amplitude. Y (resp. Z) is an underlying Poisson amplitude mark process selected defined as $Y = \ln(1 + J(Y))$.

For convenience, N_t^s (resp. N_t^v) is the standard Poisson jump counting process with jump intensity λ^s (resp. λ^v) and expected value $\lambda^s dt$ (resp. $\lambda^v dt$). Also, the symbolic jump term for the asset price and volatility respectively are

$$J^s(Y)dN_t^s = \sum_{i=1}^{dN_t^s} J^s(Y_i),$$

$$J^v(Z)dN_t^s = \sum_{j=1}^{dN_t^v} J^v(Z_j).$$

Here Y_i (resp. Z_j) is the ith (resp. jth) jump-amplitude random variable taken from a set of independent, identically distributed random variables. We assume that the density of both jump-amplitude Y and Z are receptively log-normal and exponential distributed (see [15])

$$\phi_Y(y) \sim \mathcal{N}(\ln(1 + \mu) - \frac{1}{2}\delta^2, \delta^2), \tag{3}$$

$$\phi_Z(z) \sim \exp(\frac{1}{\zeta}). \tag{4}$$

3. European Call Option Price

In this section, we want to price a European call option by using the PDE approach which is by now quite standard in literature (see e.g. Heston [10], Bates [3], Bakshi [1]). We let C denote the price at time t of a European style call option on S_t with strike price K and expiration time $T = t + \tau$. Using the fact that a terminal payoffs of an European call option on the underlying asset S_t with strike price K is $\max(S_t - K, 0)$ and assuming that the short-term interest rate r is constant over the lifetime of the option. European call price at time t is computed as discounted risk-neutral conditional expectations of the terminal payoffs

$$C(t, S_t, V_t) = e^{-r(T-t)} E^Q[\max(S_t - K, 0)|\mathcal{F}_t], \tag{5}$$

where E^Q (resp. $Q(.)$) is the conditional expectation with respect to the risk-neutral probability measure Q (resp. $Q(.)$ is the conditional probability density function of terminal asset price S_T).

Through the definition of contingent claim value, we cannot express the call value in term of closed formula since the probability density function of the underlying asset under SVJJ model is not available. Luckily, there is a very useful theorem 2-dim Dynkin's formula (see Hanson [9]) known as partial integro-differential equations which transforms a stochastic differential system to a partial differential system and solve with respect to Riemann integral.

3.1. Partial Integro-Differential Equations

The same treatment framework was done by Bates [3]and Heston [10], here we will elaborate 2-dim Dynkin's theorem to derive the partial integro-differential equation (PIDE) satisfied by the value of an option. Let us define X_t^1 and X_t^2 as

$$\begin{aligned} dX_t^1 &= f_1(t, X_t^1, X_t^2)dt + g_1(t, X_t^1, X_t^2)dB_t^1 + h_1(t, X_t^1)dP^1(t; X_t^1, t), \\ dX_t^2 &= f_2(t, X_t^1, X_t^2)dt + g_2(t, X_t^1, X_t^2)dB_t^1 + h_2(t)dP^2(t). \end{aligned}$$

The Dynkin theorem states that the conditional expectation, where T is the terminal time,

$$u(t, x_1, x_2) = E^Q[U(X_T^1, X_T^2)|X_t^1 = x_1, X_t^2 = x_2],$$

is the solution of the PIDE

$$\begin{aligned} 0 = \frac{\partial u}{\partial t} + A[u] + \lambda^s \int_Q (u(t, x_1 + y, x_2) - u(t, x_1, x_2))\phi_Y(y)dy \\ + \lambda^v \int_Q (u(t, x_1, x_2 + z) - u(t, x_1, x_2))\phi_Z(z)dz, \end{aligned} \tag{6}$$

with

$$\begin{aligned} A[u] = f_1(t, x_1, x_2)\frac{\partial u(t, x_1, x_2)}{\partial x_1} + f_2(t, x_1, x_2)\frac{\partial u(t, x_1, x_2)}{\partial x_2} \\ + \frac{1}{2}g_1^2(t, x_1, x_2)\frac{\partial^2 u(t, x_1, x_2)}{\partial x_1^2} + \frac{1}{2}g_2^2(t, x_1, x_2)\frac{\partial^2 u(t, x_1, x_2)}{\partial x_2^2} \\ + \rho g_1(t, x_1, x_2)g_2(t, x_1, x_2)\frac{\partial^2 u(t, x_1, x_2)}{\partial x_1 x_2}. \end{aligned}$$

3.2. Characteristic Function Formulation for Solution

In 1993, Heston [10] derive the characteristic function from Heston model, this was done in a Gil-Palaez inversion framework [8]. Here we present the derivation in a Carr & Madan setting. To start with, before introduce the lemma which present partial Integro-differential equations of contingent claim value, we simplify the pricing PIDE in equation (6) by defining respectively, forward option price $\tilde{C}(t, X_t, V_t)$ and change of variable X_t as

$$\tilde{C}(t, X_t, V_t) \equiv e^{r(T-t)} C(t, S_t, V_t), \tag{7}$$

$$X_t = \ln\left(\frac{e^{r(T-t)} S_t}{K}\right). \tag{8}$$

Corollary 1. *The forward option price in equation (7) satisfies the partial integro-differential equation (PIDEs)*

$$\begin{aligned} 0 = \frac{\partial \tilde{C}}{\partial t} + A[\tilde{C}] + \lambda^s \int_Q (\tilde{C}(t, x_1 + y, x_2) - \tilde{C}(t, x_1, x_2))\phi_Y(y)dy \\ + \lambda^v \int_Q (\tilde{C}(t, x_1, x_2 + z) - \tilde{C}(t, x_1, x_2))\phi_Z(z)dz, \end{aligned} \tag{9}$$

with

$$\begin{aligned} A[\tilde{C}] = (r - \lambda^s\kappa - \frac{1}{2}v)\frac{\partial \tilde{C}(t,x,v)}{\partial x} + \epsilon(\bar{v} - v)\frac{\partial \tilde{C}(t,x,v)}{\partial v} \\ + \frac{1}{2}v\frac{\partial^2 \tilde{C}(t,x,v)}{\partial x^2} \\ + \frac{1}{2}\sigma^2 v\frac{\partial^2 \tilde{C}(t,x,v)}{\partial v^2} + \rho\sigma v\frac{\partial^2 \tilde{C}(t,x,v)}{\partial x \partial v} - r\tilde{C}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} V_t &= v, \\ X_t &= x. \end{aligned}$$

Proof. Through Itô's chain rule and under a risk-neutral probability measure Q, the log-return process $(\ln S_t)_{t\in[0,T]}$ satisfies the SDE

$$d\ln S_t = (r - \lambda^s\kappa - \frac{1}{2}v)dt + \sqrt{v}dB_t^s + \ln(y_t + 1)dN_t^s,$$

changing variable

$$X(t) = \ln\left(\frac{e^{r(T-t)}S_t}{K}\right),$$

and applying 2-dim Dynkin's formula to the stochastic differential system

$$\begin{aligned} dX_t &= (r - \lambda^s\kappa - \frac{1}{2}V_t)dt + \sqrt{V_t}dB_t^s + J^s(Y)S_{t-}dN_t^s - r, \\ dV_t &= \epsilon(\bar{v} - V_t)dt + \sigma\sqrt{V_t}dB_t^v + J^v(Z)dN_t^v, \end{aligned} \tag{11}$$

we tend to the desired result. □

Next, we present three lemmas that culminate in the characteristic function of SVJJ model. Before presenting the lemmas, we need to refer to advanced results from Duffie, Pan & Singleton [6]. They indicated that for diffusion processes the characteristic function $\exp[e^{i\omega x_\tau}]$ of x_τ is of the form

$$f(\tau, x, v, \omega) = \exp\{A(\omega, \tau) + vB(\omega, \tau) + C(\omega, \tau)x\}. \tag{12}$$

The characteristic function must satisfy the initial condition $f(0, x, v, \omega) = \exp(i\omega x_T)$, which in turn implies that $A(\omega, 0) = B(\omega, 0) = 0$ and $C(\omega, 0) = i\omega$ for all ω. Since $f(\tau, x, v, \omega) \equiv E^Q[e^{i\omega x_\tau}]$, where Q is as defined above, substituting (12) into (10) and using the initial condition for $C(\omega, \tau)$ one can easily show that

$$f(\tau, x, v, \omega) = \exp\{A(\omega, \tau) + vB(\omega, \tau) + i\omega x\}. \tag{13}$$

The solutions to $A(\omega, \tau)$ and $B(\omega, \tau)$ are provided by the next two lemmas.

Lemma 2. *The functions $A(\omega, \tau)$ and $B(\omega, \tau)$ in (13) with the initial conditions $A(\omega, 0) = B(\omega, 0) = 0$, satisfy the following system of ordinary differential equations(ODEs) for all $\omega \in R$*

$$\frac{dA}{d\tau} = \alpha B + \beta, \tag{14}$$

$$\frac{dB}{d\tau} = aB^2 - bB + c. \tag{15}$$

Proof. As indicated above, the ch.f $f(\tau, x, v, \omega)$ satisfies PIDE. Substituting (13) into (10) yields

$$\begin{aligned} 0 = \frac{\partial f}{\partial t} + A[f] + \lambda^s \int_R (f(t, x+y, v) - f(t, x, v))\phi_Y(y)dy \\ + \lambda^v \int_R (f(t, x, v+z) - f(t, x, v))\phi_Z(z)dz. \end{aligned} \tag{16}$$

First we compute

$$\begin{aligned}
\frac{\partial f}{\partial t} &= (\frac{\partial A}{\partial \tau} + v\frac{\partial B}{\partial \tau})f, \\
\frac{\partial f}{\partial x} &= i\omega f, \\
\frac{\partial^2 f}{\partial x^2} &= \omega^2 f \\
\frac{\partial f}{\partial v} &= Bf, \\
\frac{\partial^2 f}{\partial v^2} &= B^2 f, \\
\frac{\partial f^2}{\partial x \partial v} &= i\omega B f,
\end{aligned}$$

and

$$\begin{aligned}
f(x+y,v,\tau) - f(x,v,\tau) &= \exp\{A(\omega,\tau) + vB(\omega,\tau) + i\omega(x+y)\} \\
&\quad - \exp\{A(\omega,\tau) + vB(\omega,\tau) + i\omega x\}, \\
&= [\exp i\omega y - 1]f(x,v,\tau),
\end{aligned}$$

$$\begin{aligned}
f(x,v+z,\tau) - f(x,v,\tau) &= \exp\{A(\omega,\tau) + (v+z)B(\omega,\tau) + i\omega x\} \\
&\quad - \exp\{A(\omega,\tau) + vB(\omega,\tau) + i\omega x\}, \\
&= [\exp zB - 1]f(x,v,\tau).
\end{aligned}$$

We substitute all terms above into equation (16) and get

$$\begin{aligned}
0 = &-\frac{\partial A}{\partial \tau} + \epsilon\bar{v}B + (r - \lambda^s\kappa)i\omega + \lambda^s \int_R (e^{i\omega y} - 1)\phi_Y(y)dy \\
&+ \lambda^v \int_R (e^{zB} - 1)\phi_Z(z)dz + v[-\frac{\partial B}{\partial \tau} + \frac{1}{2}(-i\omega + \omega^2) \\
&+ (-\epsilon + \rho\sigma i\omega)B + \frac{1}{2}\sigma^2 B^2],
\end{aligned} \tag{17}$$

which is simplified to

$$0 = -\frac{\partial A}{\partial \tau} + \alpha B + \beta + v[-\frac{\partial B}{\partial \tau} + c - bB + aB^2]. \tag{18}$$

Separating the order v in terms to reduce the equation (17) to two ordinary differential equations (ODEs),

$$\begin{aligned}
\frac{\partial A}{\partial \tau} &= \alpha B + \beta, \\
\frac{\partial B}{\partial \tau} &= aB^2 - bB + c,
\end{aligned} \tag{19}$$

with

$$\begin{aligned}
\alpha &= \epsilon\bar{v},\\
\beta &= (r-\lambda^s\kappa)i\omega + \lambda^s\int_R (e^{i\omega y}-1)\phi_Y(y)dy + \lambda^v\int_R (e^{zB}-1)\phi_Z(z)dz,\\
a &= \frac{1}{2}\sigma^2,\\
b &= \epsilon - \rho\sigma i\omega,\\
c &= \frac{1}{2}(i\omega+\omega^2).
\end{aligned} \tag{20}$$

□

Lemma 3. *The solution to the system of ODEs as specified in first lemma is given by*

$$\begin{aligned}
A(\omega,\tau) &= \frac{\epsilon\bar{v}}{\sigma^2}\left((\epsilon-\rho\sigma i\omega-\Delta)\tau - 2\ln\left(\frac{1-B_0e^{-\Delta\tau}}{1-B_0}\right)\right) + i\omega(r-\lambda^s\kappa)\tau\\
&\quad + \lambda^s\tau\int_R (e^{i\omega y}-1)\phi_Y(y)dy + \lambda^v\tau\int_R (e^{zB}-1)\phi_Z(z)dz,\\
B(\omega,\tau) &= \frac{1}{\sigma^2}\left(\frac{1-e^{-\Delta\tau}}{1-B_0e^{-\Delta\tau}}\right)(\epsilon-\rho\sigma i\omega-\Delta),\\
C(\omega,\tau) &= i\omega,
\end{aligned} \tag{21}$$

where

$$\Delta = \sqrt{(\epsilon-\rho\sigma i\omega)^2 - (i\omega+\omega^2)\sigma^2},$$
$$B_0 = \frac{\epsilon-\rho\sigma i\omega-\Delta}{\epsilon-\rho\sigma i\omega+\Delta}.$$

Proof. First we solve for $B(\omega,\tau)$

$$\frac{dB}{d\tau} = aB^2 - bB + c = a(B-B_1) + (B-B_2),$$

with

$$B_j = \frac{b\pm\Delta}{2a}, \qquad j=1,2,$$
$$\Delta = \sqrt{b^2-4ac}.$$

Separating variables gives

$$\frac{1}{a(B-B_1)(B-B_2)}dB = d\tau,$$

which is equivalent to

$$\frac{\frac{1}{(B_1-B_2)}}{a(B-B_1)} - \frac{\frac{1}{(B_1-B_2)}}{a(B-B_2)}dB = d\tau,$$

integrating on both sides gives

$$\ln(\frac{B-B_1}{B-B_2}) = \Delta\tau + \Delta c_B,$$

using initial condition $B(\omega, 0) = 0$, we have

$$c_B = \frac{1}{\Delta}\left(\ln\frac{B_1}{B_2}\right).$$

Solving for B in equation (15) yields

$$B(\omega,\tau) = \frac{1}{\sigma^2}\left(\frac{1-e^{-\Delta\tau}}{1-B_0e^{-\Delta\tau}}\right)(\epsilon - \rho\sigma i\omega - \Delta).$$

Now we are able to solve for $A(\omega, \tau)$

$$\begin{aligned} A(\omega,\tau) &= \int (\alpha B + \beta)\, d\tau, \\ &= \alpha \int B d\tau + \beta\tau + c_A, \\ &= \alpha[B_2\tau - \frac{1}{a}\ln\left(1 - B_0e^{-\Delta\tau}\right)] + \beta\tau + c_A, \end{aligned}$$

from the initial condition $A(\omega, \tau) = 0$, the constant of integration c_A is as follows

$$c_A = \frac{\alpha}{a}\ln(1 - B_0),$$

with

$$B_2 = \frac{\epsilon - \rho\sigma i\omega - \Delta}{\sigma^2},$$
$$\alpha = \epsilon\bar{v}, a = \frac{\sigma^2}{2},$$

$$
\begin{aligned}
A(\omega,\tau) &= \alpha\left[B_2\tau - \frac{1}{a}\ln\left(\frac{1-B_0e^{-\Delta\tau}}{1-B_0}\right)\right] + \beta\tau,\\
&= \epsilon\bar{v}\left[\frac{\epsilon-\rho\sigma i\omega-\Delta}{\sigma^2}\tau - \frac{2}{\sigma^2}\ln\left(\frac{1-B_0e^{-\Delta\tau}}{1-B_0}\right)\right] + \beta\tau,\\
&= \frac{\epsilon\bar{v}}{\sigma^2}\left[(\epsilon-\rho\sigma i\omega-\Delta)\tau - 2\ln\left(\frac{1-B_0e^{-\Delta\tau}}{1-B_0}\right)\right] + \beta\tau,\\
&= \frac{\epsilon\bar{v}}{\sigma^2}\left[(\epsilon-\rho\sigma i\omega-\Delta)\tau - 2\ln\left(\frac{1-B_0e^{-\Delta\tau}}{1-B_0}\right)\right] + i\omega(r-\lambda^s\kappa)\tau\\
&+ \lambda^s\tau\int_R (e^{i\omega y}-1)\phi_Y(y)dy + \lambda^v\tau\int_R (e^{zB}-1)\phi_Z(z)dz.
\end{aligned}
$$

□

Consequently, we arrived to the desired result by replacing β by its value.

Lemma 4. *In Stochastic Volatility double Jump model (SVJJ), the characteristic function $\phi_T(\omega)$ of log-terminal asset price $\ln S_T$ is given by*

$$
\begin{aligned}
\phi_T(\omega) = \exp&\left(\frac{\epsilon\bar{v}}{\sigma^2}\left[(\epsilon-\rho\sigma i\omega-\Delta)T - 2\ln\left(\frac{1-B_0e^{-\Delta T}}{1-B_0}\right)\right]\right)\\
&+ i\omega(r-\lambda^s\kappa)T + \lambda^s T\left((1+\mu)^{i\omega}e^{-\frac{1}{2}\delta^2 i\omega(i\omega-1)}-1\right)\\
&+ \lambda^v T\left(\frac{1}{1-\zeta B}-1\right)\\
&\times \exp\left(\frac{v_0}{\sigma^2}\left(\frac{1-e^{-\Delta T}}{1-B_0e^{-\Delta T}}\right)(\epsilon-\rho\sigma i\omega-\Delta)\right)\\
&\times \exp\left(iw[\ln S_0 + rT]\right),
\end{aligned}
\tag{22}
$$

where Δ and B_0 are as defined in the previous lemma.

Proof. On the one hand, at maturity the characteristic function $\exp[e^{i\omega X_t}]$ of X_t is as follows

$$
\begin{aligned}
E^Q\left[e^{i\omega X_t}\right] &= E^Q\left[e^{i\omega\ln(\frac{e^{r(T-t)}S_t}{K})}\right]\\
&= E^Q[\exp^{i\omega\ln S_t}\exp^{i\omega r(T-t)}\exp^{-i\omega\ln K}].
\end{aligned}
$$

At the maturity, substituting $t = T$ in the characteristic function of X_t gives

$$\begin{aligned} E^Q\left[e^{i\omega X_T}\right] &= E^Q[\exp^{i\omega \ln S_T}\, e^{-i\omega \ln K}] \\ &= e^{-i\omega \ln K} E^Q[\exp^{i\omega \ln S_T}], \end{aligned}$$

hence

$$E^Q\left[e^{i\omega X_T}\right] = \exp[A(\omega, T) + vB(\omega, T) + i\omega X_T],$$

the characteristic function of log-terminal asset price S_T is as follows

$$\begin{aligned} E^Q\left[e^{i\omega \ln S_T}\right] &= e^{i\omega \ln K} E^Q[\exp(i\omega X_T)] \\ &= \exp(i\omega \ln K) \exp[A(\omega, T) + vB(\omega, T) + i\omega \ln \frac{S_T}{K}] \\ &= \exp\left(\frac{\epsilon\bar{v}}{\sigma^2}\left[(\epsilon - \rho\sigma i\omega - \Delta)T - 2\ln\left(\frac{1 - B_0 e^{-\Delta T}}{1 - B_0}\right)\right]\right) \\ &+ i\omega(r - \lambda^s\kappa)T + \lambda^s T \int_R (e^{i\omega y} - 1)\phi_Y(y)dy \\ &+ \lambda^v T \int_R (e^{zB} - 1)\phi_Z(z)dz \\ &\times \exp\left(\frac{v_0}{\sigma^2}\left(\frac{1 - e^{-\Delta T}}{1 - B_0 e^{-\Delta T}}\right)(\epsilon - \rho\sigma i\omega - \Delta)\right) \\ &\times \exp\left(iw[\ln S_0 + rT]\right). \end{aligned}$$

On the other hand, applying the log-normal distribution of jump-amplitude Y in our general formulas, leads to the following integral

$$\begin{aligned} \int_{-\infty}^{\infty} e^{i\omega y}\phi_Y(y)dy &= \int_{-\infty}^{\infty} e^{i\omega y}\frac{1}{\sqrt{2\pi}\delta}\exp[\frac{-(\ln(y+1) - (\ln(\mu+1) - \frac{\delta^2}{2}))^2}{\frac{\delta^2}{2}}]dy, \\ &= \int_{-\infty}^{\infty} \exp[\frac{-1}{2\delta^2}(-(\delta^2 i\omega)^2 - 2\delta^2 i\omega(\ln(1+\mu) - \frac{\delta^2}{2})] \\ &\frac{1}{\sqrt{2\pi}\delta}\exp[\frac{-[(\ln(y+1) - (\ln(\mu+1) - \frac{\delta^2}{2} - 2\delta^2 i\omega))^2]}{\frac{\delta^2}{2}}]dy, \\ &= (1+\mu)^{i\omega}\exp[\frac{1}{2}\delta^2 i\omega(i\omega - 1)], \end{aligned}$$

with $\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\delta}\exp[\frac{-(y-\mu)^2}{\frac{\delta^2}{2}}]dy = 1$.

Then applying exponential distribution to the jump-amplitude of stochastic volatility yields

$$\int_{-\infty}^{\infty} e^{zB}\phi_Z(z)dz = \int_0^{\infty} e^{zB}\frac{1}{\zeta}e^{-\frac{1}{\zeta}z}dz$$
$$= \frac{1}{\zeta}\int_0^{\infty} e^{-z(\frac{1}{\zeta}+B)}dz = \frac{1}{1-\zeta B}.$$

Summarizing the above, we get the desired results. □

Remark 5. *The integral* $\int_{-\infty}^{\infty} e^{i\omega y}\phi_Y(y)dy$ *(resp.* $\int_{-\infty}^{\infty} e^{zB}\phi_Z(z)dz$*) represents how small variation in log-terminal price of underlying asset (resp. volatility) impacts the value of option.*

4. Fourier Transform Inversion

4.1. Discretization

In 1999, Carr & Madan [5], developed a different method designed to use the fast Fourier transform pricing options. They introduced a new technique to calculate the Fourier transform of a modified call option price with respect to the logarithmic strike price so that the fast Fourier transform can be applied to calculate the integrals. Consider a European call with the maturity T and the strike price $\ln K \equiv k$, which is written on a stock whose price process is $\ln S_T \equiv s$, under a risk-neutral probability density function $Q(\ln S_T|\mathcal{F}_t)$. Consider the price of a call option in $t = 0$ without lost of generality,

$$\begin{aligned} C(T,k) &= e^{-rT}E^Q[(e^{s_T} - e^k)^+|\mathcal{F}_0] \\ &= e^{-rT}\int_k^{\infty}(e^{s_T} - e^k)Q(s_T|\mathcal{F}_0)ds_T. \end{aligned} \tag{23}$$

Note that $C_T(k) \to S_0$ as $k \to \infty$ and the function $C_T(k)$ is not square-integrable. Thus, we cannot express the Fourier transform in strike in terms of the characteristic function $\phi_T(\omega)$ of s_T and then find a range of strikes by Fourier inversion to obtain a square-integrable function we consider the modified call price

$$C_{mod}(T,k) = e^{\alpha k}C(T,k),$$

for some $\alpha > 0$, which is chosen to improve the integrability. Carr & Madan [5], showed that a sufficient condition for square-integrability of $C(T,k)$ is given by $\int |C_{mod}(T,k)|dk$. Now consider the Fourier transform of $C_{mod}(T,k)$

$$\begin{aligned}\psi_T(\omega) &= \mathcal{F}\left(C_{mod}(T,k)\right)(\omega),\\ &= \int_{-\infty}^{\infty} e^{i\omega k} C_{mod}(T,k)dk.\end{aligned} \tag{24}$$

A call price can be obtained by an inverse Fourier transform of $\psi_T(\omega)$ as

$$C(T,k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega k}\psi_T(\omega)d\omega. \tag{25}$$

Substitute equation (25) into equation (24) and interchanging integrals yields

$$\psi_T(\omega) = \frac{e^{-rT\phi_T(\omega-(\alpha+1)i)}}{\alpha^2+\alpha-\omega^2+i(2\alpha+1)\omega}. \tag{26}$$

Thus, a call pricing function is obtained by substituting equation (26) into equation (25)

$$C(T,k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega k}\frac{e^{-rT\phi_T(\omega-(\alpha+1)i)}}{\alpha^2+\alpha-\omega^2+i(2\alpha+1)\omega}d\omega. \tag{27}$$

Now this integral can be evaluated by the numerical approximation using the Simpson rule

$$\begin{aligned}C(T,k_p) \approx \frac{e^{-\alpha k_p}}{\pi}\frac{\Delta\omega_n}{3}&\\ Re\{\sum_{1}^{N} e^{\frac{-i2\pi pn}{N}} e^{i\omega_n b}\psi_T(\omega_n)(3+(-1)^n-\delta_{n-1})d\omega_n\}.&\end{aligned} \tag{28}$$

Here, k_p and ω_n are the step size of the summation grid. Denote the Kronecker delta function that equals one whenever $n = 1$. Where, $\frac{1}{3}(3+(-1)^n-\delta_{n-1})$ is the weight implementing a choice of Simpson's summation. The constant $b \in R$ can be tuned such that the grid is laid around at-the-money strikes, since we are mainly interested in option prices with these particular strikes. To apply the algorithm of FFT (see [7]), we define the grid points as follow

$$\omega_n = n\Delta\omega_n, \qquad n = 1,2,\cdots,N, \tag{29}$$

$$k_p = -b + p\Delta k_p, \qquad p = 1,2,\cdots,N, \tag{30}$$

step sizes $\Delta\omega_n$ and Δk_p, satisfy the Nyquist relation given by

$$\Delta\omega_n\Delta k_p = \frac{2\pi}{N}.$$

4.2. Numerical Results

In this section we present the numerical results for the SVJJ model, which includes jumps in both SDE, log terminal asset price and volatility. As our goal is to price European call options using the methods considered in the previous section we implement FFT method at different levels strike price near at-the-money (ATM). Then, we compare the result to traditional valuation methods Monte Carlo simulation.

For this model, all the graphics and results corresponding to an European call option with the following conditions: $S_0 = 20.0, V_0 = 0.1, \sigma = 0.1, r = 0.05, \epsilon = 3.0, v_0 = 0.25, \rho = 0.6, \lambda^s = 0.75, \lambda^v = 0.25, \mu = 0.5, \delta = 2.5, \zeta = 1.2, \alpha = 1.0$ While ω_n (resp. k_p), varies from $\omega_n = 1, .., N$ (resp. in the range $(-b, b)$) and which we assume to be equal in length. Considering the maturity $T = 1.0$ and the expected value of relative price jump size $\kappa = E[Y - 1] = \mu - 1$. Furthermore, the model has been implemented considering $I = 10000$ simulation with $M = 50$ time grid.

We implement SVJJ model in PYTHON. The illustration below shows dynamic of stock price and volatility in market free-arbitrage.

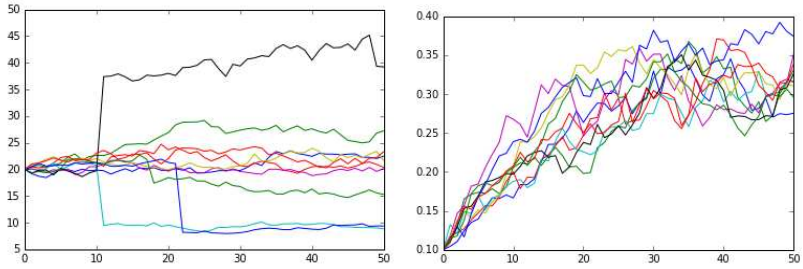

Figure 1: The graph on the left-hand side (resp. right hand side) shows how distribution of Asset price (resp. stochastic volatility) look like under SVJJ model.

As expected, the evolution of call price at different level strike looks more closely to the Black & Scholes than Monte Carlo simulation.

5. Conclusion

It is important to point out that the numerical results represent an empirical support to prove or disprove the effectiveness of model. Fortunately, FFT algorithm is more accurate and faster than the other models, it provide at the

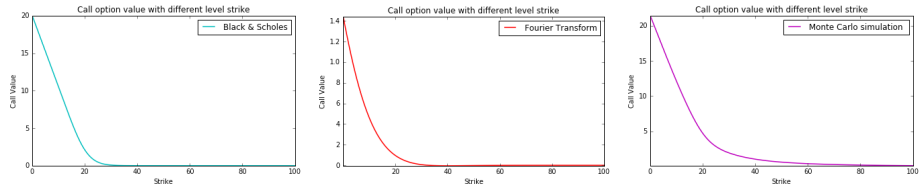

Figure 2: The graphs display the relation ship between call value and strike under different models B&S, SVJJ and M.C simulation.

same time prices for about 2^{10} strikes, while Monte Carlo simulations provide only single strike price. Therefore, all the analyzes performed show that Fourier transform pricing option under SVJJ model is versatile enough to describe precisely the value of call option since they are usually traded near-the-money. For this reason, we strongly recommend the solution techniques FFT as a flexible and promising framework to be applied at general derivative security pricing problem and it is not only limited to SVJJ case.

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