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ON SINGULAR PRODUCTS OF DISTRIBUTIONS IN COLOMBEAU ALGEBRA

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Abstract: Results on singular products of Schwartz distributions on the Euclidean space $\mathbb{R}^m$ are derived when the products are so 'balanced' that they exist in the distribution space. The results follow the idea of a known distributional product published by Jan Mikusiński and are obtained in Colombeau algebra of generalized functions. This algebra contains the distributions and the notion of 'association' permits obtaining results on the level of distributions.

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The Schwartz distributions are widely employed in natural sciences and many mathematical fields. Since products of distributions with coinciding singularities often appear in them, the problem of their multiplication has been objective of studies for a long time. On the other hand, the differential algebra $\mathcal{G}$ of generalized functions of J.F. Colombeau has become popular since it has almost optimal properties for tackling non-linear problems of distributions: they are linearly embedded in $\mathcal{G}$ and the multiplication is compatible with differentiation and products with smooth functions. The so-called 'association' in $\mathcal{G}$,

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being a faithful generalization of the equality of distributions, makes it possible to obtain results on distributional level. This is why the generalized functions of Colombeau have application to various mathematical fields. Here we follow this approach: a product of distributions with coinciding point singularities is evaluated, as they are embedded in $\mathcal{G}$ and their product admits an associated distribution.

Recall now the famous result of Jan Mikusiński published in [5]:

$$x^{-1} \,.\, x^{-1} \;-\; \pi^2\, \delta(x)\,.\,\delta(x) \;=\; x^{-2}, \quad x \in \mathbb{R}. \tag{1}$$

Although, neither of the products on the left-hand side here exists, their difference is so *'balanced'* that it has a correct meaning. Formulas of this kind can be found in the mathematical and physical literature. In [2] we derived generalizations of (1) in Colombeau algebra $\mathcal{G}(\mathbb{R})$, while in [3] and [4] further balanced singular products of distributions were obtained. In the present paper, new results on singular products of the distributions $x_\pm^a$ and $\delta^{(p)}(x)$ $(x \in \mathbb{R}^m)$ are derived in the algebra $\mathcal{G}(\mathbb{R}^m)$. We will remind first some basic notions of Colombeau theory.

Notation 1: (a). If $\mathbb{N}_0$ stands for the nonnegative integers and $p = (p_1, p_2, ..., p_m)$ is a multiindex in $\mathbb{N}_0^m$, we let $|p| = \sum_{i=1}^m p_i$ and $p! = p_1!...p_m!$. If $x = (x_1, ..., x_m) \in \mathbb{R}^m$, we denote

$$x^p = (x_1^{p_1}, x_2^{p_2}, ..., x_m^{p_m}) \text{ and } \partial^p = \partial^{|p|}/\partial x_1^{p_1} ... \partial x_m^{p_m}.$$

(b). If $q \in \mathbb{N}_0$, we put $A_q(\mathbb{R}) = \{\varphi(x) \in \mathcal{D}(\mathbb{R}) : \int_{\mathbb{R}} x^j\, \varphi(x)\, dx = \delta_{0j}$ for $0 \le j \le q$, where $\delta_{00} = 1, \delta_{0j} = 0$ for $j > 0\}$. This also extends to $\mathbb{R}^m$ as an m-fold product: $A_q(\mathbb{R}^m) = \{\varphi(x) \in \mathcal{D}(\mathbb{R}^m) : \varphi(x_1, ..., x_m) = \prod_{i=1}^m \chi(x_i)$ for some χ in $A_q(\mathbb{R})\}$. Finally, we denote $\varphi_\varepsilon = \varepsilon^{-m}\varphi(\varepsilon^{-1}x)$ for φ in $A_q(\mathbb{R}^m)$ and $\varepsilon > 0$.

(c). For $\varphi \in A_0(\mathbb{R})$ and $\varepsilon > 0$, we will use the following notation: $\varphi_\varepsilon = \varepsilon^{-1}\varphi(\varepsilon^{-1}x)$ and $s \equiv s(\varphi) := sup|x| : \varphi(x) \ne 0$. Since $s(\varphi_\varepsilon) = \varepsilon s(\varphi)$, denoting $\sigma \equiv \sigma(\varphi, \varepsilon) := s(\varphi_\varepsilon) > 0$, we have $\sigma := \varepsilon s = O(\varepsilon)$, as $\varepsilon \to 0$, for each $\varphi \in A_0(\mathbb{R})$.

Definition 1. Let $\mathcal{E}\,[\mathbb{R}^m]$ be the algebra of functions $f(\varphi, x) : A_q(\mathbb{R}^m) \times \mathbb{R}^m \to \mathbb{C}$ that are infinitely differentiable, by a fixed 'parameter' φ. The generalized functions of Colombeau are elements of the quotient algebra $\mathcal{G} \equiv \mathcal{G}(\mathbb{R}^m) = \mathcal{E}_{\rm M}[\mathbb{R}^m] \,/\, \mathcal{I}\,[\mathbb{R}^m]$; here $\mathcal{E}_{\rm M}[\mathbb{R}^m]$ is the subalgebra of functions such that for each compact subset K of $\mathbb{R}^m$ and $p \in \mathbb{N}_0^m$ there is a $q \in \mathbb{N}$ so that for any

$\varphi \in A_q(\mathbb{R}^m)$, $\sup_{x\in K} |\partial^p f(\varphi_\varepsilon, x)| = O(\varepsilon^{-q})$, as $\varepsilon \to 0_+$. The ideal $\mathcal{I}[\mathbb{R}^m]$ of $\mathcal{E}_{\mathrm{M}}[\mathbb{R}^m]$ is the set of functions that for each compact subset K of $\mathbb{R}^m$ and $p \in \mathbb{N}_0^m$ there is a $q \in \mathbb{N}$ so that for every $r \geq q$ and $\varphi \in A_r(\mathbb{R}^m)$, $\sup_{x\in K} |\partial^p f(\varphi_\varepsilon, x)| = O(\varepsilon^{r-q})$, as $\varepsilon \to 0_+$.

The algebra $\mathcal{G}$ contains the distributions canonically embedded as a $\mathbb{C}$-vector subspace by the map $i : \mathcal{D}'(\mathbb{R}^m) \to \mathcal{G} : u \mapsto \widetilde{u} = \{\widetilde{u}(\varphi, x) = (u * \check{\varphi})(x)\}$, where $\check{\varphi}(x) = \varphi(-x)$ and $\varphi \in A_q(\mathbb{R}^m)$.

Definition 2. (a) The functions $f, g \in \mathcal{G}$ are 'associated', which is denoted as $f \approx g$, if for some $f(\varphi_\varepsilon, x)$ and $g(\varphi_\varepsilon, x)$ and arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R}^m)$ there is some $q \in \mathbb{N}_0$ so that, for any $\varphi(x) \in A_q(\mathbb{R})$, it holds: $\lim_{\varepsilon\to 0_+} \int_{\mathbb{R}} [f(\varphi_\varepsilon, x\, dx) - g(\varphi_\varepsilon, x)]\psi(x)\, dx = 0$.

(b) A function $f \in \mathcal{G}$ admits some $u \in \mathcal{D}'(\mathbb{R}^m)$ as 'associated distribution', denoted $f \approx u$, if for some representative $f(\varphi_\varepsilon, x)$ of f and arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R}^m)$ there is a $q \in \mathbb{N}_0$ such that, for any $\varphi(x) \in A_q(\mathbb{R})$, it holds: $\lim_{\varepsilon\to 0_+} \int_{\mathbb{R}} f(\varphi_\varepsilon, x)\psi(x)\, dx = \langle u, \psi\rangle$.

The distribution associated, if it exists, is unique and its image is associated with the former [1], the association thus being a faithful generalization of equality in $\mathcal{D}'(\mathbb{R}^m)$. Then, by product of distributions in the algebra $\mathcal{G}$ is meant the product of their embeddings, whenever the result admits an associated distribution.

Now, if $\widetilde{x^{-p}}$ and $\widetilde{\delta^{(q)}}(x)$ are embeddings in $\mathcal{G}(\mathbb{R})$ of the corresponding distributions, the following generalization of (1) for any $p, q \in \mathbb{N}_0$ was proved in [2]:

$$\widetilde{x^{-p}}\,.\,\widetilde{x^{-q}} - \frac{(-1)^{p+q}\pi^2}{(p-1)!\,(q-1)!}\widetilde{\delta^{(p-1)}}(x)\,.\,\widetilde{\delta^{(q-1)}}(x) \approx x^{-p-q},\ x \in \mathbb{R}.$$

Below we shall need this general property of balanced products of distributions on $\mathbb{R}^m$ with tensor-product structure [4].

Theorem 1. *Let u_k, v_k, $k = 1, 2$ be distributions in $\mathcal{D}'(\mathbb{R}^m)$ such that $u_k(x) = \prod_{i=1}^m u_k^i(x_i)$, $v_k(x) = \prod_{i=1}^m v_k^i(x_i)$, where all u_k^i, v_k^i are distributions in $\mathcal{D}'(\mathbb{R})$, and their embeddings in $\mathcal{G}(\mathbb{R})$ satisfy: $\widetilde{u_1^i}\,.\,\widetilde{v_1^i} - \widetilde{u_2^i}\,.\,\widetilde{v_2^i} \approx 0$, $i = 1, ..., m$. Then it holds for the embeddings in $\mathcal{G}(\mathbb{R}^m)$ of the tensor-product distributions u_k, v_k:*

$$\widetilde{u_1}\,.\,\widetilde{v_1} - \widetilde{u_2}\,.\,\widetilde{v_2} \approx 0. \tag{2}$$

We proceed further to evaluating some balanced products of the distributions $x_\pm^a$ and $\delta^{(p)}(x)$ in $\mathcal{G}(\mathbb{R}^m)$.

Notation 2: (a). Let $a = (a_1, \ldots, a_m)$ be ordered m-tuples in $\mathbb{R}^m$ with vector operations and $k \in \mathbb{Z}$. Then $a + k$ stands for $(a_1 + k, \ldots, a_m + k)$, $a > k$ denotes $a_i > k$ for $i = 1, ..., m$. Define now by $x^a := (x_1^{a_1}, \ldots, x_m^{a_m})$ and $\Gamma(a) = \prod_{i=1}^m \Gamma(a_i)$. Then, $\Omega = \{a \in \mathbb{R} : a \neq -1, -2, \ldots\}$ and Ω^m is its m-fold tensor product.

(b). Denote further the 'normed' powers of $x \in \mathbb{R}^m$, $a \in \Omega^m$, and supported only in one quadrant of $\mathbb{R}^m$, as follows:

$$\chi_\pm^a \equiv \chi_\pm^a(x) = \{\frac{\pm x^a}{\Gamma(a+1)} \quad \text{for } x > 0/\ x < 0, \quad = 0 \ \text{ elsewhere } \}.$$

By Notation 1 (a), $x^a \,/\, \Gamma(a+1) \ = \prod_{i=1}^m x_i^{a_i} \,/\, \Gamma(a_i)$.

We will prove now the following.

Theorem 2. *For any function $f(x) \in C_d^2(\mathbb{R} \setminus \{0\})$, its imbedding $\widetilde{f}(x)$ in $\mathcal{G}(\mathbb{R})$ satisfies the equation*

$$\widetilde{f}(x) \,.\, \widetilde{\delta}'(x) \approx -(h_0 + m_1)\ \delta \ + \ m_0\ \delta', \tag{3}$$

where $h_0 = f(0_+) - f(0_-), m_0 = (f(0_-) + f(0_+))/2, m_1 = (f'(0_-) + f'(0_+))/2.$

Proof. Denote $T(\sigma) := \langle\ \widetilde{f}_\sigma(x) \,.\, \widetilde{\delta}'_\sigma(x), \psi(x) \rangle$, $\psi(x) \in \mathcal{D}(\mathbb{R})$. On changing the variables $y = \sigma v + x\, x = -\sigma u$, we get:

$$T(\sigma) = \frac{-1}{\sigma} \int_{-l}^{l} du\, \psi(-\sigma u)\widetilde{\delta}'(x) \left(\int_{-l}^{u} \tau\widetilde{\delta}\, dv + \int_{u}^{l} f(\tau)\ \widetilde{\delta}\, dv \right).$$

Here the notations $\tau := \sigma v - \sigma u$ and $\widetilde{\delta} := \widetilde{\delta}(u, v)$ are temporarily used for shortness. Applying the Taylor theorem to the test-function ψ and changing the order of integration we get:

$$\begin{aligned} T(\sigma) \ &= \ \frac{-\psi(0)}{\sigma} \int_{-l}^{l} dv\, \widetilde{\delta} \left(\int_{v}^{l} f(\tau)\, \widetilde{\delta}'\, du + \int_{-l}^{v} f(\tau)\, \widetilde{\delta}'\, du \right) \\ &+ \ \psi'(0) \int_{-l}^{l} dv\, \widetilde{\delta} \left(\int_{v}^{l} f(\tau) u\, \widetilde{\delta}' du + \int_{-l}^{v} f(\tau)\, u\, \widetilde{\delta'(x)}\, du \right) \\ &+ \ O(\sigma) =: \ \psi(0)\ T_1 \ + \ \psi'(0)\ T_2 + O(\sigma). \end{aligned}$$

The above asymptotic evaluation is obtained taking into account that the third term in the Taylor expansion is multiplied by definite integrals, majorizable by constants. Integrating further by parts with respect to the variable u, applying the Lebesque theorem on bounded convergence, and taking into account Notation 3, we obtain for the main part of T_1:

$$T_1 = f(0_-) - f(0_+) - \frac{1}{2}\,\big[\, f'(0_-) + f'(0_+)\,\big] + O(\sigma) = -\,(h_0 + m_1).$$

Here the Taylor theorem is applied to the function $f' \in C^1(\mathbb{R}_\pm)$ up to second order to get the above expansion about the point $\sigma(v-u)$; which is respectively > 0 or < 0. All that finally gives for $v > u$, resp. $v < u$:

$$f'(\sigma v - \sigma u) = f'(0_\pm) + \sigma(v-u)f''(0_\pm) + O(\sigma^2) = f'(0_\pm) + O(\sigma).$$

Proceeding similarly further, we obtain for the main part of T_2:

$$T_2 = -\frac{1}{2}\,\big[\, f(0_-) + f'(0_+)\,\big] = -\,m_0\, +\, O(\sigma).$$

We have used that $\widetilde{\delta}$ satisfies

$$\int_{-l}^{l} v\,\widetilde{\delta}^2\,dv = 0 \quad \text{and} \quad \int_{-l}^{l} dv\,\widetilde{\delta}\int_{-l}^{v}\widetilde{\delta}\,du = \frac{1}{2}.$$

Thus

$$T(\sigma) = \langle \widetilde{f}.\widetilde{\delta}\psi(x)\,\rangle = -(h_0 + m_1)\,\langle\,\delta,\psi\,\rangle + m_0\,\langle\,\delta',\psi\,\rangle + \; O(\sigma).$$

Passing finally to the limit as $\sigma \to 0_+$ and applying Definition 2 (b), we obtain equation (3). The proof is complete. □

Remarks. **(a)** The result obtained is related to singularities at the point $x = 0$, but the calculations are clearly valid for any $x \in \mathbb{R}$.

(b) In the particular case of infinitely-differentiable functions, equation (3) is in consistence with the corresponding products obtained in the classical Distribution theory. Indeed, for $f \in C^\infty(\mathbb{R})$ and $\psi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle\, f\,.\,\delta',\,\psi\,\rangle = \langle\,\delta',\,f\,\psi\,\rangle = -\partial x\,(f\,\psi)$$

at the point $x = 0$, or else

$$f\,.\,\delta' \;= -f'(0)\delta + f(0)\delta'.$$

This coincides with equation (3) since in that case $m_0 = f(0), m_1 = f'(0), h_0 = 0$.

Example. The following equation is obtained on replacing the model $F(x)$ of a function $f(x) \in C^2_d(\mathbb{R}\setminus\{0\})$ with the model H of the θ-function in equation (3):

$$H\,.\,D' \;\approx\; -\,\delta + \frac{1}{2}\,\delta'.$$

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