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HOMOGENIZATION OF A DEGENERATE PDE WITH NON LINEAR WENTZELL BOUNDARY CONDITION

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Abstract: We develop homogenization results of a degenerate semilinear PDE with a Wentzell-type boundary condition. The second order operator is also degenerate. Our approach is entirely probabilistic, and extends the result of Diakhaby and Ouknine [3].

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1. Introduction

We study the solution of a semi-linear partial differential equation (PDE) in the half-space $\boldsymbol{D} = \left\{(x_1, x_2, \ldots, x_d) \in R^d \;:\; x_1 > 0\right\}$ with additional boundary condition on $\partial\boldsymbol{D}$. For each $\varepsilon > 0$, we consider

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$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) = L_\varepsilon u^\varepsilon(t,x) + f\left(\dfrac{x}{\varepsilon}, u^\varepsilon(t,x)\right) + \dfrac{1}{\varepsilon} e\left(\dfrac{x}{\varepsilon}, u^\varepsilon(t,x)\right), \\ \quad x \in \boldsymbol{D},\ 0 < t, \\ \Gamma_\varepsilon u^\varepsilon(t,x) + h\left(\dfrac{x}{\varepsilon}, u^\varepsilon(t,x)\right) + \dfrac{1}{\varepsilon} l\left(\dfrac{x}{\varepsilon}, u^\varepsilon(t,x)\right) = 0, \\ x \in \partial\boldsymbol{D},\ 0 \le t, \\ u^\varepsilon(0,x) = g(x), \qquad x \in \boldsymbol{D}, \end{cases} \tag{1}$$

where

- $e : R^d \times R \longrightarrow R$ is a measurable mapping, which is periodic, of period one in each direction in the first argument, continuous in the second argument uniformly with respect to the first, and satisfies:

$$\int_{T^d} e(x,y) m(dx) = 0, \qquad \forall y \in R, \tag{2}$$

  where m is the unique invariant measure on the torus T^d.

  We suppose that e is twice continuously differentiable in y, uniformly with respect to x, and there exists a constant K such that:

$$|e(x,y)| + \left|\frac{\partial}{\partial y} e(x,y)\right| + \left|\frac{\partial^2}{\partial y^2} e(x,y)\right| \le K, \quad \forall x \in R^d,\ y \in R. \tag{3}$$

- $l : R^{d-1} \times R \longrightarrow R$ is a measurable mapping, which is periodic, of period one in each direction in the first argument, continuous in the second argument uniformly with respect to the first, and satisfies:

$$\int_{T^{d-1}} l(x,y) m_0(dx) = 0, \qquad \forall y \in R, \tag{4}$$

  where m_0 denotes the invariant measure on the torus T^{d-1}. We suppose that l is twice continuously differentiable in y, uniformly with respect to x, and there exists a constant K such that:

$$|l(x,y)| + \left|\frac{\partial}{\partial y} l(x,y)\right| + \left|\frac{\partial^2}{\partial y^2} l(x,y)\right| \le K, \quad \forall x \in R^{d-1},\ y \in R. \tag{5}$$

- $f : R^d \times R \longrightarrow R$, $g : R^d \longrightarrow R$ and $h : R^d \times R \longrightarrow R$ are sufficiently smooth functions. Equivalently the coefficients can be seen as periodic functions with respect to the first variable with period one in each direction on R^d and are such that for some $c > 0$, $p > 0$, $\mu \in R$, $\beta < 0$, and all $x \in R^d$, $y, y' \in R$:

$$|g(x)| \leq c\,(1 + |x|^p) \tag{6}$$

$$|f(x,y)| \leq c\left(1 + |y|^2\right) \tag{7}$$

$$(y - y')\,[f(x,y) - f(x,y')] \leq \mu\,|y - y'| \tag{8}$$

$$(y - y')\,[h(x,y) - h(x,y')] \leq \beta\,|y - y'| \tag{9}$$

$$|h(x,y)| \leq c\left(1 + |y|^2\right). \tag{10}$$

Assumptions and Definitions

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which two Brownian motions of $\left(B^i\right)_{1\leq i\leq d}$ and $\left(N^i\right)_{1\leq i\leq d}$ are defined (see [3]). Let E the corresponding expectation operator.

The differential operator L_ε inside $\boldsymbol{D}$ and Γ_ε in ∂D are given by:

$$L_\varepsilon = \frac{1}{2}\sum_{i,j}^{d} a_{i,j}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2}{\partial x_i x_j} + \frac{1}{\varepsilon}\sum_{i=1}^{d} b_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i} + \sum_{i=1}^{d} c_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i}, \tag{11}$$

$$\Gamma_\varepsilon = \frac{1}{2}\sum_{i,j}^{d} \tau_{i,j}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2}{\partial x_i x_j} + \frac{1}{\varepsilon}\sum_{i=1}^{d} \beta_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i} + \sum_{i=1}^{d} \gamma_i\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_i}. \tag{12}$$

These operators are the generators of the reflected $(L_\varepsilon, \Gamma_\varepsilon)$-diffusion (see Tanaka [6]):

$$\begin{cases} dX_t^\varepsilon = \sigma\left(\dfrac{X_t^\varepsilon}{\varepsilon}\right)dB_t + \dfrac{1}{\varepsilon}b\left(\dfrac{X_t^\varepsilon}{\varepsilon}\right)dt + c\left(\dfrac{X_t^\varepsilon}{\varepsilon}\right)dt + \tau\left(\dfrac{X_t^\varepsilon}{\varepsilon}\right)dN_t \\ +\dfrac{1}{\varepsilon}\beta\left(\dfrac{X_t^\varepsilon}{\varepsilon}\right)d\varphi_t^\varepsilon + \gamma\left(\dfrac{X_t^\varepsilon}{\varepsilon}\right)d\varphi_t^\varepsilon, \quad 0 < t, \\ X_t^{1,\varepsilon} \geq 0,\ \varphi^\varepsilon \text{ is continuous and increasing}, \\ \displaystyle\int_0^t X_s^{1,\varepsilon}d\varphi_s^\varepsilon = 0, \quad 0 < t, \\ X_0^\varepsilon = x, \end{cases} \tag{13}$$

where $X^{1,\varepsilon}$ denotes the first component of the process X^ε. We recall that $\boldsymbol{D} = R_+^* \times R^{d-1}$, so that X^ε lives is $\boldsymbol{D}$, that is, $X^{1,\varepsilon}$ remains non-negative and φ^ε increases when and only when $X^{1,\varepsilon}$ is zero, just to keep it non-negative.

The functions

$$\tau: \ \partial \boldsymbol{D} \left(\cong R^{d-1}\right) \longrightarrow R^{(d-1)\times(d-1)}, \beta: \ \partial \boldsymbol{D} \left(\cong R^{d-1}\right) \longrightarrow R^{d-1}$$

and the function $\gamma: \ \partial \boldsymbol{D} \left(\cong R^{d-1}\right) \longrightarrow R^d$ with $\gamma^1(x) = 1$, are smooth and periodic of period one in each direction. The matrix $\boldsymbol{\alpha} = \boldsymbol{\tau}\boldsymbol{\tau}^*$ is degenerate and satisfies the hypo-elliptic Hörmander condition.

We suppose that the functions

$$\sigma : R^d \longrightarrow R^{d\times d}, \quad b: \ R^d \longrightarrow R^d, \quad c: \ R^d \longrightarrow R^d$$

are smooth and periodic of period one in each direction. The novelty here (in comparison with the work of Diakhaby and Ouknine [3]), is that assume that the matrix $\boldsymbol{a} = \boldsymbol{\sigma}\boldsymbol{\sigma}^*$ is degenerate and satisfies hypo-elliptic Hömander condition.

Definition 1.1 (**Lie bracket**). The Lie bracket between the vectors fields A_j and A_k is defined by:

$$[A_j, A_k] := A_j^{\bigtriangledown} A_k - A_k^{\bigtriangledown} A_j, \tag{14}$$

where $A_j^{\bigtriangledown} A_k := A_j^l \partial_l A_k^i \dfrac{\partial}{\partial x_i}$.

Definition 1.2 (**Strong Hömander Condition (SHC)**). Let $H(n, x)$ be the set of the Lie brackets of the vector field $(m_j(x))_{1\leq j\leq d}$ of order lower than n at the point $x \in \mathcal{X}$. We say that matrix m satisfies the (SHC) if for all $x \in \mathcal{X}$, there exists $n_x \in N$ such that $H(n_x, x)$ generates $\mathcal{X}$.

Let us define $\tilde{X}_t^\varepsilon := \dfrac{1}{\varepsilon} X_{\varepsilon^2 t}^\varepsilon$ and $\tilde{\varphi}_t^\varepsilon := \dfrac{1}{\varepsilon}\varphi_{\varepsilon^2 t}^\varepsilon$, then we get with two news standard d-dimensional Brownian motions $\{B_t^\varepsilon : \ t \geq 0\}$, and $\{N_t^\varepsilon : \ t \geq 0\}$ which in fact depends on ε:

$$\begin{cases} d\tilde{X}_t^\varepsilon = \sigma\left(\tilde{X}_t^\varepsilon\right) dB_t^\varepsilon + b\left(\tilde{X}_t^\varepsilon\right) dt + \varepsilon c\left(\tilde{X}_t^\varepsilon\right) dt + \tau\left(\tilde{X}_t^\varepsilon\right) dN_t^\varepsilon \\ + \beta\left(\tilde{X}_t^\varepsilon\right) d\tilde{\varphi}_t^\varepsilon + \varepsilon\gamma\left(\tilde{X}_t^\varepsilon\right) d\tilde{\varphi}_t^\varepsilon, \quad 0 < t, \\ \tilde{X}_t^{1,\varepsilon} \geq 0, \ \tilde{\varphi} \text{ is continuous and increasing}, \\ \displaystyle\int_0^t \tilde{X}_s^{1,\varepsilon} d\tilde{\varphi}_s^\varepsilon = 0, \quad 0 < t, \\ \tilde{X}_0^\varepsilon = \dfrac{x}{\varepsilon}. \end{cases} \tag{15}$$

We set L_0 and Γ_0 the two operators defined as:

$$L_0 := \frac{1}{2}\sum_{i,j}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \quad x \in T^d, \tag{16}$$

$$\Gamma_0 := \frac{1}{2}\sum_{i,j}^{d} \alpha_{i,j}(x)\frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^{d} \beta_i(x)\frac{\partial}{\partial x_i}, \quad x \in T^{d-1}. \tag{17}$$

Hypothesis (H): We suppose that σ and α satisfies the SHC.

2. Weak Convergence (SDE and BSDE)

With **(H)** and the boundary condition $\Gamma_0 u = 0$, according to Tanaka [6] (see Pardoux and Diedhiou [2] in the case that $\partial D = \emptyset$), the process $\tilde{X}^\varepsilon$ is ergodic and has a unique invariant measure whose density is strictly positive. Thereafter, we set m the invariant measure associated of L_0 on T^d and we set m_0 the invariant measure associated of the differential operator Γ_0 on T^{d-1}.

Throughout the paper, we suppose that

$$\int_{T^d} b(x)m(dx) = 0, \tag{18}$$

$$\int_{T^{d-1}} \beta(x)m_0(dx) = 0. \tag{19}$$

Remark 2.1. τ and β are regarded as $d \times d$-matrix and d-vector-valued functions respectively, with the convention that:

$$\tau^{i1} = \tau^{1l} = \beta^1 = 0.$$

Let $\hat{b}$ be the solution of Poisson equation : $L_0\hat{b} = -b$ in $\boldsymbol{D}$, and $\Gamma_0\hat{b} = -\beta$ in $\partial\boldsymbol{D}$ let us introduce the process $\hat{X}_t^\varepsilon$ defined as:

$$\begin{aligned}\hat{X}_t^\varepsilon &= X_t^\varepsilon + \varepsilon\left[\hat{b}\left(\frac{X_t}{\varepsilon}\right) - \hat{b}\left(\frac{x}{\varepsilon}\right)\right] \\ &= x + \int_0^t \left(I + \nabla\hat{b}\right)\sigma\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dB_s + \int_0^t \left(I + \nabla\hat{b}\right)c\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds \\ &\quad + \int_0^t \left(I + \nabla\hat{b}\right)\tau\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dN_s + \int_0^t \left(I + \nabla\hat{b}\right)\gamma_\varepsilon\left(\frac{X_s^\varepsilon}{\varepsilon}\right)d\varphi_s^\varepsilon.\end{aligned} \tag{20}$$

With (**H**) we know that the pair (L_0, Γ_0) is hypo-elliptic on $\overline{\boldsymbol{D}}$. Then there exists a bounded and smooth solution η of the PDE with Wentzell-type boundary condition:

$$\begin{cases} L_0\eta = 0 & \text{in } \boldsymbol{D} \\ \Gamma_0\eta = \left(I + \nabla\hat{b}\right)\gamma - \int_{T^{d-1}} \left(I + \nabla\hat{b}\right)(x)\gamma(x)m_0(dx) & \text{on } \boldsymbol{\partial D}. \end{cases} \tag{21}$$

Taking such a solution η, we have by Itô:

$$\begin{aligned} \varepsilon^2\left[\eta\left(\frac{X_t^\varepsilon}{\varepsilon}\right) - \eta\left(\frac{x}{\varepsilon}\right)\right] &= \varepsilon\int_0^t \nabla\eta\sigma\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dB_s + \varepsilon\int_0^t \nabla\eta\tau\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dN_s \\ &+ \varepsilon\int_0^t \nabla\eta c\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds + \varepsilon\int_0^t \nabla\eta\gamma\left(\frac{X_s^\varepsilon}{\varepsilon}\right)d\phi_s^\varepsilon \\ &+ \int_0^t \left(I + \nabla\hat{b}\right)\gamma\left(\frac{X_s^\varepsilon}{\varepsilon}\right)d\varphi_s^\varepsilon - \varphi_t^\varepsilon\int_{T^{d-1}}\left(I + \nabla\hat{b}\right)(x)\gamma(x)m_0(dx). \end{aligned} \tag{22}$$

Putting (22) into (20) we have

$$\begin{aligned} \hat{X}_t^\varepsilon &= x + \int_0^t \left(I + \nabla\hat{b}\right)\sigma\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dB_s + \int_0^t \left(I + \nabla\hat{b}\right)\tau\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dN_s \\ &+ \int_0^t \left(I + \nabla\hat{b}\right)c\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds + \varphi_t^\varepsilon\int_{T^{d-1}}\left(I + \nabla\hat{b}\right)\gamma(x)m_0(dx) \\ &- \varepsilon\int_0^t \nabla\eta\gamma\left(\frac{X_s^\varepsilon}{\varepsilon}\right)d\phi_s^\varepsilon - \varepsilon\int_0^t \nabla\eta\sigma\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dB_s \\ &- \varepsilon\left(\int_0^t \nabla\eta\tau\left(\frac{X_s^\varepsilon}{\varepsilon}\right)dN_s - \int_0^t \nabla\eta c\left(\frac{X_s^\varepsilon}{\varepsilon}\right)ds + \varepsilon\left[\eta\left(\frac{X_t}{\varepsilon}\right) - \eta\left(\frac{x}{\varepsilon}\right)\right]\right). \end{aligned} \tag{23}$$

Before proceeding, we introduce some definitions:

$$\begin{aligned} a_0 &= \int_{T^d}\left(I + \nabla\hat{b}\right)(x)a(x)\left(I + \nabla\hat{b}\right)^*(x)m(dx), \\ \alpha_0 &= \int_{T^d}\left(I + \nabla\hat{b}\right)(x)\alpha(x)\left(I + \nabla\hat{b}\right)^*(x)m(dx), \\ L &= \frac{1}{2}\sum_{i,j=1}^d a_0^{ij}\partial_i\partial_j + \sum_{i=1}^d c_0^i\partial_i, \quad \text{with} \\ c_0 &= \int_{T^d}\left(I + \nabla\hat{b}\right)(x)c(x)m_0(dx), \\ \Gamma &= \frac{1}{2}\sum_{i,j=2}^d \alpha_0^{ij}\partial_i\partial_j + \sum_{i=1}^d \gamma_0^i\partial_i, \quad \text{with} \end{aligned}$$

$$\gamma_0 = \int_{T^{d-1}} \left(I + \nabla\hat{b}\right)(x)\gamma(x)m_0(dx).$$

As in [3], we have the following theorem.

Theorem 2.2. *Under (**H**) with conditions (17) and (18), the $(L_\varepsilon, \Gamma_\varepsilon)$-reflected diffusion process X^ε converges in law to the (L, Γ)-reflected diffusion process X as $\varepsilon \downarrow 0$. Moreover, on the space $C\left([0,T], R^{2d+1}\right)$ equipped with the sup-norm topology,*

$$\left(X^\varepsilon, M_t^{X^\varepsilon}, \varphi^\varepsilon\right) \longrightarrow \left(X, M^X, \varphi\right),$$

where:

- *M^X is the martingale part of X,*
- *φ (resp. φ^ε) is the local time of X^1 (resp. $X^{1,\varepsilon}$).*

Let $\overline{X}$ denote the unique diffusion process with values in the d-dimensional torus T^d, whose generator is the operator L_0.

We now consider a type of BSDE which has been introduced in Pardoux and Zhang [5]. For each fixed $(t,x) \in [0,T] \times \overline{\boldsymbol{D}}$, let $\left\{ (Y_s^\varepsilon, U_s^\varepsilon); 0 \le s \le T^d \right\}$ be the solution of the BSDE

$$\begin{aligned} Y_s^\varepsilon = & g\left(X_t^\varepsilon\right) + \int_s^t f\left(\frac{X_r^\epsilon}{\varepsilon}, Y_r^\varepsilon\right) dr + \frac{1}{\varepsilon}\int_s^t e\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\epsilon\right) dr \\ & + \int_s^t h\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon\right) d\varphi_r^\varepsilon + \frac{1}{\varepsilon}\int_s^t l\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon\right) d\varphi_r^\varepsilon - \int_s^t U_r^\varepsilon dM_r^{X^\varepsilon}. \end{aligned} \tag{24}$$

For each fixed $y \in R$, let set $\hat{e}$ be the solution of the Poisson equation:

$$L_0\hat{e}(x,y) + e(x,y) = 0,\ x \in T^d,\ y \in R. \tag{25}$$

More precisely by (2), $\hat{e}$ is centered with respect to the invariant measure m and is given by the formula

$$\hat{e}(x,y) = \int_0^\infty E^x e\left(\overline{X}_t, y\right) dt. \tag{26}$$

Note that, see [4], $\hat{e} \in C^{0,2}\left(T^d, R\right)$ and $\hat{e}(.,y)$, $\frac{\partial}{\partial y}\hat{e}(.,y)$, $\frac{\partial^2}{\partial y^2}\hat{e}(.,y) \in W^{2,p}\left(T^d\right)$, for any $p \ge 1$ there exists K' such that for all $y \in R$

$$\|\hat{e}(.,y)\|_{W^{2,p}(T^d)} + \left\|\frac{\partial}{\partial y}\hat{e}(.,y)\right\|_{W^{2,p}(T^d)} + \left\|\frac{\partial^2}{\partial y^2}\hat{e}(.,y)\right\|_{W^{2,p}(T^d)} \le K'. \tag{27}$$

For each fixed $y \in R^d$, let set $\hat{l}$ be the solution of the Poisson equation which satisfies (4) :

$$\Gamma_0 \hat{l}(x,y) + l(x,y) = 0,\ x \in T^d,\ y \in R. \tag{28}$$

Note that, see [4], $\hat{l} \in C^{0,2}\left(T^{d-1}, R\right)$ and $\hat{l}(.,y), \dfrac{\partial}{\partial y}\hat{l}(.,y), \dfrac{\partial^2}{\partial y^2}\hat{l}(.,y) \in W^{2,p}\left(T^d\right)$, for any $p \geq 1$ there exists K' such that for all $y \in R$

$$\left\|\hat{l}(.,y)\right\|_{W^{2,p}\left(T^d\right)} + \left\|\frac{\partial}{\partial y}\hat{l}(.,y)\right\|_{W^{2,p}\left(T^d\right)} + \left\|\frac{\partial^2}{\partial y^2}\hat{l}(.,y)\right\|_{W^{2,p}\left(T^d\right)} \leq K'. \tag{29}$$

We introduce the notations:

$$M_t^\varepsilon = \int_0^t U_r^\varepsilon dM_r^{X^\varepsilon} \quad \text{and} \quad M_t = \int_0^t U_r dM_r^X, \quad 0 \leq t \leq T,$$

and we consider the quintuple $\left(X, M^X, \varphi, Y, M\right)$ (resp. $\left(X^\varepsilon, M^{X^\varepsilon}, \varphi^\varepsilon, Y^\varepsilon, M^\varepsilon\right)$) as a random element of the space $C\left([0,t], R^{2d+1}\right) \times D\left([0,t], R^2\right)$, where we equip the first factor with the sup-norm topology, and the second factor with the S-topology of Jakubowski (see [1]).

Considering the SDE and the BSDE satisfying respectively by X and Y:

$$\begin{aligned} X_t &= x + c_0 t + \int_0^t b_0(Y_s)ds + \sqrt{a_0}B_t + \int_0^t d_0(Y_s)d\varphi_t \\ &\quad + \sqrt{\alpha_0}N_t + \gamma_0\varphi_t, \\ Y_t &= g(X_T) + \int_t^T f_0(Y_s)ds + \int_t^T h_0(Y_s)d\varphi_s + M_t - M_T, \end{aligned} \tag{30}$$

where

$$b_0(y) = \int_{T^d} (I + \nabla\hat{b})a(x)\frac{\partial^2 \hat{e}}{\partial x \partial y}(x,y)m(dx),$$

$$c_0 = \int_{T^d} \left(I + \nabla\hat{b}\right)c(x)m(dx),$$

$$f_0(y) = \int_{T^d} (\left[\left\langle \frac{\partial \hat{e}}{\partial x}, c(x)\right\rangle - (\frac{\partial \hat{e}}{\partial y} \times e) + \frac{\partial^2 \hat{e}}{\partial x \partial y}a(x)\left(\frac{\partial \hat{e}}{\partial x}\right)^*\right])(x,y)m(dx),$$
$$+ \int_{T^d} f(x,y)m(dx),$$

$$h_0 = \int_{T^{d-1}} \left(h(.,y) + \left\langle \frac{\partial \hat{e}}{\partial x}(.,y), \gamma\right\rangle\right)(x)m_0(dx),$$

$$d_0(y) = \int_{T^d} \left(I + \nabla\hat{b}\right)\alpha(x)\frac{\partial^2 \hat{e}}{\partial x \partial y}(x,y)m(dx).$$

Then we have the following theorem.

Theorem 2.3. *Under (**H**), (2), (4) and the conditions (6),..., (10), we have*

$$Y_0^\varepsilon \longrightarrow Y_0 \ \ in \ \ R.$$

Proof. We adopt the same techniques as in [3]. □

3. Main Result

For each $(t,x) \in R_+ \times \overline{\boldsymbol{D}}$, the solution of (1) is into the form

$$u^\varepsilon(t,x) := Y_0^\varepsilon, \tag{31}$$

where Y^ε denotes the solution of the BSDE considered in the previous section. Now, let us consider the following homogenized system:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = Lu(t,x) + f_0\big(u(t,x)\big) + b_0\big(u(t,x)\big)\nabla u(t,x)\ ,\ x \in \boldsymbol{D}, \\ \Gamma u(t,x) + d_0\big(u(t,x)\big)\nabla u(t,x) + h_0\big(u(t,x)\big) = 0\ ,\ x \in \partial\boldsymbol{D},\ t \geq 0, \\ u(0,x) = g(x)\ ,\ x \in \boldsymbol{D}. \end{cases} \tag{32}$$

We shall assume w.l.o.g. that the orthogonal basis of R^d has been chosen in such a way that the matrix a_0 is of the form

$$a_0 = \begin{pmatrix} a_0' & 0 \\ 0 & 0 \end{pmatrix},$$

where a_0' is a $d' \times d'$ positive definite matrix, with $d' \leq d$.

We set $R^d = E_{d'} \oplus E_{d-d'}$, where $E_{d'}$ is the subspace of R^d of dimension d' generated by the vectors e_i, $i := 1, \ldots, d'$ after a new arrangement of the basis vectors of R^d so we can obtain the wished form of a_0.

Define the space

$$H_{a_0}(\boldsymbol{D}) = \left\{ v \in L^2(\overline{\boldsymbol{D}}) : \ \ \text{s.t.} \ \ \sqrt{a_0} \cdot \nabla v \in \big(L^2(\overline{\boldsymbol{D}})\big)^d \text{ and } v\Big|_{\partial\boldsymbol{D}} = 0 \right\},$$

which will be associated to the norm:

$$\|v\|_{H_{a_0}(\boldsymbol{D})} = \Big(\|v\|^2_{L^2(\overline{\boldsymbol{D}})} + \|\sqrt{a_0}\nabla v\|^2_{(L^2(\overline{\boldsymbol{D}}))^d} \Big)^{1/2}.$$

By our assumptions, we have the *a priori* estimates

$$\|f_0(v)\|_{L^2(\overline{\boldsymbol{D}})} + \|b_0(v)\nabla v\|_{L^2(\overline{\boldsymbol{D}})} \leq C\left(1 + \|v\|_{H_{a_0}(\boldsymbol{D})}\right).$$

Thus, we can show the following theorem.

Theorem 3.1. *Under (**H**), (2), (4), (18), (19) and the conditions (6),...,(10); the system (32) has a unique solution u in $L^2\left([0,T], H_0^1(\overline{\boldsymbol{D}})\right)$, such that for all $1 \leq k \leq d$*

$$\left\langle a_0 \nabla u_k, \nabla u_k \right\rangle \in L^1\left([0,T] \times \boldsymbol{D}\right),$$

$$\left\langle \alpha_0 \nabla u_k, \nabla u_k \right\rangle \in L^1\left([0,T] \times \partial\boldsymbol{D}\right), \quad \text{with } \; u_k = \frac{\partial u}{\partial x_k}.$$

Moreover

$$u \in C\big(R_+ \times \overline{\boldsymbol{D}}\big)$$

and we have for all $t \geq 0$, for all $x \in R^d$

$$u^\varepsilon(t,x) \to u(t,x)\ , \ \text{when } \; \varepsilon \to 0,$$

where $u^\varepsilon(t,x)$ is the solution of the PDE system (1).

Proof. We adopt similar tools as in [2].
∗ *Step 1*:
We first assume that the matrix a_0 is elliptic and we look for a solution

$$u \in L^2\Big(\big((0,T); H_0^1(\overline{\boldsymbol{D}})\big) \bigcap C\Big([0,T]; L^2(\overline{\boldsymbol{D}})\Big).$$

Let us prove the existence and uniqueness of the solution of the PDE. Set $F(\boldsymbol{D}) = L^2\Big((0,T); H_0^1(\overline{\boldsymbol{D}})\Big)$ and consider the map:

$$\Psi : F(\boldsymbol{D}) \longrightarrow F(\boldsymbol{D})$$

Let us show that Ψ is a contraction. For $\overline{v} \in F$, let $\overline{u} = \Psi(\overline{v})$ where $\overline{u} = u - u'$ and $\overline{v} = v - v'$. Denote by ν the ellipticity constant of a_0. For any $\alpha > 0$,

$$\begin{aligned}
&\frac{1}{2} e^{-\alpha t}\|\overline{u}_t\|^2_{L^2(\overline{\boldsymbol{D}})} + \nu \int_0^t e^{-\alpha s}\|\nabla \overline{u}_s\|^2_{\left(L^2(\overline{\boldsymbol{D}})\right)^d} ds \\
&\quad \leq -\frac{\alpha}{2}\int_0^t e^{-\alpha s}\|\overline{u}_s\|^2_{L^2(\overline{\boldsymbol{D}})} ds + \int_0^t e^{-\alpha s}\left\langle h_0\big(v_s\big) - h_0\big(v_s'\big), \overline{u}_s\right\rangle_{L^2(\overline{\boldsymbol{D}})} d\varphi_s \\
&\qquad + \int_0^t e^{-\alpha s}\left\langle f_0\big(v_s\big) - f_0\big(v_s'\big), \overline{u}_s\right\rangle_{L^2(\overline{\boldsymbol{D}})} ds.
\end{aligned}$$

Remark that

$$(h_0(v) - h_0(v'))\overline{u} \le \beta \|\overline{v}\|_{L^2(\overline{\boldsymbol{D}})} \|\overline{u}\|_{L^2(\overline{\boldsymbol{D}})} \quad \text{(as a reminder } \beta < 0)$$
$$(f_0(v) - f_0(v'))\overline{u} \le \mu \|\overline{v}\|_{L^2(\overline{\boldsymbol{D}})} \|\overline{u}\|_{L^2(\overline{\boldsymbol{D}})}.$$

From this, we have

$$\begin{aligned}&\frac{1}{2}e^{-\alpha t}\|\overline{u}_t\|^2_{L^2(\overline{\boldsymbol{D}})} + \nu\int_0^t e^{-\alpha s}\|\nabla\overline{u}_s\|^2_{(L^2(\overline{\boldsymbol{D}}))^d}ds \\ &\quad \le -\frac{\alpha}{2}\int_0^t e^{-\alpha s}\|\overline{u}_s\|^2_{L^2(\overline{\boldsymbol{D}})}ds + \mu\int_0^t e^{-\alpha s}\|\overline{v}_s\|_{L^2(\overline{\boldsymbol{D}})}\|\overline{u}_s\|_{L^2(\overline{\boldsymbol{D}})}ds.\end{aligned}$$

By the fact that:

$$(\nu X - \mu Y)^2 \ge 0 \Rightarrow XY \le \frac{\nu}{2\mu}X^2 + \frac{\mu}{2\nu}Y^2,$$

we have

$$\begin{aligned}&\frac{1}{2}e^{-\alpha t}\|\overline{u}_t\|^2_{L^2(\overline{\boldsymbol{D}})} + \nu\int_0^t e^{-\alpha s}\|\nabla\overline{u}(s)\|^2_{(L^2(\overline{\boldsymbol{D}}))^d}ds \\ &\qquad + \frac{\alpha}{2}\int_0^t e^{-\alpha s}\|\overline{u}(s)\|^2_{L^2(\overline{\boldsymbol{D}})}ds \\ &\qquad \le \frac{\nu}{2}\int_0^t e^{-\alpha s}\Big(\|\overline{v}(s)\|^2_{L^2(\overline{\boldsymbol{D}})} + \underbrace{\|\nabla\overline{v}(s)\|^2_{(L^2(\overline{\boldsymbol{D}}))^d}}_{\text{we add this term}}\Big)ds \\ &\qquad\quad + \frac{\mu^2}{2\nu}\int_0^t e^{-\alpha s}\|\overline{u}(s)\|^2_{L^2(\overline{\boldsymbol{D}})}ds.\end{aligned}$$

Thereby,

$$\begin{aligned}&\nu\int_0^t e^{-\alpha s}\|\nabla\overline{u}(s)\|^2_{(L^2(\overline{\boldsymbol{D}})^d}ds + \left(\frac{\alpha}{2} - \frac{\mu^2}{2\nu}\right)\int_0^t e^{-\alpha s}\|\overline{u}(s)\|^2_{L^2(\overline{\boldsymbol{D}})}ds \\ &\quad \le \frac{\nu}{2}\int_0^t e^{-\alpha s}\Big(\|\overline{v}(s)\|^2_{L^2(\overline{\boldsymbol{D}})} + \|\nabla\overline{v}(s)\|^2_{(L^2(\overline{\boldsymbol{D}}))^d}\Big)ds.\end{aligned}$$

Choose $\alpha = 2\nu + \dfrac{\mu^2}{\nu}$, then we have

$$\begin{aligned}&\int_0^t e^{-\alpha s}\|\nabla\overline{u}(s)\|^2_{(L^2(\overline{\boldsymbol{D}}))^d}ds + \int_0^t e^{-\alpha s}\|\overline{u}(s)\|^2_{L^2(\overline{\boldsymbol{D}})}ds \\ &\quad \le \frac{1}{2}\int_0^t e^{-\alpha s}\Big(\|\overline{v}(s)\|^2_{L^2(\overline{\boldsymbol{D}})} + \|\nabla\overline{v}(s)\|^2_{(L^2(\overline{\boldsymbol{D}}))^d}\Big)ds.\end{aligned}$$

There follows that Ψ is a contraction on $F(\boldsymbol{D})$ with the norm:

$$\|\overline{u}\|_\alpha = \left(\int_0^t e^{-\alpha s} \left(\|\overline{u}(s)\|^2_{L^2(\overline{\boldsymbol{D}})} + \|\nabla \overline{u}(s)\|^2_{(L^2(\overline{\boldsymbol{D}}))^d} \right) ds \right)^{\frac{1}{2}}.$$

$*$ *Step 2*:

Consider the perturbed matrix $A^n = a_0 + \frac{1}{n}\boldsymbol{I}_d$, where a_0 can be degenerate. Let u^n be the unique solution of (32) after substituting a_0 to A^n. Multiply equations of (32) by u^n then,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\boldsymbol{D}} |u^n(t,x)|^2 \, dx + \frac{1}{2}\int_{\boldsymbol{D}} \langle A^n \nabla u^n(t,x), \nabla u^n(t,x) \rangle \, dx \\
&= \int_{\boldsymbol{D}} b_0 \left(u^n(t,x)\right) \cdot \nabla \left(u^n(t,x)\right) u^n(t,x) dx + \int_{\boldsymbol{D}} f_0 \left(u^n(t,x)\right) u^n(t,x) dx \\
&\quad + \int_{\partial \boldsymbol{D}} \Gamma \left(u^n(t,x)\right) u^n(t,x) d\varsigma + \int_{\partial \boldsymbol{D}} h_0 \left(u^n(t,x)\right) u^n(t,x) d\varsigma \\
&\quad + \frac{1}{2}\int_{\boldsymbol{D}} c_0 \cdot \frac{\partial}{\partial x} \left(u^n(t,x)^2\right) dx \\
&\quad + \int_{\partial \boldsymbol{D}} b_0 \left(u^n(t,x)\right) \cdot \nabla \left(u^n(t,x)\right) u^n(t,x) dx,
\end{aligned}$$

where ς is the $(d-1)$-dimensional volume element on $\partial \boldsymbol{D}$.

First we note that,

$$\begin{aligned}
&\int_{\boldsymbol{D}} c_0 \cdot \frac{\partial}{\partial x} \left(u^n(t,x)^2\right) dx = 0 \quad t \ \text{ a.e.,} \qquad \text{and} \\
&\int_{\partial \boldsymbol{D}} \gamma_0 \nabla \left(u^n(t,x)\right) u^n(t,x) d\varsigma = 0.
\end{aligned}$$

Second, by the boundedness of σ, τ and $\frac{\partial^2 \hat{e}}{\partial x \partial y}$, one can easy show that

$$\begin{aligned}
&\left| \int_{\boldsymbol{D}} b_0 \left(u^n(t,x)\right) \cdot \nabla \left(u^n(t,x)\right) u^n(t,x) dx \right| \\
&\leq K \int_{\boldsymbol{D}} \|\sqrt{a_0} \nabla u^n(t,x)\| \, |u^n(t,x)| \, dx \\
&\left| \int_{\partial \boldsymbol{D}} d_0 \left(u^n(t,x)\right) \cdot \nabla \left(u^n(t,x)\right) u^n(t,x) dx \right| \\
&\leq K \int_{\partial \boldsymbol{D}} \|\sqrt{\alpha_0} \nabla u^n(t,x)\| \, |u^n(t,x)| \, d\varsigma.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\boldsymbol{D}} |u^n(t,x)|^2\,dx + \frac{1}{2}\int_{\boldsymbol{D}} \langle A^n \nabla u^n(t,x), \nabla u^n(t,x)\rangle\,dx \\
&+ \frac{1}{2}\int_{\partial \boldsymbol{D}} \langle \alpha_0 \nabla u^n(t,x), \nabla u^n(t,x)\rangle\,d\varsigma \\
&\le \mu \int_{\boldsymbol{D}} |u^n(t,x)|^2\,dx + K\int_{\boldsymbol{D}} \left\|\sqrt{A^n}\nabla u^n(t,x)\right\| |u^n(t,x)|\,dx \\
&\le \left(\mu + \frac{K^2}{2\delta}\right)\int_{\boldsymbol{D}} |u^n(t,x)|^2\,dx \\
&\quad + \frac{\delta}{2}\left(\int_{\boldsymbol{D}} \langle A^n\nabla u^n, \nabla u^n\rangle (t,x)dx + \int_{\partial\boldsymbol{D}} \langle \alpha_0 \nabla u^n, \nabla u^n\rangle (t,x) d\varsigma\right).
\end{aligned}
$$

Choosing $\delta = \dfrac{1}{2}$, we deduct by Gronwall's lemma

$$
\int_{\boldsymbol{D}} |u^n(t,x)|^2\,dx \le K' \mathrm{e}^{K't},
$$

and

$$
\begin{aligned}
&\int_0^T \int_{\boldsymbol{D}} \langle A \nabla u^n(t,x), \nabla u^n(t,x)\rangle\,dxdt \\
&+ \int_0^T \int_{\partial\boldsymbol{D}} \langle \alpha \nabla u^n(t,x), \nabla u^n(t,x)\rangle\,d\varsigma d\varphi_t \le k(T).
\end{aligned}
$$

Now we differentiate equations of (32) for u^n with respect to x_k. Then $u_k^n = \dfrac{\partial u^n}{\partial x_k}$ satisfies:

$$
\begin{cases}
\dfrac{\partial u_k^n}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} A_{ij}^n \dfrac{\partial^2 u_k^n}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^{d} c_0^i \dfrac{\partial u_k^n}{\partial x_i}(t,x) \\
\quad + f_0'\left(u^n(t,x)\right) u_k^n(t,x) + b_0'\left(u^n(t,x)\right) u_k^n(t,x)\nabla u^n(t,x) \\
\quad + b_0\left(u^n(t,x)\right)\nabla u_k^n(t,x), \qquad x \in \boldsymbol{D} \\
\Gamma\left[u_k^n(t,x)\right] + d_0'\left(u^n(t,x)\right) u_k^n(t,x)\nabla u^n(t,x) + d_0\left(u^n(t,x)\right)\nabla u_k^n(t,x) \\
\quad + h_0'\left(u^n(t,x)\right) u_k^n(t,x) = 0, \qquad x \in \partial\boldsymbol{D} \\
u_k^n(0,x) = \dfrac{\partial g}{\partial x}(x).
\end{cases}
\tag{33}
$$

Note that

$$
\int_{\boldsymbol{D}} c_0 \cdot \frac{\partial}{\partial x}(u_k^n(t,x)^2)dx = 0,\ t \text{ a.e, and } \int_{\partial\boldsymbol{D}} \gamma_0 \nabla\left(u_k^n(t,x)\right) u_k^n(t,x) d\varsigma = 0.
$$

From this, multiplying equations of (33) by u_k^n, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\boldsymbol{D}} |u_k^n(t,x)|^2\,dx + \frac{1}{2}\int_{\boldsymbol{D}} \langle A^n \nabla u_k^n(t,x), \nabla u_k^n(t,x)\rangle\,dx \\
&+ \frac{1}{2}\int_{\boldsymbol{D}}\int_{\boldsymbol{\partial D}} \langle \alpha_0 \nabla u_k^n(t,x), \nabla u_k^n(t,x)\rangle\,d\varsigma \\
= &\int_{\boldsymbol{D}} \left[b_0'\left(u^n(t,x)\right)\nabla u^n(t,x)\left(u_k^n(t,x)\right)^2 + b_0\left(u^n(t,x)\right)\nabla u_k^n(t,x) u_k^n(t,x)\right] dx \\
&\int_{\boldsymbol{D}} \left[d_0'\left(u^n(t,x)\right)\nabla u^n(t,x)\left(u_k^n(t,x)\right)^2 + d_0\left(u^n(t,x)\right)\nabla u_k^n(t,x) u_k^n(t,x)\right] d\varsigma \\
&+ \int_{\boldsymbol{D}} f_0'\left(u^n(t,x)\right)\left(u_k^n(t,x)\right)^2 dx + \int_{\partial \boldsymbol{D}} h_0'\left(u^n(t,x)\right)\left(u_k^n(t,x)\right)^2 d\varsigma.
\end{aligned}$$

Remark 3.2.

- $\left\langle b_0'\left(u^n(t,x)\right), \nabla u^n(t,x)\right\rangle \left(u_k^n(t,x)\right)^2 = -b_0'\left(u^n(t,x)\right) u^n(t,x)(\nabla u_k^n) u_k^n(t,x)$
- $h_0'\left(u^n(t,x)\right)\left(u_k^n(t,x)\right)^2 \leq \beta\,|u^n(t,x)|\,|u_k^n(t,x)|^2 \leq \beta'\,|u_k^n(t,x)|^2 \quad \left(\beta' < 0\right)$
- $f_0'\left(u^n(t,x)\right)\left(u_k^n(t,x)\right)^2 \leq \mu\,|u^n(t,x)|\,|u_k^n(t,x)|^2 \leq \mu'\,|u_k^n(t,x)|^2.$

Thereafter,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\boldsymbol{D}} |u_k^n(t,x)|^2\,dx + \frac{1}{2}\int_{\boldsymbol{D}} \langle A^n \nabla u_k^n(t,x), \nabla u_k^n(t,x)\rangle\,dx \\
&+ \frac{1}{2}\int_{\partial \boldsymbol{D}} \langle \alpha_0 \nabla u_k^n(t,x), \nabla u_k^n(t,x)\rangle\,d\varsigma \\
&\leq \mu' \int_{\boldsymbol{D}} |u_k^n(t,x)|^2\,dx + K\int_{\boldsymbol{D}} \left\|\sqrt{A^n}\nabla u_k^n(t,x)\right\| |u_k^n(t,x)|\,dx \\
&\leq \left(\mu' + \frac{K^2}{2\delta}\right)\int_{\boldsymbol{D}} |u_k^n(t,x)|^2\,dx \\
&\quad + \frac{\delta}{2}\left(\int_{\boldsymbol{D}} \langle A^n \nabla u_k^n, \nabla u_k^n\rangle (t,x) dx + \int_{\boldsymbol{\partial D}} \langle \alpha_0 \nabla u_k^n, \nabla u_k^n\rangle (t,x) d\varsigma\right).
\end{aligned}$$

By an appropriate choice of δ, we have using Gronwall's lemma

$$\int_{\boldsymbol{D}} |u_k^n(t,x)|^2\,dx \leq K\mathrm{e}^{Kt}.$$

We have proved that u^n is bounded in $L^\infty\left([0,T];H_0^1(\overline{\boldsymbol{D}})\right)$, and also that each u_k^n is bounded $L^2\left([0,T];H_{a_0}(\boldsymbol{D})\right)$.

$*$ *Step 3*:

Let us show that u^n is a Cauchy sequence in $L^2\left([0,T];H_{a_0}(\boldsymbol{D})\right)$

$$\begin{aligned}\frac{\partial\left(u^n-u^m\right)}{\partial t}(t,x) &= \frac{1}{2}\sum_{i,j=1}^{d}(a_0)_{ij}\frac{\partial^2\left(u^n-u^m\right)}{\partial x_i\partial x_j}(t,x)\\ &+\frac{1}{2n}\sum_{i,j=1}^{d}\frac{\partial^2 u^n}{\partial x_i\partial x_j}(t,x)\\ &-\frac{1}{2m}\sum_{i,j=1}^{d}\frac{\partial^2 u^m}{\partial x_i\partial x_j}(t,x)+\sum_{i=1}^{d}c_0^i\frac{\partial\left(u^n-u^m\right)}{\partial x_i}(t,x)\\ &+f_0\left(u^n(t,x)\right)-f_0\left(u^m(t,x)\right)+b_0\left(u^n(t,x)\right)\nabla u^n(t,x)\\ &-b_0\left(u^m(t,x)\right)\nabla u^m(t,x)+\Gamma\left[\left(u^n-u^m\right)(t,x)\right]\\ &+h_0\left(u^n(t,x)\right)-h_0\left(u^m(t,x)\right)+d_0\left(u^m(t,x)\right)\nabla u^m(t,x)\\ &-d_0\left(u^m(t,x)\right)\nabla u^m(t,x).\end{aligned}$$

Then by multiplying this equation by (u^n-u^m), we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|u^n-u^m\right\|^2(t)+\frac{1}{2}\int_{\boldsymbol{D}}\left\langle a_0\nabla\left(u^n-u^m\right),\nabla\left(u^n-u^m\right)\right\rangle(t,x)dx\\ &\frac{1}{2}\int_{\boldsymbol{D}}\left\langle\left(\frac{1}{n}\nabla u^n-\frac{1}{m}\nabla u^m\right),\nabla\left(u^n-u^m\right)\right\rangle(t,x)dx\\ &=\frac{1}{2}\int_{\boldsymbol{D}}\sum_{i=1}^{d}c_0^i\frac{\partial\left[\left(u^n-u^m\right)^2\right]}{\partial x_i}(t,x)dx\\ &+\int_{\boldsymbol{D}}\left\langle f_0\left(u^n\right)-f_0\left(u^m\right),\left(u^n-u^m\right)\right\rangle(t,x)dx\\ &\quad+\int_{\boldsymbol{D}}\left\langle b_0\left(u^n\right)\nabla u^n-b_0\left(u^m\right)\nabla u^m,\left(u^n-u^m\right)\right\rangle(t,x)dx\\ &+\int_{\partial\boldsymbol{D}}\left(\Gamma\left[\left(u^n-u^m\right)\right]\left(u^n-u^m\right)+\left\langle h_0\left(u^n\right)-h_0\left(u^m\right),\left(u^n-u^m\right)\right\rangle\right)(t,x)d\varsigma\\ &+\int_{\partial\boldsymbol{D}}\left\langle d_0\left(u^n\right)\nabla u^n-d_0\left(u^m\right)\nabla u^m,\left(u^n-u^m\right)\right\rangle(t,x)d\varsigma.\end{aligned}$$

Observe that

$$\int_{\boldsymbol{D}}c_0\cdot\frac{\partial}{\partial x}\left[\left(u^n-u^m\right)^2\right]dx=0\quad t\text{ a.e.},$$

and

$$\int_{\partial \boldsymbol{D}} \gamma_0 \nabla \left[(u^n - u^m) \right] (u^n - u^m) (t, x) d\varsigma = 0.$$

And integrating with respect to t, we have

$$\begin{aligned}
&\frac{1}{2} \|u^n - u^m\|^2 (t) + \frac{1}{2} \int_0^t \int_{\boldsymbol{D}} \langle a_0 \nabla (u^n - u^m), \nabla (u^n - u^m) \rangle (s, x) dx ds \\
&+ \frac{1}{2} \int_0^t \int_{\boldsymbol{D}} \langle \alpha_0 \nabla (u^n - u^m), \nabla (u^n - u^m) \rangle (s, x) d\varsigma d\varphi_s \\
&+ \frac{1}{2} \int_0^t \int_{\boldsymbol{D}} \left\langle \left(\frac{1}{n} \nabla u^n - \frac{1}{m} \nabla u^m \right), \nabla (u^n - u^m) \right\rangle (s, x) dx ds \\
&= \quad \int_0^t \int_{\boldsymbol{D}} \langle f_0 (u^n) - f_0 (u^m), (u^n - u^m) \rangle (s, x) dx ds \\
&\quad + \int_0^t \int_{\boldsymbol{D}} \langle b_0 (u^n) \nabla u^n - b_0 (u^m) \nabla u^m, (u^n - u^m) \rangle (s, x) dx ds \\
&\quad + \int_0^t \int_{\partial \boldsymbol{D}} \langle d_0 (u^n) \nabla u^n - d_0 (u^m) \nabla u^m, (u^n - u^m) \rangle (s, x) d\varsigma d\varphi_s \\
&\quad + \int_0^t \int_{\partial \boldsymbol{D}} \langle h_0 (u^n) - h_0 (u^m), (u^n - u^m) \rangle (s, x) d\varsigma d\varphi_s.
\end{aligned}$$

Since ∇u^n and ∇u^m are bounded in $L^2\big([0,T]; \overline{\boldsymbol{D}}\big)^d$,

$$\frac{1}{2} \int_0^T \int_{\boldsymbol{D}} \left\langle \left(\frac{1}{n} \nabla u^n - \frac{1}{m} \nabla u^m \right), \nabla (u^n - u^m) \right\rangle (t, x) dx dt$$

tends to zero whenever n and m tend to infinity.

For $\varepsilon > 0$, there exists $N_\varepsilon \in N$ such that for $n, m \geq N_\varepsilon$, all $\delta > 0$:

$$\begin{aligned}
&\frac{1-\delta}{2} \int_0^t \int_{\boldsymbol{]0,t[\times D}} \langle a_0 (\nabla u^n - \nabla u^m), (\nabla u^n - \nabla u^m) \rangle (s, x) dx ds \\
&+ \frac{1}{2} \|u^n - u^m\|^2 (t) \\
&\quad + \frac{1-\delta}{2} \int_0^t \int_{\partial \boldsymbol{D}} \langle \alpha_0 (\nabla u^n - \nabla u^m), (\nabla u^n - \nabla u^m) \rangle (s, x) d\varsigma d\varphi_s \\
&\qquad \leq \varepsilon + \left(\mu' + \frac{K^2}{2\delta} \right) \int_0^t \int_{\boldsymbol{D}} \|u^n - u^m\|^2 (s, x) dx ds.
\end{aligned}$$

Hence choosing $\delta = \frac{1}{2}$ and exploiting Gronwall's lemma, we have

$$\begin{aligned}
&\frac{1}{2}\|u^n - u^m\|^2(t) + \frac{1}{4}\int_0^t \int_{\boldsymbol{D}} \langle a_0(\nabla u^n - \nabla u^m), (\nabla u^n - \nabla u^m)\rangle (s,x)dxds \\
&\qquad + \frac{1}{4}\int_0^t \int_{\partial \boldsymbol{D}} \langle \alpha_0(\nabla u^n - \nabla u^m), (\nabla u^n - \nabla u^m)\rangle (s,x)d\varsigma d\varphi_s \\
&\qquad\qquad \leq \varepsilon \mathrm{e}^{Kt}, \quad \forall n, m \geq N_\varepsilon, \quad t \in [0,T].
\end{aligned}$$

There follows that u^n is a Cauchy sequence in $L^2([0,T]; H_{a_0}(\boldsymbol{D}))$, and there exists $u \in L^2([0,T]; H_{a_0}(\boldsymbol{D}))$ such that

$$u^n \longrightarrow u \quad \text{in} \quad L^2([0,T]; H_{a_0}(\boldsymbol{D})).$$

Moreover, since

$$\begin{aligned}
&\int_0^T \int_{\boldsymbol{D}} \langle f_0(u^n) - f_0(u), (u^n - u)\rangle (t,x)dxdt \\
&\quad + \int_0^T \int_{\boldsymbol{D}} \langle b_0(u^n)\nabla u^n - b_0(u)\nabla u, (u^n - u)\rangle (t,x)dxdt \\
&\leq K \int_0^T \int_{\boldsymbol{D}} \Big\{ \|u^n - u\|^2(t,x) + \langle a_0(\nabla u^n - \nabla u), \nabla u^n - \nabla u\rangle (s,x) \Big\} dxdt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \int_{\partial \boldsymbol{D}} |\langle h_0(u^n) - h_0(u), (u^n - u)\rangle| \, d\varsigma d\varphi_t \\
&\quad + \int_0^T \int_{\partial \boldsymbol{D}} \langle d_0(u^n)\nabla u^n - d_0(u)\nabla u, (u^n - u)\rangle (t,x)d\varsigma d\varphi_t \\
&\leq K' \int_0^T \int_{\partial \boldsymbol{D}} \Big\{ \|u^n - u\|^2 + \langle \alpha_0(\nabla u^n - \nabla u), (\nabla u^n - \nabla u)\rangle (t,x) \Big\} d\varsigma d\varphi_t.
\end{aligned}$$

Then,

$$\begin{aligned}
&f_0(u^n) + b_0(u^n)\nabla u^n \longrightarrow f_0(u) + b_0(u)\nabla u \quad \text{in} \quad L^2\big([0,T]; H_{a_0}(\boldsymbol{D})\big) \\
&h_0(u^n) + d_0(u^n)\nabla u^n \longrightarrow h_0(u) + d_0(u)\nabla u \quad \text{in} \quad L^2\big([0,T]; \partial \boldsymbol{D}\big).
\end{aligned}$$

Moreover the sequence $\{u^n\}$ is bounded in $F(\boldsymbol{D})$, hence u is in $F(\boldsymbol{D})$.

By similar arguments, one can easy show the uniqueness of the solution u in $F(\boldsymbol{D})$. □

Remark 3.3. We can drop the hypothesis that the matrix α_0 is degenerate without changing the conclusions.

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