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INVERSE PROBLEM STABILITY OF A CONTINUOUS-IN-TIME FINANCIAL MODEL

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Abstract: In this work, we study the inverse problem stability of the continuous-in-time model which is designed to be used for the finances of public institutions. We discuss this study with determining the Loan Measure from Algebraic Spending Measure in Radon measure space $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max}])$, and in Hilbert space $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$ when they are density measures. For this inverse problem we prove the uniqueness theorem, obtain a procedure for constructing the solution and provide necessary and sufficient conditions for the solvability of the inverse problem in $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$.

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1. Introduction

In the last two decades, the theory and practice of inverse problems have been developed in many scientific domains. Consequently, it is rapidly growing, if not exploding. Moreover, document [4] shows how much researchers contribute to this field. Many inverse problems arising in scientific domains present numerical

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instability: the noise affecting the data may produce arbitrarily large errors in the solutions. In other words, these problems are ill-posed in the sense of Hadamard. The concept of ill-posedness was introduced by Hadamard [2] in the field of partial differential equations. We mention the book on the mathematics of ill-posed problems by Tikhonov and Arsenin, [3].

We constructed in previous work [8] the continuous-in-time model which is based on using the mathematical tools such convolution and integration. Indeed, this model uses measures over time interval to describe loan scheme, reimbursement scheme and interest payment scheme. The model contains some financial quantities. For instance, the Repayment Pattern Measure $\tilde{\gamma}$ is a non-negative measure with total mass which equals 1, the Algebraic Spending Measure $\tilde{\sigma}$ is defined such that the difference between spendings and incomes required to satisfy the current needs between times t_1 and t_2 is:

$$\int_{t_1}^{t_2} \tilde{\sigma}, \tag{1}$$

and the Loan Measure $\tilde{\kappa}_E$ is defined such that the amount borrowed between times t_1 and t_2 is:

$$\int_{t_1}^{t_2} \tilde{\kappa}_E. \tag{2}$$

The measure $\tilde{\gamma}$ is absolutely continuous with respect to the Lebesgue measure dt. This means that it reads $\gamma(t)dt$, where t is the variable in $\mathbb{R}$. The work [8] proposes the resolution of the inverse problem over the space of square-integrable functions when density γ is equal to $\frac{1}{\Theta_\gamma}$ over time period $[0, \Theta_\gamma]$ and to 0 elsewhere. In the paper [18], we use a mathematical framework to discuss an inverse problem of determining the Loan Measure $\tilde{\kappa}_E$ from Algebraic Spending Measure $\tilde{\sigma}$. This inverse problem is used in [8] on simplified examples in order to show its capability to be used to forecast a financial strategy.

In this paper we provide some general results on this inverse problem for any density γ. We show the stability of the inverse problem. We continue to extend some results of this inverse problem in measure space with proving its stability. In other words, we describe a more complete numerical study for the inverse problem. The main result of this paper is the existence and uniqueness of solutions for the system modeling the financial multiyear planning. This result ensures the mathematical well-posedness under the balanced equation assumption.

The paper is organized as follows. Section 2 describes the inverse problem of the model in $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$. Section 3 shows the inverse problem of the model in $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max}])$.

2. Inverse Problem of the Model in $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$

Denote by $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$ the space of square-integrable functions over $\mathbb{R}$ having their support in $[t_{\mathtt{I}}, \Theta_{\max}]$ and by $\mathbb{L}^2([0, \Theta_\gamma])$ the space of square-integrable functions over $\mathbb{R}$ having their support in $[0, \Theta_\gamma]$. We state the Repayment Pattern Density γ as follows:

$$\gamma \in \mathbb{L}^2([0, \Theta_\gamma]), \tag{3}$$

where Θ_γ is a positive number such that:

$$\Theta_\gamma < \Theta_{\max} - t_{\mathtt{I}}. \tag{4}$$

We justify relation (3) because the support of convolution of two compactly supported densities κ_E in $[t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma]$ and γ in $[0, \Theta_\gamma]$ is included in $[t_{\mathtt{I}}, \Theta_{\max}]$.

Lemma 1. *The linear operator $\mathcal{L}$ acting on Loan Density $\kappa_E \in \mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])$ and defined as:*

$$\mathcal{L}[\kappa_E](t) = \kappa_E(t) - (\kappa_E \star \gamma)(t) - \alpha \int_{t_{\mathtt{I}}}^{t} (\kappa_E - \kappa_E \star \gamma)(s)\, ds, \tag{5}$$

is compact operator from $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])$ to $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$.

Proof. The definition (5) of the operator $\mathcal{L}$ gives that for any two Loan Densities κ_{E_1} and κ_{E_2} the following equality:

$$\begin{aligned} \mathcal{L}[\kappa_{E_2}](t) - \mathcal{L}[\kappa_{E_1}](t) &= \kappa_{E_2}(t) - \kappa_{E_1}(t) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(t) \\ &- \alpha \int_{t_{\mathtt{I}}}^{t} \kappa_{E_2}(s) - \kappa_{E_1}(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s)\, ds. \end{aligned} \tag{6}$$

Taking norm $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$ and applying triangle inequality to relation (6), we obtain the following inequality:

$$\begin{aligned} &\|\mathcal{L}[\kappa_{E_2}] - \mathcal{L}[\kappa_{E_1}]\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])} \\ &\leq \|\kappa_{E_2} - \kappa_{E_1} - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])} \\ &+ \mid \alpha \mid \times C_{E_1^2}, \end{aligned} \tag{7}$$

where $C_{E_1^2}$ is defined and is increased as follows:

$$\begin{aligned} C_{E_1^2} &= \left\| \int_{t_{\mathtt{I}}}^{t} (\kappa_{E_2} - \kappa_{E_1})(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s)\, ds \right\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])} \\ &= \sqrt{\int_{t_{\mathtt{I}}}^{\Theta_{\max}} \left(\int_{t_{\mathtt{I}}}^{t} (\kappa_{E_2} - \kappa_{E_1})(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s)\, ds \right)^2 dt} \\ &\leq \sqrt{\Theta_{\max} - t_{\mathtt{I}}} \times \|(\kappa_{E_2} - \kappa_{E_1}) - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^1([t_{\mathtt{I}}, \Theta_{\max}])}. \end{aligned} \tag{8}$$

We use the Cauchy-Schwartz inequality to obtain:

$$\begin{aligned} &\|(\kappa_{E_2} - \kappa_{E_1}) - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^1([t_{\mathtt{I}}, \Theta_{\max}])} \\ &\quad = \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \mid (\kappa_{E_2} - \kappa_{E_1} - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma))(s) \mid \; ds \\ &\quad \leq \sqrt{\int_{t_{\mathtt{I}}}^{\Theta_{\max}} 1^2\, ds} \times \\ &\quad \sqrt{\int_{t_{\mathtt{I}}}^{\Theta_{\max}} \left((\kappa_{E_2} - \kappa_{E_1})(s) - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)(s) \right)^2 ds} \\ &\quad \leq \sqrt{\Theta_{\max} - t_{\mathtt{I}}} \; \times \\ &\quad \|(\kappa_{E_2} - \kappa_{E_1}) - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])}. \end{aligned} \tag{9}$$

Thanks to the properties (8) and (9), we get:

$$C_{E_1^2} \leq (\Theta_{\max} - t_{\mathtt{I}}) \times \|(\kappa_{E_2} - \kappa_{E_1}) - (\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])}. \tag{10}$$

From this and according to relation (7), we get:

$$\begin{aligned} \|\mathcal{L}[\kappa_{E_2}] - \mathcal{L}[\kappa_{E_1}]\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])} &\leq (1 + \mid \alpha \mid \times (\Theta_{\max} - t_{\mathtt{I}})) \\ &\times \|\kappa_{E_2} - \kappa_{E_1} - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])}. \end{aligned} \tag{11}$$

The triangle and the Young inequalities imply that:

$$\begin{aligned}
&\|\kappa_{E_2} - \kappa_{E_1} - ((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])} \\
&\leq \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])} \\
&+ \|((\kappa_{E_2} - \kappa_{E_1}) \star \gamma)\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])} \\
&\leq \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])} \\
&+ \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])} \times \|\gamma\|_{\mathbb{L}^1([0,\Theta_\gamma])} \\
&\leq (1 + \|\gamma\|_{\mathbb{L}^1([0,\Theta_\gamma])}) \\
&\times \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])}.
\end{aligned} \tag{12}$$

From this and according to relation (11), we get:

$$\begin{aligned}
&\|\mathcal{L}[\kappa_{E_2}] - \mathcal{L}[\kappa_{E_1}]\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])} \\
&\leq (1+ \mid \alpha \mid \times(\Theta_{\max} - t_{\mathtt{I}})) \times (1 + \|\gamma\|_{\mathbb{L}^1([0,\Theta_\gamma])}) \\
&\times \|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])}.
\end{aligned} \tag{13}$$

Consequently, linear operator $\mathcal{L}$ is uniformly bounded and is a Hilbert-Schmidt operator in $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$ of constant $(1+ \mid \alpha \mid \times(\Theta_{\max} - t_{\mathtt{I}})) \times (1+\|\gamma\|_{\mathbb{L}^1([0,\Theta_\gamma])})$, achieving the proof of the lemma. □

Lemma 2. *Linear operator $\mathcal{L}$ given by relation* (5) *is Fredholm operator such that:*

$$codim\ Im(\mathcal{L}) = dim\ Ker(\mathcal{L}) < \infty. \tag{14}$$

Proof. We consider $\mathbf{L}$ the space of continuous linear applications of $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])$ in $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$. We define operator K in $\mathbf{L}$ as an integral operator:

$$\forall \kappa_E \in \mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma]), K[\kappa_E](x) = \int_{t_{\mathtt{I}}}^{\Theta_{\max}} F(x, y)\kappa_E(y)\, dy. \tag{15}$$

We want to show that linear operator $\mathcal{L}$ is a difference between the identity application from space $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])$ to $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$ and the compact operator K given by (15) as the following form:

$$\mathcal{L}[\kappa_E](x) = Id_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma]) \to \mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])}[\kappa_E](x) - K[\kappa_E](x), \tag{16}$$

where, F which defines operator K, is a function in $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma] \times [t_{\mathtt{I}}, \Theta_{\max}])$ to be determined. In what to follows, the definition (5) of linear operator $\mathcal{L}$ is used in order to introduce the indicator function $\mathbb{1}_{\{y\leq x\}}$. It allows to simplify integrals because of using the Fubini-Tonelli theorem to interchange the integrals. These simplifications can be detailed as follows:

$$\begin{aligned}
&\mathcal{L}[\kappa_E](x) \qquad (17)\\
&= \kappa_E(x) - (\kappa_E \star \gamma)(x) - \alpha \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \mathbb{1}_{\{y\leq x\}}(\kappa_E - \kappa_E \star \gamma)(y)\, dy\\
&= \kappa_E(x) - (\kappa_E \star \gamma)(x) - \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \alpha \kappa_E(y) \mathbb{1}_{\{y\leq x\}}\, dy\\
&+ \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \alpha \mathbb{1}_{\{y\leq x\}}(\kappa_E \star \gamma)(y)\, dy\\
&= \kappa_E(x) - \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \kappa_E(y)\gamma(x-y)\, dy - \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \alpha \kappa_E(y) \mathbb{1}_{\{y\leq x\}}\, dy\\
&+ \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \alpha \kappa_E(y) \left(\int_{t_{\mathtt{I}}}^{\Theta_{\max}} \mathbb{1}_{\{t\leq x\}}\gamma(t-y)\, dt \right) dy\\
&= \kappa_E(x) - \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \kappa_E(y) \Bigg(\gamma(x-y) + \alpha \mathbb{1}_{\{y\leq x\}}\\
&- \alpha \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \mathbb{1}_{\{t\leq x\}}\gamma(t-y)\, dt \Bigg)\, dy.
\end{aligned}$$

From this, we get the expression of function F:

$$F(x,y) = \gamma(x-y) + \alpha \mathbb{1}_{\{y\leq x\}} - \alpha \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \mathbb{1}_{\{t\leq x\}}\gamma(t-y)\, dt. \qquad (18)$$

In order to show that function F is square-integrable over $[t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma] \times [t_{\mathtt{I}}, \Theta_{\max}]$, we will show that three functions they are. Indeed, these functions are $(x,y) \to \gamma(x-y)$, $(x,y) \to \mathbb{1}_{\{y\leq x\}}$ and $(x,y) \to \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \mathbb{1}_{\{t\leq x\}}\gamma(t-y)\, dt$. Otherwise, since we have:

$$\begin{aligned}
&\|\mathbb{1}_{\{y\leq x\}}\|^2_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma]\times[t_{\mathtt{I}},\Theta_{\max}])} = \int_{t_{\mathtt{I}}}^{\Theta_{\max}-\Theta_\gamma} \left(\int_{t_{\mathtt{I}}}^{\Theta_{\max}} \mathbb{1}^2_{\{y\leq x\}}\, dy \right) dx \qquad (19)\\
&= \int_{t_{\mathtt{I}}}^{\Theta_{\max}-\Theta_\gamma} (x - t_{\mathtt{I}})\, dx\, \frac{(\Theta_{\max} - \Theta_\gamma - t_{\mathtt{I}})^2}{2},
\end{aligned}$$

we have $(x,y) \to \mathbb{1}_{\{y\leq x\}} \in \mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma] \times [t_{\mathtt{I}}, \Theta_{\max}])$. In what to follows, we will show that function $\int_{t_{\mathtt{I}}}^{\Theta_{\max}} \mathbb{1}_{\{t\leq x\}}\gamma(t-y)\, dt$ is square-integrable over

$[t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma] \times [t_{\mathtt{I}}, \Theta_{\max}]$. For that, we set the following inequality:

$$\left\| \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \mathbb{1}_{\{t \le x\}} \gamma(t-y)\, dt \right\|^2_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}-\Theta_\gamma] \times [t_{\mathtt{I}}, \Theta_{\max}])} \tag{20}$$
$$= \int_{t_{\mathtt{I}}}^{\Theta_{\max}-\Theta_\gamma} \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \left| \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \mathbb{1}_{\{t \le x\}} \gamma(t-y)\, dt \right|^2 dx\, dy$$
$$\le \int_{t_{\mathtt{I}}}^{\Theta_{\max}-\Theta_\gamma} \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \left(\int_{t_{\mathtt{I}}}^{x} |\gamma(t-y)|\, dt \right)^2 dx\, dy.$$

We use the Cauchy-Schwarz inequality to obtain:

$$\forall x \in [t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma], \left(\int_{t_{\mathtt{I}}}^{x} |\gamma(t-y)|\, dt \right)^2 \le (x - t_{\mathtt{I}}) \times \int_{t_{\mathtt{I}}}^{x} |\gamma(t-y)|^2\, dt. \tag{21}$$

Since the density γ satisfies the relation (3), there exists a constant C_γ such that relation (21) gives:

$$\left(\int_{t_{\mathtt{I}}}^{x} |\gamma(t-y)|\, dt \right)^2 \le (x - t_{\mathtt{I}}) \times C_\gamma. \tag{22}$$

According to relations (20) and (22), we get the following inequality:

$$\left\| \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \mathbb{1}_{\{t \le x\}} \gamma(t-y)\, dt \right\|^2_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}-\Theta_\gamma] \times [t_{\mathtt{I}}, \Theta_{\max}])} \le C_\gamma \times (\Theta_{\max} - t_{\mathtt{I}}) \times \frac{(\Theta_{\max} - \Theta_\gamma - t_{\mathtt{I}})^2}{2}, \tag{23}$$

which proves that function $(x, y) \to \int_{t_{\mathtt{I}}}^{\Theta_{\max}} \mathbb{1}_{\{t \le x\}} \gamma(t-y)\, dt$ is square-integrable over the interval $[t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma] \times [t_{\mathtt{I}}, \Theta_{\max}]$. Consequently, we get:

$$F \in \mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma] \times [t_{\mathtt{I}}, \Theta_{\max}]). \tag{24}$$

It is concluded that linear operator $\mathcal{L}$ can be written in the form which is shown in relation (16). We showed that operator $\mathcal{L}$ is Fredholm operator (see Th. 3.45, p. 206 in [5], or the Fredholm Alternative Theorem 1.3.1 in [19], p. 13). In addition, the Fredholm Alternative Theorem (Lemma 4.45, p. 160 in [6]) is used to get that linear operator $\mathcal{L}$ has a finite codimension, a closed image and a finite dimension of its kernel. Thus the lemma is proved. □

Lemma 3. *Linear operator $\mathcal{D}$ acting on Initial Debt Repayment Density $\rho_{\mathcal{K}}^{\mathtt{I}} \in \mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$ defined as:*

$$\mathcal{D}[\rho_{\mathcal{K}}^{\mathtt{I}}](t) = -\alpha \int_t^{\Theta_{\max}} \rho_{\mathcal{K}}^{\mathtt{I}}(s)\, ds - \rho_{\mathcal{K}}^{\mathtt{I}}(t), \tag{25}$$

is compact operator from $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$ to $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$.

Proof. The triangle inequality is applied to definition (25) of operator $\mathcal{D}$ to get:

$$\begin{aligned} \|\mathcal{D}[\rho_{\mathcal{K}}^{\mathtt{I}}]\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])} \leq \mid \alpha \mid \times & \left\| \int_t^{\Theta_{\max}} \rho_{\mathcal{K}}^{\mathtt{I}}(s)\, ds \right\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])} \\ & + \|\rho_{\mathcal{K}}^{\mathtt{I}}\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])}. \end{aligned} \tag{26}$$

Since $\left\| \int_t^{\Theta_{\max}} \rho_{\mathcal{K}}^{\mathtt{I}}(s)\, ds \right\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])} \leq (\Theta_{\max} - t_{\mathtt{I}}) \times \|\rho_{\mathcal{K}}^{\mathtt{I}}\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])}$ and according to relation (26), we have:

$$\|\mathcal{D}[\rho_{\mathcal{K}}^{\mathtt{I}}]\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])} \leq (\mid \alpha \mid \times (\Theta_{\max} - t_{\mathtt{I}}) + 1) \times \|\rho_{\mathcal{K}}^{\mathtt{I}}\|_{\mathbb{L}^2([t_{\mathtt{I}},\Theta_{\max}])}. \tag{27}$$

□

We can decompose the Algebraic Spending Density σ defined in relation (1) as a sum of operators $\mathcal{L}$ and $\mathcal{D}$:

$$\sigma(t) = \mathcal{L}[\kappa_E](t) + \mathcal{D}[\rho_{\mathcal{K}}^{\mathtt{I}}](t). \tag{28}$$

Lemma 4. *The singular point of function $1 - \mathcal{F}(\gamma)$ is zero for any constant and affine density γ.*

Proof. If the Repayment Pattern Density γ is a constant function given by:

$$\gamma(t) = \frac{1}{\Theta_\gamma} \mathbb{1}_{[0,\Theta_\gamma]}(t), \tag{29}$$

we have:

$$\forall \xi \in \mathbb{R}^\star, 1 - \mathcal{F}(\gamma)(\xi) = 1 - \frac{i}{\xi \Theta_\gamma}(e^{-i\xi\Theta_\gamma} - 1). \tag{30}$$

From this, we get:

$$\begin{aligned}\forall \xi \in \mathbb{R}^\star, 1-\mathcal{F}(\gamma)(\xi)=0 \Rightarrow \xi\Theta_\gamma &= \sin(\xi\Theta_\gamma) \text{ and } \cos(\xi\Theta_\gamma)=1,\\ \Rightarrow (\xi\Theta_\gamma)^2+1&=1,\\ \Rightarrow \xi\Theta_\gamma &= 0.\end{aligned} \tag{31}$$

Since the real Θ_γ is positive, the function $\xi \to 1-\mathcal{F}(\gamma)(\xi)$ is not zero function over $\mathbb{R}^\star$. Conversely, if a real ξ is zero, function $\xi \to 1-\mathcal{F}(\gamma)(\xi)$ is also a zero function. Indeed, the Fourier Transform of any density γ at the origin is defined as:

$$\mathcal{F}(\gamma)(0)=\int_{-\infty}^{+\infty}\gamma(t)\,dt. \tag{32}$$

Furthermore, since density γ is with total mass which equals 1, relation (32) implies that:

$$\mathcal{F}(\gamma)(0)=1. \tag{33}$$

Now we will show that the singular point of function $1-\mathcal{F}(\gamma)$ is zero for an affine density γ given by:

$$\gamma(t)=\left(\frac{t}{\Theta_\gamma^2}+\frac{1}{2\Theta_\gamma}\right)\mathbb{1}_{[0,\Theta_\gamma]}(t). \tag{34}$$

The integration by parts states that:

$$\begin{aligned}&\forall \xi \in \mathbb{R}^\star,\quad 1-\mathcal{F}(\gamma)(\xi)\\ &=1-\left(\frac{1}{(\xi\Theta_\gamma)^2}+\frac{i}{2\xi\Theta_\gamma}\right)(e^{-i\xi\Theta_\gamma}-1)-\frac{i}{\xi\Theta_\gamma}e^{-i\xi\Theta_\gamma}.\end{aligned} \tag{35}$$

From this, we get the following system of equations:

$$\begin{cases}(\cos(\xi\Theta_\gamma)-1)+\frac{3\xi\Theta_\gamma}{2}\sin(\xi\Theta_\gamma)=(\xi\Theta_\gamma)^2,\\ -\sin(\xi\Theta_\gamma)+\xi\Theta_\gamma\cos(\xi\Theta_\gamma)+\frac{\xi\Theta_\gamma}{2}(\cos(\xi\Theta_\gamma)-1)=0.\end{cases} \tag{36}$$

From this, we get the following equality:

$$\xi\Theta_\gamma\left(1+\frac{3(\xi\Theta_\gamma)^2}{2}\right)-\left(1+\frac{9(\xi\Theta_\gamma)^2}{4}\right)\sin(\xi\Theta_\gamma)=0. \tag{37}$$

According to Figure 1, we state that equation (37) does not have solution on $\mathbb{R}^\star$. Consequently, the function $\xi \to 1 - \mathcal{F}(\gamma)(\xi)$ is not a zero function over $\mathbb{R}^\star$. Inversely, assuming that real ξ is zero, we get:

$$\begin{aligned} 1 - \mathcal{F}(\gamma)(0) &= 1 - \int_{-\infty}^{+\infty} \gamma \\ &= 1 - \frac{1}{\Theta_\gamma^2} \int_0^{\Theta_\gamma} t\,dt - \frac{1}{2\Theta_\gamma} \int_0^{\Theta_\gamma} dt = 0. \end{aligned} \tag{38}$$

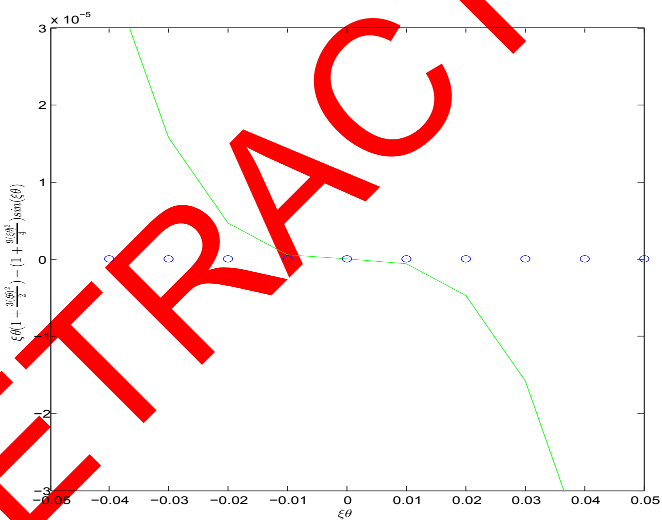

Figure 1: Graph of the function $\xi\Theta_\gamma \to \xi\Theta_\gamma\left(1 + \frac{3(\xi\Theta_\gamma)^2}{2}\right) - \left(1 + \frac{9(\xi\Theta_\gamma)^2}{4}\right)\sin(\xi\Theta_\gamma)$ over interval $[-0.05, 0.05]$. Also shown that zero is its singular point.

We conclude that function $1 - \mathcal{F}(\gamma)$ is zero at the origin for density γ given by (29) (or by (34)). In what to follows, we want to extend this conclusion for any affine density γ given by:

$$\gamma(t) = (c_2 t + c_1)\mathbb{1}_{[0,\Theta_\gamma]}(t), \tag{39}$$

where coefficients c_1 and c_2 satisfy:

$$c_1 + \frac{c_2\Theta_\gamma}{2} = \frac{1}{\Theta_\gamma}. \tag{40}$$

We obtain using the integration by parts:

$$\begin{aligned} \forall \xi \in \mathbb{R}^\star, 1 - \mathcal{F}(\gamma)(\xi) = 1 - c_2\left(\frac{e^{-i\xi\Theta_\gamma} - 1}{\xi^2} + \frac{i\Theta_\gamma e^{-i\xi\Theta_\gamma}}{\xi}\right) \\ - \frac{ic_1(e^{-i\xi\Theta_\gamma} - 1)}{\xi} = 0. \end{aligned} \tag{41}$$

Next, we obtain by separating the real and the imaginary parts of function $1 - \mathcal{F}(\gamma)$ the following system of equations:

$$\begin{cases} 1 - c_2\left(\frac{\cos(\xi\Theta_\gamma)-1}{\xi^2} + \frac{\Theta_\gamma \sin(\xi\Theta_\gamma)}{\xi}\right) - \frac{c_1\sin(\xi\Theta_\gamma)}{\xi} = 0, \\ c_2\left(-\frac{\sin(\xi\Theta_\gamma)}{\xi^2} + \frac{\Theta_\gamma\cos(\xi\Theta_\gamma)}{\xi}\right) + \frac{c_1(\cos(\xi\Theta_\gamma)-1)}{\xi} = 0. \end{cases} \tag{42}$$

Let us check the consistency of the system of equations given by (42). Indeed, if coefficients c_1 and c_2 are respectively equal to $\frac{1}{2\Theta_\gamma}$ and $\frac{1}{\Theta_\gamma^2}$, then we get relation (36). Thanks to (42), we get the following equality:

$$\begin{aligned} &\frac{c_2\Theta_\gamma}{\xi} + \xi\Theta_\gamma + \frac{c_1\xi}{c_2} \\ &= \left(\frac{c_2}{\xi^2} + \Theta_\gamma(c_2\Theta_\gamma + c_1) + \frac{c_1(c_2\Theta_\gamma + c_1)}{c_2}\right)\sin(\xi\Theta_\gamma). \end{aligned} \tag{43}$$

According to relations (42) and (43), we get:

$$\frac{c_2\Theta_\gamma}{\xi} + \xi\Theta_\gamma + \frac{c_1\xi}{c_2} = \left(\frac{c_2}{\xi^2} + 2 + \frac{c_1^2}{c_2}\right)\sin(\xi\Theta_\gamma). \tag{44}$$

We will show that relation (44) is consistent. Indeed, assuming that coefficients c_1 and c_2 are respectively equal to $\frac{1}{2\Theta_\gamma}$ and $\frac{1}{\Theta_\gamma^2}$, we get relation (37). Otherwise, relation (44) gives:

$$\sin(\xi\Theta_\gamma) - \xi\Theta_\gamma\left(\frac{c_2\Theta_\gamma^2 + (\xi\Theta_\gamma)^2(1+\frac{c_1}{c_2\Theta_\gamma})}{c_2\Theta_\gamma^2 + (\xi\Theta_\gamma)^2(2+\frac{c_1^2}{c_2})}\right) = 0. \tag{45}$$

We use equalities (45) and (40) which is multiplied by $\frac{c_1}{c_2}$ to give the following system of equations:

$$\begin{cases} \frac{c_1^2}{c_2} = -1 + \frac{1}{c_2\Theta_\gamma^2} + \frac{c_2\Theta_\gamma^2}{4}, \\ \frac{c_1}{c_2\Theta_\gamma} = \frac{1}{c_2\Theta_\gamma^2} - \frac{1}{2}. \end{cases} \tag{46}$$

Replacing quantities $\frac{c_1^2}{c_2}$ and $\frac{c_1}{c_2\Theta_\gamma}$ defined in relation (46) in equality (45), we obtain the following equality:

$$\sin(\xi\Theta_\gamma) - \xi\Theta_\gamma\left(\frac{c_2\Theta_\gamma^2 + (\xi\Theta_\gamma)^2(\frac{1}{2}+\frac{1}{c_2\Theta_\gamma^2})}{c_2\Theta_\gamma^2 + (\xi\Theta_\gamma)^2(1+\frac{1}{c_2\Theta_\gamma^2}+\frac{c_2\Theta_\gamma^2}{4})}\right) = 0. \tag{47}$$

As

$$\forall\xi \in \mathbb{R}_\star, \left|\frac{\sin(\xi\Theta_\gamma)}{\xi\Theta_\gamma}\right| \leq 1, \tag{48}$$

using inequality (48), we obtain according to (47) the following inequality:

$$\frac{1}{2} + \frac{1}{c_2\Theta_\gamma^2} \leq 1 + \frac{1}{c_2\Theta_\gamma^2} + \frac{c_2\Theta_\gamma^2}{4}, \tag{49}$$

which is simplified to give:

$$-2 \leq c_2\Theta_\gamma^2. \tag{50}$$

Figure 2 shows that equation (47) does not have solution on $\mathbb{R}^\star$. Consequently, the proof of the lemma is done. □

Many inverse problems in finance involve its study with regular singularities. For instance, paper [1] deals with inverting the differential operators on the half-line having a discontinuity in an interior. For studying the inverse problem we agree that together with function $1 - \mathcal{F}(\gamma)$ we suppose that zero is only its singularity for any density γ. This assumption gives a constructive procedure for the validity of the following theorem.

Theorem 5. *If Repayment Pattern γ satisfies relation (3) and following relation:*

$$\exists \epsilon > 0, \quad \frac{1}{1-\mathcal{F}(\gamma)}_{|]-\infty,-\epsilon[\cup]\epsilon,+\infty[} \in \mathbb{L}^{\infty}(\mathbb{R}), \tag{51}$$

where $\mathcal{F}$ stands for the Fourier Transform Operator. And if Initial Debt Repayment Density $\rho_{\mathcal{K}}^{\mathtt{I}}$ is in $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$, then for any Algebraic Spending Density σ in $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$ is satisfying the following equality:

$$\int_{t_{\mathtt{I}}}^{\Theta_{\max}} \left((\sigma - \mathcal{D}[\rho_{\mathcal{K}}^{\mathtt{I}}])(y) + \alpha \int_{t_{\mathtt{I}}}^{y} (\sigma - \mathcal{D}[\rho_{\mathcal{K}}^{\mathtt{I}}])(s) \mathrm{e}^{\alpha(y-s)}\, ds \right) dy = 0, \tag{52}$$

there exists an unique Loan Density κ_E stable in $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])$ which is

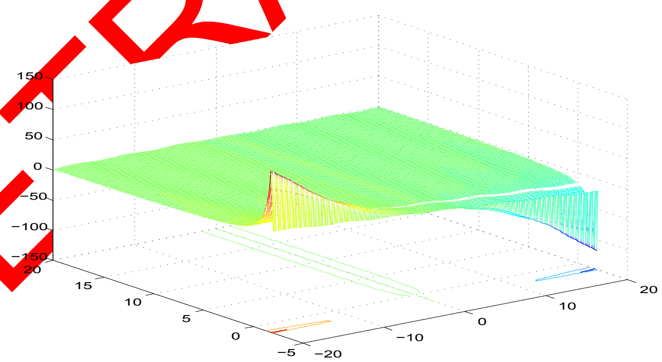

Figure 2: Graph of the function $(\xi\Theta_\gamma, c_2\Theta_\gamma^2) \to \sin(\xi\Theta_\gamma) - \xi\Theta_\gamma \left(\frac{c_2\Theta_\gamma^2 + (\xi\Theta_\gamma)^2(\frac{1}{2} + \frac{1}{c_2\Theta_\gamma^2})}{c_2\Theta_\gamma^2 + (\xi\Theta_\gamma)^2(1 + \frac{1}{c_2\Theta_\gamma^2} + \frac{c_2\Theta_\gamma^2}{4})} \right)$ over interval $[-20, 20] \times [-2, 20]$. Also shown that zero is its singular point.

given in terms of σ by:

$$\kappa_E = \mathcal{F}^{-1}\left(\frac{\mathcal{F}\Big(\sigma - \mathcal{D}[\rho_{\mathcal{K}}^{\mathtt{I}}] + \alpha \displaystyle\int_{t_{\mathtt{I}}}^{\bullet} (\sigma(s) - \mathcal{D}[\rho_{\mathcal{K}}^{\mathtt{I}}](s))\, \mathrm{e}^{\alpha(\bullet - s)}\, ds\Big)}{1 - \mathcal{F}(\gamma)}\right), \tag{53}$$

where $\mathcal{F}^{-1}$ stands for Inverse Fourier Transform, such that (28) *holds.*

Proof of Theorem 5. Lemma 3.5 in [18] shows that operator $\mathcal{L}$ given by relation (5) is a one-to-one application. From this, we obtain the uniqueness of κ_E.

If we assume that our noise (the error between measurement σ_2 and measurement σ_1) is small in the $\mathbb{L}^2$-norm, so that $\|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])} \leq \delta$, and we are happy with a small error in the parameter in the $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$ sense, then there is no problem. The reconstruction will be accurate in the sense that we have the following inequality: $\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])} \leq C_\alpha^\gamma \delta$, where C_α^γ is a real constant to be determined,

$$\kappa_{E_2} - \kappa_{E_1} = \mathcal{F}^{-1}\left(\frac{\mathcal{F}\Big(\sigma_2 - \sigma_1 + \alpha \displaystyle\int_{t_{\mathtt{I}}}^{\bullet} (\sigma_2 - \sigma_1)(s)\, \mathrm{e}^{\alpha(\bullet - s)}\, ds\Big)}{1 - \mathcal{F}(\gamma)}\right). \tag{54}$$

Since the Inverse Fourier Transform $\mathcal{F}^{-1}$ preserves the norm from $\mathbb{L}^2(\mathbb{R})$ to $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$, we obtain from relation (54) the following equality:

$$\begin{aligned} &\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])} \\ &= \left\|\frac{\mathcal{F}\Big(\sigma_2 - \sigma_1 + \alpha \displaystyle\int_{t_{\mathtt{I}}}^{\bullet} (\sigma_2 - \sigma_1)(s)\, \mathrm{e}^{\alpha(\bullet - s)}\, ds\Big)}{1 - \mathcal{F}(\gamma)}\right\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])}. \end{aligned} \tag{55}$$

Under the assumption (51) and the fact that the Fourier Transform $\mathcal{F}$ preserves the norm from $\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])$ to $\mathbb{L}^2(\mathbb{R})$, we have:

$$\begin{aligned} &\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])} \\ &\leq \sup_{\xi \in \mathbb{R}^\star} \left\{\frac{1}{\mid 1 - \mathcal{F}(\gamma)(\xi) \mid}\right\} \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])} \\ &\mid \alpha \mid \times \left\| \int_{t_{\mathtt{I}}}^{\bullet} (\sigma_2 - \sigma_1)(s)\ + \mathrm{e}^{\alpha(\bullet - s)}\, ds \right\|_{\mathbb{L}^2([t_{\mathtt{I}}, \Theta_{\max}])}. \end{aligned} \tag{56}$$

We want to increase quantity $\left\| \int_{t_I}^{\bullet} (\sigma_2 - \sigma_1)(s)\, e^{\alpha(\bullet - s)}\, ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}$ by a constant to be determined multiplied by $\|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}$:

$$\begin{aligned}
&\left\| \int_{t_I}^{\bullet} (\sigma_2 - \sigma_1)(s)\, e^{\alpha(\bullet - s)}\, ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \\
&= \sqrt{\int_{t_I}^{\Theta_{\max}} \left(\int_{t_I}^{t} (\sigma_2 - \sigma_1)(s)\, e^{\alpha(t-s)}\, ds \right)^2 dt} \\
&\le \sqrt{\int_{t_I}^{\Theta_{\max}} \left(\int_{t_I}^{\Theta_{\max}} (\sigma_2 - \sigma_1)(s)\, e^{\alpha(\Theta_{\max}-s)}\, ds \right)^2 dt} \\
&\le \sqrt{\Theta_{\max} - t_I} \times \sup_{s \in [t_I, \Theta_{\max}]} \{ e^{\alpha(\Theta_{\max}-s)} \} \\
&\times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^1([t_I, \Theta_{\max}])} \\
&\le \sqrt{\Theta_{\max} - t_I} \times e^{\alpha(\Theta_{\max}-t_I)} \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^1([t_I, \Theta_{\max}])}.
\end{aligned} \tag{57}$$

Using the Cauchy-Schwarz inequality:

$$\|\sigma_2 - \sigma_1\|_{\mathbb{L}^1([t_I, \Theta_{\max}])} \le \sqrt{\Theta_{\max} - t_I} \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}, \tag{58}$$

and according to relations (56) and (57), we get:

$$\begin{aligned}
&\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} \\
&\le \left(\sup_{\xi \in \mathbb{R}^\star} \left\{ \frac{1}{| 1 - \mathcal{F}(\gamma)(\xi) |} \right\} + | \alpha | \times (\Theta_{\max} - t_I) \times e^{\alpha(\Theta_{\max} - t_I)} \right) \\
&\times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}.
\end{aligned} \tag{59}$$

If we set:

$$C_\alpha^\gamma = \sup_{\xi \in \mathbb{R}^\star} \left\{ \frac{1}{| 1 - \mathcal{F}(\gamma)(\xi) |} \right\} + | \alpha | \times (\Theta_{\max} - t_I) \times e^{\alpha(\Theta_{\max} - t_I)}, \tag{60}$$

we get:

$$\|\kappa_{E_2} - \kappa_{E_1}\|_{\mathbb{L}^2([t_I, \Theta_{\max} - \Theta_\gamma])} \le C_\alpha^\gamma \times \|\sigma_2 - \sigma_1\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \tag{61}$$

□

3. Inverse Problem of the Model in $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max}])$

The aim of this section is to study the inverse problem in measure space. Denote by $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max}])$ the Radon measure space which is a continuous and linear form acting on continuous functions space $\mathcal{C}_c^o([t_{\mathtt{I}}, \Theta_{\max}])$ defined over a time interval $[t_{\mathtt{I}}, \Theta_{\max}]$. The usual norm on $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max}])$ is:

$$\|\mu\|_{\mathcal{M}((t_{\mathtt{I}},\Theta_{\max}))} = \sup_{\psi\in\mathcal{C}_c^o([t_{\mathtt{I}},\Theta_{\max}]),\psi\neq 0} \left\{ \frac{|\langle \mu, \psi\rangle|}{\|\psi\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}])}} \right\}, \tag{62}$$

where $\|\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}])}$ is the usual norm on $\mathcal{C}_c^o([t_{\mathtt{I}}, \Theta_{\max}])$ defined as:

$$\|\psi\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}])} = \sup_{t\in[t_{\mathtt{I}},\Theta_{\max}]} \{|\psi(t)|\}. \tag{63}$$

We set the Repayment Pattern Measure $\tilde{\gamma}$ such that:

$$\tilde{\gamma} \in \mathcal{M}([0, \Theta_\gamma]), \tag{64}$$

where Θ_γ is positive number satisfying relation (4). By relation (64), the support of convolution of two compactly supported measures $\tilde{\kappa}_E$ in $[t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma]$ and $\tilde{\gamma}$ in $[0, \Theta_\gamma]$ is included in $[t_{\mathtt{I}}, \Theta_{\max}]$. Indeed, formally:

$$Supp(\tilde{\kappa}_E \star \tilde{\gamma}) \subset Supp(\tilde{\kappa}_E) + Supp(\tilde{\gamma}). \tag{65}$$

Let $\mathcal{L}_1$ be a linear operator defined from $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])$ to $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max}])$ acting on Loan Measure $\tilde{\kappa}_E$ by

$$\mathcal{L}_1[\tilde{\kappa}_E] = \tilde{\kappa}_E - \tilde{\kappa}_E \star \tilde{\gamma} - \alpha\Big\langle \tilde{\kappa}_E - \tilde{\kappa}_E \star \tilde{\gamma}, \mathbb{1}_{|[t_{\mathtt{I}},t[}\Big\rangle dt. \tag{66}$$

Let $\mathcal{D}_1$ be an operator defined in $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max}])$ acting on Initial Debt Repayment Measure $\tilde{\rho}_{\mathcal{K}}^{\mathtt{I}}$ by

$$\mathcal{D}_1[\tilde{\rho}_{\mathcal{K}}^{\mathtt{I}}] = -\alpha\Big\langle \tilde{\rho}_{\mathcal{K}}^{\mathtt{I}}, \mathbb{1}_{|[t,\Theta_{\max}[}\Big\rangle dt - \tilde{\rho}_{\mathcal{K}}^{\mathtt{I}}. \tag{67}$$

Algebraic Spending Measure $\tilde{\sigma}$ is defined such that the difference between spendings and incomes required to satisfy the current needs. Measure $\tilde{\sigma}$ is decomposed as a sum of operators $\mathcal{L}_1$ and $\mathcal{D}_1$ given by relations (66) and (67), respectively:

$$\tilde{\sigma} = \mathcal{L}_1[\tilde{\kappa}_E] + \mathcal{D}_1[\tilde{\rho}_{\mathcal{K}}^{\mathtt{I}}]. \tag{68}$$

Theorem 6. *If Repayment Pattern Measure $\tilde{\gamma}$ is satisfying relation* (64) *and the following relation*

$$\frac{1}{1-\mathcal{F}(\tilde{\gamma})}_{|]-\infty,-\epsilon[\cup]\epsilon,+\infty[} \in \mathbb{L}^{\infty}(\mathbb{R}), \tag{69}$$

for any positive real ϵ and if Loan Measure $\tilde{\kappa}_E$ exists in $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])$ for any Initial Debt Repayment Measure $\tilde{\rho}_{\mathcal{K}}^{\mathtt{I}}$ and for any Algebraic Spending Measure $\tilde{\sigma}$ in $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max}])$ satisfying the following equality:

$$\tilde{\kappa}_E = \mathcal{F}^{-1}\left(\frac{\mathcal{F}\Big(\tilde{\sigma} - \mathcal{D}_1[\tilde{\rho}_{\mathcal{K}}^{\mathtt{I}}] + \alpha\tilde{\mathrm{e}}_\alpha\Big\langle \tilde{\sigma} - \mathcal{D}_1[\tilde{\rho}_{\mathcal{K}}^{\mathtt{I}}], \mathrm{e}_{-\alpha|[t_{\mathtt{I}},t[}\Big\rangle\Big)}{1-\mathcal{F}(\tilde{\gamma})}\right), \tag{70}$$

then, the Loan Measure $\tilde{\kappa}_E$ is unique and stable in space $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])$.

Proof. We notice that since that the inverse Fourier transform $\mathcal{F}^{-1}$ is not surjective from $\mathcal{M}(\mathbb{R})$ to $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max}])$, the solution $\tilde{\kappa}_E$ does not exist for the inverse problem. Moreover, if Loan Measure $\tilde{\kappa}_E$ is supposed satisfying relation (70), then it is unique due to the injectivity of operator $\mathcal{L}_1$ (see Lemma 3.4 in [18]).

Now we will show that the solution Loan Measure $\tilde{\kappa}_E$ is stable. Definition (66) of operator $\mathcal{L}_1$ gives that for any two Loan Densities $\tilde{\kappa}_{E_1}$ and $\tilde{\kappa}_{E_2}$ the following equality:

$$\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1} = \mathcal{F}^{-1}\left(\frac{\mathcal{F}\Big(\tilde{\sigma}_2 - \tilde{\sigma}_1 + \alpha\tilde{\mathrm{e}}_\alpha\Big\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, \mathrm{e}_{-\alpha|[t_{\mathtt{I}},t[}\Big\rangle\Big)}{1-\mathcal{F}(\tilde{\gamma})}\right). \tag{71}$$

The usual Radon norm on $\mathcal{M}([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])$ of quantity $\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}$ is defined by:

$$\begin{aligned} &\|\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}\|_{\mathcal{M}([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])} \\ &\quad = \sup_{\phi\in\mathcal{C}_c^o([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma]),\phi\neq 0} \left\{\frac{|\langle \tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}, \phi\rangle|}{\|\phi\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])}}\right\}. \end{aligned} \tag{72}$$

We replace measure $\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}$ given by (71) in right equality (72) in order to obtain the following equality:

$$\begin{aligned} \|\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}\|_{\mathcal{M}([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])} &= \frac{1}{|1-\mathcal{F}(\tilde{\gamma})|}\times \\ \sup_{\phi\in\mathcal{C}_c^o([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma]),\phi\neq 0} &\left\{\frac{|\langle \tilde{\sigma}_2 - \tilde{\sigma}_1 + \alpha\tilde{\mathrm{e}}_\alpha\Big\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, \mathrm{e}_{-\alpha|[t_{\mathtt{I}},t[}\Big\rangle, \mathcal{F}(\phi)\rangle|}{\|\phi\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])}}\right\}. \end{aligned} \tag{73}$$

By the Fourier Transform of function $\phi \in \mathcal{C}_c^o([t_{\mathtt{I}}, \Theta_{\max} - \Theta_\gamma])$, we get the following inequality:

$$
\begin{aligned}
\mid \mathcal{F}(\phi)(\xi) \mid &= \left| \int_{t_{\mathtt{I}}}^{\Theta_{\max}-\Theta_\gamma} \phi(x) e^{-ix\xi}\, dx \right| \\
&\leq \int_{t_{\mathtt{I}}}^{\Theta_{\max}-\Theta_\gamma} |e^{-ix\xi}|\, dx \times \|\phi\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])} \\
&\leq (\Theta_{\max} - \Theta_\gamma - t_{\mathtt{I}}) \times \|\phi\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])}.
\end{aligned}
\tag{74}
$$

Using relations (73) and (74), we obtain:

$$
\begin{aligned}
&\|\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}\|_{\mathcal{M}([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])} \\
&\leq (\Theta_{\max} - \Theta_\gamma - t_{\mathtt{I}}) \times \sup_{\xi \in \mathbb{R}^\star} \left\{ \frac{1}{\mid 1 - \mathcal{F}(\tilde{\gamma})(\xi) \mid} \right\} \times \\
&\sup_{\phi \in \mathcal{C}_c^o([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma]), \phi \neq 0} \left\{ \frac{\mid \langle \tilde{\sigma}_2 - \tilde{\sigma}_1 + \alpha \tilde{e}_\alpha \left\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, \mathrm{e}_{-\alpha|[t_{\mathtt{I}},t[} \right\rangle, \mathcal{F}(\phi) \rangle \mid}{\|\mathcal{F}(\phi)\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])}} \right\}.
\end{aligned}
\tag{75}
$$

Since we have:

$$
\begin{aligned}
&\mid \left\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, \mathrm{e}_{-\alpha|[t_{\mathtt{I}},t[} \right\rangle \mid \\
&= \frac{\mid \left\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, \mathrm{e}_{-\alpha|[t_{\mathtt{I}},t[} \right\rangle \mid}{\|\mathrm{e}_{-\alpha|[t_{\mathtt{I}},t[}\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])}} \times \|\mathrm{e}_{-\alpha|[t_{\mathtt{I}},t[}\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])} \\
&\leq \|\tilde{\sigma}_2 - \tilde{\sigma}_1\|_{\mathcal{M}([t_{\mathtt{I}},\Theta_{\max}])} \times \|\mathrm{e}_{-\alpha|[t_{\mathtt{I}},t[}\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])} \\
&\leq \|\tilde{\sigma}_2 - \tilde{\sigma}_1\|_{\mathcal{M}([t_{\mathtt{I}},\Theta_{\max}])} \times \mathrm{e}^{|-\alpha| t_{\mathtt{I}}},
\end{aligned}
\tag{76}
$$

we obtain:

$$
\begin{aligned}
&\mid \alpha \tilde{e}_\alpha \left\langle \tilde{\sigma}_2 - \tilde{\sigma}_1, \mathrm{e}_{-\alpha|[t_{\mathtt{I}},t[} \right\rangle, \mathcal{F}(\phi) \rangle \mid \\
&\leq \mid \alpha \mid \times (\Theta_{\max} - t_{\mathtt{I}}) \times \mathrm{e}^{|\alpha|(\Theta_{\max} - t_{\mathtt{I}})} \\
&\times \|\tilde{\sigma}_2 - \tilde{\sigma}_1\|_{\mathcal{M}([t_{\mathtt{I}},\Theta_{\max}])} \times \|\mathcal{F}(\phi)\|_{L^\infty([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])}.
\end{aligned}
\tag{77}
$$

According to relations (75) and (77), we obtain the following inequality:

$$
\begin{aligned}
&\|\tilde{\kappa}_{E_2} - \tilde{\kappa}_{E_1}\|_{\mathcal{M}([t_{\mathtt{I}},\Theta_{\max}-\Theta_\gamma])} \\
&\leq (\Theta_{\max} - \Theta_\gamma - t_{\mathtt{I}}) \times \sup_{\xi \in \mathbb{R}^\star} \left\{ \frac{1}{\mid 1 - \mathcal{F}(\tilde{\gamma})(\xi) \mid} \right\} \\
&\times (1 + \mid \alpha \mid \times (\Theta_{\max} - t_{\mathtt{I}}) \times \mathrm{e}^{|\alpha|(\Theta_{\max} - t_{\mathtt{I}})}) \times \|\tilde{\sigma}_2 - \tilde{\sigma}_1\|_{\mathcal{M}([t_{\mathtt{I}},\Theta_{\max}])}.
\end{aligned}
\tag{78}
$$

□

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