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LIMIT OF A PARABOBOLIC PDE SOLUTION DEPENDING ON TWO PARAMETERS

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Abstract: We study the behaviour of the parabolic partial differential equation (PDE) solution which depends on two parameters, when the large deviations parameter ε tends more quickly than the homogenization's one δ, to zero. In other words, we assume that

$$\lim_{\epsilon,\delta\longrightarrow 0}\frac{\delta}{\epsilon}=+\infty.$$

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1. Introduction

We study the partial differential equation (PDE) on R^d

$$\begin{cases} \dfrac{\partial u^{\varepsilon,\delta}}{\partial t}(t,x) =L_{\varepsilon,\delta}u^{\varepsilon,\delta}(t,x)+\dfrac{1}{\varepsilon}f\left(\dfrac{x}{\delta},u^{\varepsilon,\delta}(t,x)\right), \\ u^{\varepsilon,\delta}(0,x) =g(x), \qquad x\in R^d, \end{cases} \tag{1}$$

where f is a 1-periodic non-linear function such that:

- $\forall x\in R^d,\ f(x,1)=0$

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- there exists a function $c \in C\left(R^d \times R, R\right)$ bounded such that: $f(x,y) = c(x,y).y$

with

- $c(x,y) > 0, \forall x \in R^d,\ y \in (0,1)$
- $c(x,y) \leq 0,\ \forall x \in R^d, y > 1 \cup R_-^*$
- $\max_{y \geq 0} c(x,y) = c(x) > 0,$

and we consider $g \in C\left(R^d, R^+\right)$ a bounded function, we set

$$\sup_{x \in R^d} g(x) = \bar{g} < \infty.$$

Let us set $G_0 = \left\{x \in R^d :\ g(x) > 0\right\}$, since g is continuous one notes $\overline{\mathring{G}_0} = \overline{G_0}$.

Assumption and definition: Let $(\Omega, \mathcal{F}, P)$ be a probability space on which a d-dimensional Brownian motion $\left(W^1, \ldots, W^d\right)$ is defined. Let be E the corresponding expectation operator. We consider $\langle ., . \rangle$ as the Euclidean inner product on R^d and define for an inverse $R^d \times R^d$-values matrix a and $\theta \in R^d$, the norm $\|\theta\|_{a^{-1}} = \sqrt{\langle \theta, a^{-1}\theta \rangle}$.

Let us consider the Markov diffusion process $X_t^{x,\varepsilon,\delta} \in R^d$ solution of the stochastic differential equation (SDE)

$$\left\{\begin{array}{l} dX_t^{x,\varepsilon,\delta} = \sqrt{\varepsilon}\sigma\left(\dfrac{X_t^{x,\varepsilon,\delta}}{\delta}\right) dW_t + B^{\varepsilon,\delta}\left(\dfrac{X_t^{x,\varepsilon,\delta}}{\delta}\right) dt \\ X_0^{x,\varepsilon,\delta} = x,\ x \in R^d \end{array}\right. , \tag{2}$$

where $\sigma :\ R^d \longrightarrow R^{d \times d}$ and $B^{\varepsilon,\delta} :\ R^d \longrightarrow R^d$ are regular applications and 1-periodic in each coordinate of the argument.

Taking $(*)$ as the symbol of transposition, we suppose that the matrix $a = \sigma\sigma^*$ is strongly elliptic. The vector-valued function $B^{\varepsilon,\delta}$ is given by: $B^{\varepsilon,\delta} = \frac{\varepsilon}{\delta}B_0 + B_1,\ \varepsilon, \delta > 0$, where B_0 and B_1 are smooth.

The infinitesimal generator is given by

$$\boldsymbol{L_{\varepsilon,\delta}} = \frac{\varepsilon}{2}\sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\delta}\right)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} B^{\varepsilon,\delta}\left(\frac{x}{\delta}\right)\frac{\partial}{\partial x_i}. \tag{3}$$

Our aim is to study the behavior of the solution of the SDE (2). Some research has been done before, by Baldi [1], Diédhiou et al. [4], [3] and [5]. There the limit when $\lim_{\delta,\epsilon} \frac{\delta}{\epsilon} = k \in [0, +\infty[$ has been studied.

Since the two parameters δ (homogenization) and ε (large deviation) tend to zero, we consider a new defined parameter $\delta_\varepsilon = \delta$. We suppose that $\boldsymbol{\lim_{\varepsilon \downarrow 0} \frac{\delta_\varepsilon}{\varepsilon}}$

$= \infty$, thus ε tends to zero sufficiently quickly compared to δ_ε. We should first treat δ_ε as fixed and carry out these calculations for slowly varying coefficients and should then let δ_ε tend to zero in the resulting computation.

2. Large Deviation Principle

We note that $\mathbf{lim}_{\varepsilon\downarrow 0} \boldsymbol{B^{\varepsilon,\delta_\varepsilon} = B_1}$. Thereby for $\Gamma > 0$ and $\phi \in C\left([0,\Gamma];R^d\right)$, let us set V_Γ be the function on $R^d \times R^d \to [0\ ,\infty)$ defined as (see [10]):

$$\boldsymbol{V_\Gamma(y,z)} = \inf_{\substack{\phi\in C\left([0,\Gamma];\ R^d\right)\\ \phi(0)=y,\,\phi(\Gamma)=z}} \int_0^\Gamma \nu\left(\dot{\phi}\ (s), B_1(\phi(s))\right) ds, \tag{4}$$

where

$$\begin{aligned}\boldsymbol{\nu\left(\dot{\phi}, B_1(\phi)\right)} &= \frac{1}{2}\left\langle \dot{\phi} - B_1\left(\phi\right), a^{-1}(\phi)\left[\dot{\phi} - B_1\left(\phi\right)\right]\right\rangle \\ &= \frac{1}{2}\left\|\dot{\phi} - B_1\left(\phi\right)\right\|^2_{a^{-1}(\phi)}.\end{aligned} \tag{5}$$

Let $\mathcal{J}\ :\ R^d \to [0\ ,\infty)$, be defined by Freidlin et al. [9] as:

$$\boldsymbol{\mathcal{J}(z)} = \lim_{\Gamma\to+\infty} \frac{1}{\Gamma} V_\Gamma\ (\ 0, \Gamma z)\,, \quad z \in R^d. \tag{6}$$

From Freidlin et al. [9] we know that the random family $\left\{X_t^{x,\varepsilon,\delta_\varepsilon}\ :\ \varepsilon > 0\right\}$ satisfies a large deviations principle with rate function $I_{T,x}$ defined as:

$$\boldsymbol{I_{T,x}(z)} = T\mathcal{J}\left(\frac{z-x}{T}\right), \quad z \in R^d. \tag{7}$$

Remark 2.1. Freidlin et al. [9] showed that for $T > 0$ and $x \in R^d$,

$$g_{x,T}(\theta) = \lim_{\epsilon,\delta\downarrow 0} \epsilon \log E[\exp[\frac{1}{\epsilon}\langle\theta, X^\epsilon_{T,x}\rangle]] = \langle\theta, x\rangle + T\mathcal{J}(\theta),$$

where

$$\begin{aligned}\mathcal{J}(\theta) = \inf_{\varphi\in C^\infty(T^d}\ \sup_{\mu\in\mathcal{P}(T^d)} \int_{T^d} \{\frac{1}{2}\sum_{\ell=1}^{d}(\langle(I+\nabla B_1)\sigma_\ell(z)),\theta\rangle)^2 \\ + \langle\nabla B_1(z), B_1(z)+\theta\rangle\}\mu(dz), \quad \theta \in R^d,\end{aligned}$$

T^d is the $d-$ dimensional torus of size one.

Since Proposition 4.6 in [9] works for the projective limit approach (see Dembo et al. [6]), thus we adopt this approach to establish the Varadhan lemma. Then, without loss of generality, one can take $T = 1$.

Let Q be the quadratic form defined as in Priouret [14]:

$$Q(v) = \langle v, av \rangle \tag{8}$$

and let Q^* the conjugate quadratic form of Q defined as:

$$Q^*(v) = \sup \left\{ 2 \langle t, v \rangle - Q(t) : t \in R^d \right\}. \tag{9}$$

If the inverse of the matrix a exists, then

$$Q^*(v) = \left\langle v, a^{-1} v \right\rangle. \tag{10}$$

Let us define the functional $\overline{\boldsymbol{C}}: \ R^d \to R$ by:

$$\overline{\boldsymbol{C}}(\boldsymbol{z}) := c(z) - \frac{1}{2} \|B_1(z)\|^2_{a^{-1}(z)}, \quad z \in R^d.$$

Proposition 2.2. *Fix $x \in R^d$ and let c be an element of $C\left(R^d, R_+\right)$. Then for each open subset $G \subseteq R^d$*

$$\liminf_{\varepsilon \downarrow 0} \varepsilon \log E \left[1_G \left(X_1^{x,\varepsilon,\delta_\varepsilon} \right) e^{\left\{ \frac{1}{\varepsilon} \int_0^1 c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} \right) ds \right\}} \right] \geq \sup_{z \in G} \left\{ \overline{C}(z) - I_{1,x}(z) \right\}.$$

Proof. We use the Girsanov change of measure to establish the lower bound of the Varadhan lemma. Let ϕ be a function on R^d such that

$$\|\phi(x)\| + \left\| \frac{\partial \phi}{\partial x}(x) \right\|^2 + \left\| \frac{\partial^2 \phi}{\partial x^2}(x) \right\| \leq M < +\infty, \quad \forall x \in R^d. \tag{11}$$

By Itô formula on $\varepsilon^2 \phi$ we have:

$$\begin{aligned} &- \varepsilon^2 \left[\phi\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} \right) - \phi\left(\frac{x}{\delta_\varepsilon} \right) \right] \\ &= \int_0^t - \left[\frac{\varepsilon^2}{\delta_\varepsilon} \left\langle \nabla \phi, B^{\varepsilon,\delta_\varepsilon} \right\rangle + \frac{\varepsilon^3}{2\delta_\varepsilon^2} \mathrm{Tr}\left(a D^2 \phi \right) \right] \left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} \right) ds \\ &- \frac{\varepsilon^2}{\delta_\varepsilon} \sqrt{\varepsilon} \int_0^t \nabla \phi \sigma \left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} \right) dW_s. \end{aligned} \tag{12}$$

Let us introduce a measure probability $\hat{P}$ on $(\Omega, \mathcal{F})$ defined as:

$$\frac{d\hat{P}}{dP} := \exp \left\{ - \frac{\varepsilon^2}{\delta_\varepsilon} \sqrt{\varepsilon} \int_0^1 \nabla \phi \sigma \left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} \right) dW_s \right.$$

$$-\frac{\varepsilon^5}{2\delta_\varepsilon^2}\int_0^1 \|\nabla\phi\|_a^2\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)\,ds\Bigg\}. \qquad (13)$$

And let us set

$$Y_t^\varepsilon=\int_0^t c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds \quad \text{and} \quad \hat{Y}_t^\varepsilon=Y_t^\varepsilon-\varepsilon^2\left[\phi\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)-\phi\left(\frac{x}{\delta_\varepsilon}\right)\right]. \qquad (14)$$

From (14) it is easy to see that Y_t^ε and $\hat{Y}_t^\varepsilon$ have the same limit when $\varepsilon\to 0$. We switch from Y^ε to $\hat{Y}^\varepsilon$ and use the conjugate quadratic form Q^* on (9), setting

$$Z_\epsilon=c-\frac{\varepsilon^2}{\delta_\varepsilon}\left\langle\ \nabla\phi,B^{\varepsilon,\delta_\varepsilon}\right\rangle-\frac{\varepsilon^3}{2\delta_\varepsilon^2}\Big\{\mathrm{Tr}\left(\ aD^2\phi\right)-\varepsilon^2\left\|\nabla\phi\right\|_a^2\Big\},$$

we have

$$\begin{aligned}E\left[1_G e^{\frac{1}{\varepsilon}\hat{Y}_1^\varepsilon}\right]&=\hat{E}\left[1_G\exp\left(\frac{1}{\varepsilon}\int_0^1 Z_\epsilon\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right)\right]\\
&\geq \hat{E}\left[1_G\exp\left(\frac{1}{\varepsilon}\int_0^1\left[c-\frac{1}{2}Q^*\left(B^{\varepsilon,\delta_\varepsilon}\right)\right]\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right)\times\right.\\
&\left.\exp\left(-\frac{\varepsilon^2}{2\delta_\varepsilon^2}\int_0^1\left[(\varepsilon^2-\varepsilon)\sup\Big\{\ \|\nabla\phi\|_a^2\Big\}+\mathrm{Tr}\left(aD^2\phi\right)\right]\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right)\right]\\
&\geq \hat{E}\left[1_G\exp\left(\frac{1}{\varepsilon}\int_0^1\left[c-\frac{1}{2}\left\|B^{\varepsilon,\delta_\varepsilon}\right\|_{a^{-1}}^2\right]\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right.\right.\\
&\qquad\left.\left.-\frac{\varepsilon^2}{2\delta\varepsilon^2}\left(\varepsilon^2-\varepsilon+1\right)M'\right)\right]\end{aligned}$$

$M'=M\times\alpha$, where α denotes the ellipticity constant.

From this we deduce

$$\begin{aligned}\varepsilon\log E\left\{1_G\exp\left(\frac{1}{\varepsilon}\hat{Y}_1\right)\right\}&\geq\int_0^1\hat{E}\left[c-\frac{1}{2}\left\|B^{\varepsilon,\delta_\varepsilon}\right\|_{a^{-1}}^2\right]\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\\
&-\frac{\varepsilon^3}{2\delta_\varepsilon^2}\left(\varepsilon^2-\varepsilon+1\right)M'+\varepsilon\log\hat{P}\Big\{X^{x,\varepsilon,\delta_\varepsilon}\in G\Big\}.\end{aligned} \qquad (15)$$

Fix $z\in R^d$, and $\varpi>0$ and let $\varepsilon'>0$ be small enough, such that G contains the set $\Big\{z'\in R^d\ :\ \left\|z'-z\right\|\leq\varpi\delta_{\varepsilon'}\Big\}$. Let us choose a $\varphi\in C\left([0,1],R^d\right)$ such that $\varphi(0)=x$ and $\varphi(1)=z$, and take ε such that $0<\varepsilon<\varepsilon'$, then we have

$$\Big\{X^{x,\varepsilon,\delta_\varepsilon}\in G\Big\}\supseteq\Big\{\left\|X^{x,\varepsilon,\delta_\varepsilon}-z\right\|\leq\varpi\delta_\varepsilon\Big\}$$

$$\equiv \left\{ \left\| \check{X}^{x,\varepsilon,\delta_\varepsilon} \right\|_{C\left([0,1],R^d\right)} \leq \varpi \frac{\delta_\varepsilon}{\sqrt{\varepsilon}} \right\}, \tag{16}$$

where

$$\check{X}_t^{x,\varepsilon,\delta_\varepsilon} = \frac{1}{\sqrt{\varepsilon}} \left(X_t^{x,\varepsilon,\delta_\varepsilon} - \varphi(t) \right), \qquad 0 \leq t \leq 1. \tag{17}$$

We remark that

$$\left\{ \left\| \check{X}^{x,\varepsilon,\delta_\varepsilon} \right\|_{C\left([0,1],R^d\right)} \leq \varpi \frac{\delta_\varepsilon}{\sqrt{\varepsilon}} \right\} \equiv \left\{ \sup_{0 \leq t \leq 1} \left\| \frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} - \frac{\varphi(t)}{\delta_\varepsilon} \right\| \leq \varpi \right\}. \tag{18}$$

On this set we have, by the lower-semi-continuity of c:

$$\begin{aligned} \int_0^1 c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right) ds &\geq \inf_{\left\{\|\psi\|_{C\left([0,1],R^d\right)} \leq \varpi\right\}} \int_0^1 c\left(\frac{\varphi(s)}{\delta_\varepsilon} + \psi(s)\right) ds \\ &\geq \int_0^1 c\left(\frac{\varphi(s)}{\delta_\varepsilon}\right) ds - \varpi, \end{aligned} \tag{19}$$

and (see [9])

$$\begin{aligned} &\int_0^1 \left\| B^{\varepsilon,\delta_\varepsilon} \right\|_{a^{-1}}^2 \left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} \right) ds \\ &\leq \sup_{\left\{\|\psi\|_{C\left([0,1],R^d\right)} \leq \varpi\right\}} \int_0^1 \left\| B^{\varepsilon,\delta_\varepsilon} \right\|_{a^{-1}}^2 \left(\frac{\varphi(s)}{\delta_\varepsilon} + \psi(s) \right) ds \\ &\leq (1 + \kappa\varpi) \int_0^1 \|B_1\|_{a^{-1}}^2 \left(\frac{\varphi(s)}{\delta_\varepsilon} \right) ds + \Phi(\varepsilon, \varpi), \end{aligned} \tag{20}$$

where

$$\Phi(\varepsilon,\varpi) = \kappa^{'} (1 + \kappa\varpi) \times \sup_{\substack{y,y^{'} \in R^d \\ \left\|y - y^{'}\right\| \leq \varpi}} \left(\frac{\varepsilon}{\delta_\varepsilon} \|B_0(y)\| + \left\| B_1(y) - B_1(y^{'}) \right\| \right)^2. \tag{21}$$

From (15) we use the inequality (19) and (20)

$$\begin{aligned} \varepsilon \log E\left[1_G e^{\frac{1}{\varepsilon}\hat{Y}_1^\varepsilon} \right] \geq \delta_\varepsilon & \int_0^{\frac{1}{\delta_\varepsilon}} \left[c - \frac{1}{2}(1 + \kappa\varpi) \left\|B_1\right\|_{a^{-1}}^2 \right] (\psi(s))\, ds \\ & - \frac{1}{2}\Phi(\varepsilon,\varpi) - \varpi - \frac{\varepsilon^3}{2\delta_\varepsilon^2} \left(\varepsilon^2 - \varepsilon + 1 \right) M' \\ & + \varepsilon \log \hat{P}\left(X^{x,\varepsilon,\delta_\varepsilon} \in G \right). \end{aligned} \tag{22}$$

Remark that P and $\hat{P}$ have the same rate function (for the details see [9]). By (22) one can deduct that (let first $\varpi \to 0$ after let $\varepsilon \to 0$)

$$\liminf_{\varepsilon \downarrow 0} \varepsilon \log E\left[1_G\left(X_1^{x,\varepsilon,\delta_\varepsilon}\right) e^{\left\{\frac{1}{\varepsilon}\int_0^1 c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right\}}\right] \geq \sup_{z\in G}\left\{\overline{C}(z) - I_{1,x}(z)\right\}.$$

□

Let $\left\{P_{t,s}^{\varepsilon} \ : \ t < s\right\}$ be the semigroup on $\boldsymbol{B}\left(R^d\right)$ the Banach space of bounded and measurable functions of R^d, defined by the adjoint of $L_{\varepsilon,\delta_\varepsilon}$. By the assumption on the matrix a, there is a $p^{\varepsilon}(s-t,z,y)$ (called heat kernel) such that

$$\left(\boldsymbol{P_{t,s}^{\varepsilon}\phi}\right)\boldsymbol{(z)} = \int_{R^d} p^{\varepsilon}(s-t,z,y)\phi(y)dy, \quad t<s, \ z\in R^d, \ \phi \in \boldsymbol{B}\left(R^d\right). \quad (23)$$

Let us introduce $P_t^{\varepsilon,\Theta}$ the transform of $P_{t,s}^{\varepsilon}$ conjugate with a potential $\Theta \in C_b^1\left(R_+\times R^d, R\right)$, where $C_b^1\left(R_+\times R^d, R\right)$ the set of bounded and differential functions on $R\times R^d$ taking values in R, and their differentials hence bounded in R^d. Let $\theta_s = \theta(s,.)$ and consider the semigroup

$$\boldsymbol{P_{t,s}^{\varepsilon,\Theta}} = e^{-\Theta_t} P_{t,s}^{\varepsilon} e^{\Theta_s}. \quad (24)$$

Then the corresponding Feynman-Kac heat kernel of $P_{t,s}^{\varepsilon,\Theta}$ is given by

$$\boldsymbol{p^{\Theta}(s-t,z,y)} = e^{-\Theta_t(z)} p^{\varepsilon}(s-t,z,y) e^{\Theta_s(y)}. \quad (25)$$

As in [9] we can derive (for other explanation, Corollary 10.3.22, [15])

$$E\left\{1_A\left(X_t^{x,\varepsilon,\delta_\varepsilon}\right) e^{\left(\frac{1}{\varepsilon}\int_0^t c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right)}\right\} = \int_{z\in A/\delta_\varepsilon} \underbrace{p^{\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2 c}}_{\text{this is } p^{\Theta}}\left(\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2 t, z, \frac{x}{\delta_\varepsilon}\right) dz. \quad (26)$$

Let us introduce some notations taking into account to the bound of Norris et al. [11] and some calculations as in Freidlin et al. [9]:

$$a^{\Theta}(z) := \frac{1}{2}a(z); \quad b^{\Theta}(z) := \left(\frac{\delta_\varepsilon}{\varepsilon}\right) a^{-1}(z) B^{\varepsilon,\delta_\varepsilon}(z) - \nabla\Theta(z);$$

$$\hat{b}^{\Theta}(z) := -\left(\frac{\delta_\varepsilon}{\varepsilon}\right) a^{-1}(z) B^{\varepsilon,\delta_\varepsilon}(z) + a^{-1}(z)\left(\text{div a}\right)^*(z) + \nabla\Theta(z);$$

$$c^{\Theta}(z) := -\frac{1}{2}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)\left(\mathrm{div}B^{\varepsilon,\delta_\varepsilon}\right)(z) + \Theta(z) + \|\nabla\Theta(z)\|^2_{a^\Theta} + \left\langle a^\Theta(z)\left(b^\Theta - \hat{b}^\Theta\right)(z), \nabla\Theta(z)\right\rangle .$$

Next define the quasi potential $\mathcal{E}^{\varepsilon,\Theta}(s-t,z,y)$ for $t<s$ (see [11]):

$$\boldsymbol{\mathcal{E}^{\varepsilon,\Theta}(s-t,z,y)} := \inf_{\substack{\phi\in C\left([0,t],R^d\right)\\ \phi(t)=z\\ \phi(0)=y}} \frac{1}{4}\int_t^s \left\|\dot{\phi}\,(s) - \left(a^\Theta\left[b^\Theta - \hat{b}^\Theta\right]\right)(\phi(s))\right\|^2_{(a^\Theta)^{-1}(\phi(s))} ds.$$

Proposition 2.3. *Fix* $x\in R^d$ *and let* c *be an element of* $C\left(R^d, R_+\right)$. *Then for each closed subset* $F\subseteq R^d$

$$\limsup_{\varepsilon\downarrow 0}\varepsilon\log E\left[1_G\left(X_1^{x,\varepsilon,\delta_\varepsilon}\right)e^{\left\{\frac{1}{\varepsilon}\int_0^1 c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right\}}\right] \le \sup_{z\in F}\left\{\overline{C}(z) - I_{1,x}(z)\right\}.$$

Proof. By Theorem 2.7 from Stroock et al. [11], there exist constants $K, K_1, K_2 > 0$ such that for all $t<0$ and for all $y,z\in R^d$,

$$\begin{aligned}
p^\Theta(-t,z,y) \le\ & K e^{\left\{-\mathcal{E}^{\varepsilon,\Theta}(t,z,y) - t\sup_{z\in R^d}\left(\Theta + \|\nabla\Theta\|^2_{a^\Theta} + \left\langle a^\Theta\left(b^\Theta - \hat{b}^\Theta\right),\nabla\Theta\right\rangle\right)(z)\right\}}\\
&\times\left(\frac{1 - t\sup_{z\in R^d}\left\{\|b^\Theta\|^2_{a^\Theta} + \|\hat{b}^\Theta\|^2_{a^\Theta} + |c^\Theta|\right\}(z) + \mathcal{E}^{\varepsilon,\Theta}(t,z,y)}{-t}\right)^{\frac{d}{2}}\\
&\times e^{\left\{-t\sup_{z\in R^d}\left(\frac{1}{4}\|b^\Theta + \hat{b}^\Theta\|^2_{a^\Theta} - \frac{1}{2}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)\left(\mathrm{div}B^{\varepsilon,\delta_\varepsilon}\right)\right)(z)\right\}}\\
\le & K_1 A_0\\
&\times e^{\left\{-\mathcal{E}^{\varepsilon,\Theta}(t,z,y) - t\sup_{z\in R^d}\left(\Theta + \|\nabla\Theta\|^2_{a^\Theta} + \left\langle a^\Theta\left(b^\Theta - \hat{b}^\Theta\right),\nabla\Theta\right\rangle\right)(z)\right\}}\\
&\times e^{\left\{-K_2\left(\frac{\delta_\varepsilon}{\varepsilon}+1\right)t\right\}},
\end{aligned} \tag{27}$$

where

$$-tA_0^{\frac{d}{2}} = 1 - t\frac{\delta_\varepsilon}{\varepsilon}\left(\frac{\delta_\varepsilon}{\varepsilon}+1\right) + \mathcal{E}^{\varepsilon,\Theta}(t,z,y)$$
$$- t\left[\sup_{z\in R^d}\left|\Theta + \|\nabla\Theta\|^2_{a^\Theta} + \left\langle a^\Theta\left(b^\Theta - \hat{b}^\Theta\right), \nabla\Theta\right\rangle\right|(z)\right].$$

From (26), by scaling we observe that

$$E\left[1_F\left(X_1^{x,\varepsilon,\delta_\varepsilon}\right)\exp\left\{\frac{1}{\varepsilon}\int_0^1 c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right\}\right]$$
$$= \delta_\varepsilon^{-d}\int_F p^{\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2 c}\left(\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2, \frac{z}{\delta_\varepsilon}, \frac{x}{\delta_\varepsilon}\right)dz. \tag{28}$$

We are going to proceed as in regime 3 (Freidlin et al. [9]) and we note here that the Feynman-Kac heat kernel p^Θ is not a density, contrary to the heat kernel p. However we are going to observe that the calculations of the bound of Norris and Stroock in (27) are independent of the potential gradient $\nabla\Theta$ which is bounded:

$$\varepsilon\mathcal{E}^{\varepsilon,\Theta}\left(-\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2, \frac{z}{\delta_\varepsilon}, \frac{x}{\delta_\varepsilon}\right) =$$
$$\frac{\varepsilon}{2}\int_0^{\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2}\left\|\dot{\phi}(s) - (\frac{\delta_\varepsilon}{\varepsilon}B^{\varepsilon,\delta_\varepsilon} + a\nabla\Theta - \frac{1}{2}(\text{div}a)^*)(\phi(s))\right\|^2_{a^{-1}(\phi(s))}ds$$
$$= \left(\frac{\delta_\varepsilon}{\sqrt{2\varepsilon}}\right)^2\int_0^{\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2}\|A(s)\|^2_{a^{-1}(\phi(s))}ds \tag{29}$$
$$= \frac{1}{2}\delta_\varepsilon\int_0^{\frac{1}{\delta_\varepsilon}}\left\|\dot{\psi}(s) - B^{\varepsilon,\delta_\varepsilon} - \frac{\varepsilon}{\delta_\varepsilon}\left\{a\nabla\Theta - \frac{1}{2}(\text{div}a)^*\right\}(\psi(s))\right\|^2_{a^{-1}(\psi(s))}ds,$$

where $A(s) = \frac{\varepsilon}{\delta_\varepsilon}\dot{\phi}(s) - B^{\varepsilon,\delta_\varepsilon} - \frac{\varepsilon}{\delta_\varepsilon}\left(a\nabla\Theta - \frac{1}{2}(\text{div}a)^*\right)(\phi(s)$.

Then we have

$$\lim_{\varepsilon\to 0}\varepsilon\mathcal{E}^{\varepsilon,\Theta}\left(-\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2, \frac{z}{\delta_\varepsilon}, \frac{x}{\delta_\varepsilon}\right) = I_{1,x}(z). \tag{30}$$

By the boundedness of $\nabla\Theta$ and a, there is a constant K_1' such that:

$$
\begin{aligned}
&\left(\Theta + \|\nabla\Theta\|^2_{a^\Theta} + \left\langle a^\Theta(z)\left(b^\Theta - \hat{b}^\Theta\right), \nabla\Theta\right\rangle\right)(z) \\
&= \left(\Theta + \frac{1}{2}\|\nabla\Theta\|^2_a - \left\langle\left(\frac{\delta_\varepsilon}{\varepsilon}B^{\varepsilon,\delta_\varepsilon} + a\nabla\Theta - \frac{1}{2}(\mathrm{div}a)^*\right), \nabla\Theta\right\rangle\right)(z) \\
&\leq \left(\Theta + \frac{1}{2}\|\nabla\Theta\|^2_a - \frac{\delta_\varepsilon}{\varepsilon}\left\langle B^{\varepsilon,\delta_\varepsilon}, \nabla\Theta\right\rangle\right)(z) + K_1' \\
&\leq \left(\Theta + \frac{1}{2}\inf_{\nabla\Theta\in R^d}\left(\|\nabla\Theta\|^2_a - 2\frac{\delta_\varepsilon}{\varepsilon}\left\langle B^{\varepsilon,\delta_\varepsilon}, \nabla\Theta\right\rangle\right)\right)(z) + K_1' \\
&\leq \left(\Theta - \frac{1}{2}\sup_{\nabla\Theta\in R^d}\left(2\frac{\delta_\varepsilon}{\varepsilon}\left\langle B^{\varepsilon,\delta_\varepsilon}, \nabla\Theta\right\rangle - \|\nabla\Theta\|^2_a\right)\right)(z) + K_1' \\
&\left(\Theta - \frac{1}{2}Q^*\left(\frac{\delta_\varepsilon}{\varepsilon}B^{\varepsilon,\delta_\varepsilon}\right)\right)(z) + K_1' \\
&\leq \left(\Theta - \frac{1}{2}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2\left\|B^{\varepsilon,\delta_\varepsilon}\right\|^2_{a^{-1}}\right)(z) + K_1'.
\end{aligned}
\tag{31}
$$

We replace in (27) t and $\Theta(z)$ respectively by $-\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2$ and $\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^2 c(z)$ and then we have

$$
\begin{aligned}
&\limsup_{\varepsilon\downarrow 0}\varepsilon\log E\left[1_F\left(X_1^{x,\varepsilon,\delta_\varepsilon}\right)e^{\left\{\frac{1}{\varepsilon}\int_0^1 c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right\}}\right] \\
&\leq \sup_{z\in F}\left\{\overline{C}(z) - I_{1,x}(z)\right\}.
\end{aligned}
$$

$\square$

Define

$$
\boldsymbol{S_{0,T}(\phi)} = \begin{cases} \int_0^T \mathcal{J}\left(\dot{\phi}(s)\right)ds & \text{if } \phi \text{ differentiable and } \phi(0) = x, \\ +\infty & \text{if not.} \end{cases}
$$

The following expresses the path space large deviations principle.

Theorem 2.4. *Let D be a Borel subset on $C\left([0,T]; R^d\right)$ and c be an element of $C\left(R^d, R_+\right)$. Then we have*

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log E\left[1_D\left(X_T^{x,\varepsilon,\delta_\varepsilon}\right) e^{\left\{\frac{1}{\varepsilon}\int_0^T c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right\}}\right]$$
$$\geq \sup_{\phi\in\overset{\circ}{D}}\left\{\int_0^T \overline{C}(\phi(s))ds - S_{0,T}(\phi)\right\},$$
$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log E\left[1_D\left(X_T^{x,\varepsilon,\delta_\varepsilon}\right) e^{\left\{\frac{1}{\varepsilon}\int_0^T c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)ds\right\}}\right]$$
$$\leq \sup_{\phi\in\overline{D}}\left\{\int_0^T \overline{C}(\phi(s))ds - S_{0,T}(\phi)\right\}.$$

3. Convergence of $u^{\varepsilon,\delta_\varepsilon}$

The Feynman-Kac formula implies that the solution of (1) obeys the equation

$$\boldsymbol{u^{\varepsilon,\delta_\varepsilon}(t,x)} = E\left[g\left(X_t^{x,\varepsilon,\delta_\varepsilon}\right) e^{\frac{1}{\varepsilon}\int_0^t c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}, Y_s^{x,\varepsilon,\delta_\varepsilon}\right)ds}\right]. \tag{32}$$

Here $Y^{x,\varepsilon,\delta_\varepsilon}$ is the progressive measurable solution associated of the BSDE introduced by Pardoux et al. [12]:

$$\begin{cases} Y_t^{x,\varepsilon,\delta_\varepsilon} = g\left(X_t^{x,\varepsilon,\delta_\varepsilon}\right) + \frac{1}{\varepsilon}\int_s^t f\left(\frac{X_r^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}, Y_r^{x,\varepsilon,\delta_\varepsilon}\right)dr \\ -\frac{1}{\sqrt{\varepsilon}}\int_s^t Z_r^{x,\varepsilon,\delta_\varepsilon} dW_r \\ E\left\{\int_s^t \left|Z_r^{x,\varepsilon,\delta_\varepsilon}\right|^2 dr\right\} < \infty. \end{cases}, \quad 0 \leq s \leq t. \tag{33}$$

Since

$$Y_0^{x,\varepsilon,\delta_\varepsilon} = u^{\varepsilon,\delta_\varepsilon}(t,x) \quad \text{and} \quad 0 < Y_0^{x,\varepsilon,\delta_\varepsilon} < 1 \vee \bar{g}.$$

Let us introduce $\boldsymbol{v^{\varepsilon,\delta_\varepsilon}(t,x)} = \varepsilon \log u^{\varepsilon,\delta_\varepsilon}(t,x)$. As in Pradeilles [13], we observe that $v^{\varepsilon,\delta_\varepsilon}(t,x)$ is a viscosity solution of:

$$\begin{cases} \dfrac{\partial v^{\varepsilon,\delta_\varepsilon}}{\partial t}(t,x) = L_{\varepsilon,\delta_\varepsilon} v^{\varepsilon,\delta_\varepsilon}(t,x) + \dfrac{1}{2}\left\|\nabla v^{\varepsilon,\delta_\varepsilon}(t,x)\sigma\left(\dfrac{x}{\delta_\varepsilon}\right)\right\|^2 \\ \qquad + c\left(\dfrac{x}{\delta_\varepsilon}, u^{\varepsilon,\delta_\varepsilon}(t,x)\right) \\ v^{\varepsilon,\delta_\varepsilon}(0,x) = \varepsilon \log(g(x)), \qquad x \in G_0 \\ \lim_{t\downarrow 0} v^{\varepsilon,\delta_\varepsilon}(t,x) = -\infty, \qquad x \in R^d \backslash G_0. \end{cases} \tag{34}$$

Let us define a distance in $R_+ \times R^d$, for $(t,x),(s,y) \in R_+ \times R^d$:

$$\boldsymbol{d\{(t,x),(s,y)\}} = \max\{|t-s|, \|x-y\|\}.$$

And let us set

$$\overline{\boldsymbol{v}}\boldsymbol{(t,x)} = \limsup_{\eta \to 0} \left\{ v^{\varepsilon,\delta_\varepsilon}(s,y) : \ (s,y) \in \mathcal{B}((t,x),\eta) \right\},$$

$$\underline{\boldsymbol{v}}\boldsymbol{(t,x)} = \liminf_{\eta \to 0} \left\{ v^{\varepsilon,\delta_\varepsilon}(s,y) : \ (s,y) \in \mathcal{B}((t,x),\eta) \right\}.$$

For $x, \theta \in R^d$, set $\boldsymbol{H}(x,\theta)$ to be the Hamiltonian associated of the Lagrangian $\boldsymbol{L(x,\theta)}$, defined as:

$$\boldsymbol{L(x,\theta)} := \frac{1}{2}\Big\langle \theta, \boldsymbol{a}(x)\theta \Big\rangle + \Big\langle \boldsymbol{B_1}(x), \theta \Big\rangle + \overline{\boldsymbol{C}}(x).$$

We remark that:

- $\boldsymbol{L}(x, \nabla\psi)$ is the limit when $\varepsilon \to 0$ of the following operator

$$\boldsymbol{L}^{\varepsilon,\delta_\varepsilon}(x,\nabla\psi) = \varepsilon e^{-\frac{1}{\varepsilon}\psi(.,z)} \left[L_{\varepsilon,\delta_\varepsilon} + \frac{1}{\varepsilon} c\left(\frac{x}{\delta_\varepsilon}\right)\right] e^{\frac{1}{\varepsilon}\psi(.,z)}, \quad \forall z \in R^d,$$

- the corresponding eigenfunction of $\boldsymbol{L}(x,\nabla\psi)$ for the eigenvalue $\boldsymbol{H}(x,\nabla\psi)$ can be chosen to be strictly positive.

Theorem 3.1. *$\overline{v}$ and $\underline{v}$ are sub and super viscosity solutions of:*

$$\begin{cases} \max\left(w, \dfrac{\partial w}{\partial t}(t,x) - H(x, \nabla w) \right) = 0, & x \in R^d,\ t > 0 \\ w(0,x) \ = 0, \quad x \in G_0 \\ \lim\limits_{t \to 0} w(t,x) \ = \ -\infty, \quad x \in R^d \backslash G_0. \end{cases}$$

Proof. We adopt the techniques as in Pradeilles [13].

We first consider $\overline{v}$. Let $\Phi \geq \overline{v}$ be a smooth function such that $\Phi(t_0,x_0) = \overline{v}(t_0,x_0)$, and (t_0,x_0) is a strict local minimum of $\Phi - \overline{v}$.

Let $\psi > 0$ be the eigenfunction corresponding to the largest eigenvalue $H(x, \nabla\Phi(t_0,x_0))$. Now consider the perturbed test function:

$$\Phi^\varepsilon(t_\varepsilon,x_\varepsilon) = \Phi(t_\varepsilon,x_\varepsilon) - \varepsilon \ln\left(\psi\left(\frac{x_\varepsilon}{\delta_\varepsilon}\right)\right) \tag{35}$$

which limit is

$$\lim_{\varepsilon \downarrow 0} \Phi^\varepsilon(t_\varepsilon,x_\varepsilon) = \overline{v}(t_0,x_0).$$

There exists a sequence $(t_\varepsilon,x_\varepsilon)$ that locally minimizes $\Phi^\varepsilon - v^{\varepsilon,\delta_\varepsilon}$ and converges towards (t_0,x_0). From (35) we have

$$
\begin{aligned}
\frac{\partial \Phi^{\varepsilon}(t,x)}{\partial t} &= \frac{\partial \Phi(t,x)}{\partial t} \\
D\Phi^{\varepsilon}(t,x) &= D\Phi(t,x) - \frac{\varepsilon}{\delta_\varepsilon}\frac{D\psi\left(\frac{x}{\delta_\varepsilon}\right)}{\psi\left(\frac{x}{\delta_\varepsilon}\right)} \\
D^2\Phi^{\varepsilon}(t,x) &= D^2\Phi(t,x) - \left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2 \frac{D^2\psi\left(\frac{x}{\delta_\varepsilon}\right)}{\psi\left(\frac{x}{\delta_\varepsilon}\right)} \\
&\quad + \left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2 \frac{D\psi\left(\frac{x}{\delta_\varepsilon}\right) D\psi\left(\frac{x}{\delta_\varepsilon}\right)}{\psi^2\left(\frac{x}{\delta_\varepsilon}\right)}.
\end{aligned} \tag{36}
$$

By (34), one sees

$$
\begin{aligned}
\frac{\partial \Phi^{\varepsilon}}{\partial t}(t_\varepsilon, x_\varepsilon) = \ & L_{\varepsilon,\delta_\varepsilon}\Phi^{\varepsilon}(t_\varepsilon,x_\varepsilon) + \frac{1}{2}\left\|\nabla\Phi^{\varepsilon}(t_\varepsilon,x_\varepsilon)\sigma\left(\frac{x_\varepsilon}{\delta_\varepsilon}\right)\right\|^2 \\
&+ c\left(\frac{x_\varepsilon}{\delta_\varepsilon}, u^{\varepsilon,\delta_\varepsilon}(t_\varepsilon,x_\varepsilon)\right).
\end{aligned} \tag{37}
$$

From (36), we have

$$
\begin{aligned}
L_{\varepsilon,\delta_\varepsilon}\Phi^{\varepsilon}(t,x) &= B^{\varepsilon,\delta_\varepsilon}\left(\frac{x}{\delta_\varepsilon}\right) D\Phi\left(t,x\right) - \frac{1}{2}\left(\frac{\varepsilon}{\delta_\varepsilon}\right)^2 \frac{\operatorname{Tr}\left[a\left(\frac{x}{\delta_\varepsilon}\right) D^2\psi\left(\frac{x}{\delta_\varepsilon}\right)\right]}{\psi\left(\frac{x}{\delta_\varepsilon}\right)} \\
&+ \frac{\varepsilon}{2}\operatorname{Tr}\left[a\left(\frac{x}{\delta_\varepsilon}\right) D^2\Phi\left(t,x\right)\right] \\
&+ \frac{1}{2}\left(\frac{\varepsilon}{\delta_\varepsilon}\right)^2 \frac{\left\langle D\psi\left(\frac{x}{\delta_\varepsilon}\right), a\left(\frac{x}{\delta_\varepsilon}\right) D\psi\left(\frac{x}{\delta_\varepsilon}\right)\right\rangle}{\psi^2\left(\frac{x}{\delta_\varepsilon}\right)} - \frac{\varepsilon}{\delta_\varepsilon}\frac{\left\langle B^{\varepsilon,\delta_\varepsilon}\left(\frac{x}{\delta_\varepsilon}\right), D\psi\left(\frac{x}{\delta_\varepsilon}\right)\right\rangle}{\psi\left(\frac{x}{\delta_\varepsilon}\right)}.
\end{aligned} \tag{38}
$$

And we have

$$
\begin{aligned}
\frac{1}{2}\left\|\nabla\Phi^{\varepsilon}(t,x)\sigma\left(\frac{x}{\delta_\varepsilon}\right)\right\|^2 &= \frac{1}{2}\left\|\sigma\left(\frac{x}{\delta_\varepsilon}\right)\left[D\Phi(t,x) - \frac{\varepsilon}{\delta_\varepsilon}\frac{D\psi\left(\frac{x}{\delta_\varepsilon}\right)}{\psi\left(\frac{x}{\delta_\varepsilon}\right)}\right]\right\|^2 \\
&= \frac{1}{2}\left\|\sigma\left(\frac{x}{\delta_\varepsilon}\right) D\Phi(t,x)\right\|^2 + o(1).
\end{aligned} \tag{39}
$$

Set $y_\varepsilon = \dfrac{x_\varepsilon}{\delta_\varepsilon}$ and $\theta_\varepsilon := \dfrac{\varepsilon}{\delta_\varepsilon}\dfrac{D\psi\left(y_\varepsilon\right)}{\psi\left(y_\varepsilon\right)}$, insert (38) and (39) in (37), then:

$$
\begin{aligned}
&\frac{\partial \Phi}{\partial t}(t_\varepsilon, x_\varepsilon) \\
&\leq \frac{\frac{1}{2}\left\|\sigma\left(y_\varepsilon\right) D\Phi(t_\varepsilon, x_\varepsilon)\right\|^2 \psi(y_\varepsilon) + \left\langle B^{\varepsilon,\delta_\varepsilon}\left(y_\varepsilon\right), D\Phi\left(t_\varepsilon, x_\varepsilon\right)\right\rangle \psi(y_\varepsilon)}{\psi(y_\varepsilon)} \\
&\quad + \frac{c(y_\varepsilon)\psi(y_\varepsilon) + \left\{\frac{1}{2}\left\langle a\left(y_\varepsilon\right)\theta_\varepsilon, \theta_\varepsilon\right\rangle - \left\langle B^{\varepsilon,\delta_\varepsilon}\left(y_\varepsilon\right), \theta_\varepsilon\right\rangle\right\}\psi(y_\varepsilon)}{\psi(y_\varepsilon)} \\
&+ o(1) \\
&\leq \frac{\frac{1}{2}\left\|\sigma\left(y_\varepsilon\right) D\Phi(t_\varepsilon, x_\varepsilon)\right\|^2 \psi(y_\varepsilon) + \left\langle B^{\varepsilon,\delta_\varepsilon}\left(y_\varepsilon\right), D\Phi\left(t_\varepsilon, x_\varepsilon\right)\right\rangle \psi(y_\varepsilon)}{\psi(y_\varepsilon)} \\
&\quad + \frac{c(y_\varepsilon)\psi(y_\varepsilon) - \frac{1}{2}\sup_{\theta_\varepsilon \in R^d}\left\{2\left\langle B^{\varepsilon,\delta_\varepsilon}\left(y_\varepsilon\right), \theta_\varepsilon\right\rangle - \left\langle a\left(y_\varepsilon\right)\theta_\varepsilon, \theta_\varepsilon\right\rangle\right\}\psi(y_\varepsilon)}{\psi(y_\varepsilon)} \\
&+ o(1) \\
&\leq \frac{\left(\frac{1}{2}\left\|\sigma\left(y_\varepsilon\right) D\Phi(t_\varepsilon, x_\varepsilon)\right\|^2 + \left\langle B^{\varepsilon,\delta_\varepsilon}\left(y_\varepsilon\right), D\Phi\left(t_\varepsilon, x_\varepsilon\right)\right\rangle + \overline{C}(y_\varepsilon)\right)\psi(y_\varepsilon)}{\psi(y_\varepsilon)} \\
&+ o(1).
\end{aligned} \tag{40}
$$

Thus for $(t_\varepsilon, x_\varepsilon)$ close to (t_0, x_0), we have

$$
\begin{aligned}
\frac{\partial \Phi}{\partial t}(t_0, x_0) &\leq \frac{L\left(x, D\Phi\left(t_0, x_0\right)\right)\psi(y_0)}{\psi(y_0)} \\
&\leq H\left(x, D\Phi(t_0, x_0)\right).
\end{aligned}
$$

Let us now consider $\underline{v}$. Let $(t_0, x_0) \in R_+^* \times R^d$ such that $\underline{v}(t_0, x_0) < 0$. Let $\Phi \leq \underline{v}$ be a smooth function such that $\Phi(t_0, x_0) = \underline{v}(t_0, x_0)$, and (t_0, x_0) is a strict local maximum of $\Phi - \underline{v}$.

We consider the same perturbed function test Φ^ε as above. Hence, there exists a sequence $(t_\varepsilon, x_\varepsilon)$ locally maximizes $\Phi^\varepsilon - v^{\varepsilon,\delta_\varepsilon}$ and converges towards (t_0, x_0). By equivalent arguments we have

$$
\frac{\partial \Phi}{\partial t}(t_0, x_0) \geq H\left(x, D\Phi(t_0, x_0)\right).
$$

□

Let $\mathcal{O}$ be a open subset in $R{\times}R^d$, define the function τ on $R{\times}C\left([0,\infty]\times R^d\right)$ values into $[0,\infty]$ as:

$$\boldsymbol{\tau}=\tau_{\mathcal{O}}(t,\phi)=\inf\left\{s:(t-s,\phi(s))\in\mathcal{O}\right\}.$$

Take $\boldsymbol{\Theta}$ the set of Markov functions τ and let $\boldsymbol{V(t,x)}$ be the function:

$$\boldsymbol{V(t,x)}=\inf_{\tau\in\Theta}\sup_{\left\{\phi\in C\left([0,t],R^d\right),\phi(0)=x,\phi(t)\in G_0\right\}}\left\{\int_0^{\tau}\overline{C}(\phi(s))ds-S^3_{0,\tau}(\phi)\right\}.$$

Consider the partitions $\mathcal{M}$ and $\mathcal{E}$ of $R_+\times R^d$:

$$\boldsymbol{\mathcal{M}}=\left\{(t,x)\in R^+\times R^d;V(t,x)=0\right\}$$
$$\boldsymbol{\mathcal{E}}=\left\{(t,x)\in R^+\times R^d;V(t,x)<0\right\}.$$

Remark 3.2.

$$\lim_{\boldsymbol{\varepsilon\downarrow 0}}\boldsymbol{u^{\varepsilon,\delta_\varepsilon}(t,x)}=\begin{cases}0 \text{ uniformly from any compact } \mathcal{K} \text{ of } \mathcal{E}\\ 1 \text{ uniformly from any compact } \mathcal{K}' \text{ of } \overset{\circ}{\mathcal{M}}.\end{cases}.$$

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