

PERIPHERY BEHAVIOUR OF SERIES IN MITTAG-LEFFLER TYPE FUNCTIONS, II

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Abstract: This is a survey on a part of author’s recent results on the subject. It is devoted to different systems of the Mittag-Leffler functions and their 3-parametric generalizations. First, asymptotic formulae necessary for obtaining the main results, are provided. Series defined by means of these systems are further studied. Starting with their domains of convergence, the behaviours of such series on the peripheries of their convergence domains are investigated and analogues of the classical results for the power series are proposed.

This serves as Part II, of our previous paper [16].

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1. Introduction

The special function $E_{\alpha, \beta}^{\gamma}$, defined in the whole complex plane \mathbb{C} by the power

series

$$E_{\alpha, \beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \quad \operatorname{Re}(\alpha) > 0, \quad (1)$$

where $(\gamma)_k$ is the Pochhammer symbol ([1], Section 2.1.1)

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma + 1) \dots (\gamma + k - 1),$$

arises as a natural generalization of the Mittag-Leffler functions E_{α} and $E_{\alpha, \beta}$. It was introduced by Prabhakar in 1971 in his paper [17]. For $\gamma = \beta = 1$ and $\gamma = 1$ this function coincides respectively with the classical Mittag-Leffler functions E_{α} and $E_{\alpha, \beta}$. The first was introduced by Mittag-Leffler (1902-1905) who investigated some of its properties, while the other first appeared in a paper of Wiman (1905).

In the previous papers [9, 10], the author considered series in systems of Mittag-Leffler type functions and, resp. in [13], series in the multi-index ($2m$ -indices) analogues of the Mittag-Leffler functions and some of their special cases, as representatives of the Special Functions of Fractional Calculus [4]. Their convergence in the complex plane \mathbb{C} is studied and Cauchy-Hadamard, Abel and Tauberian type theorems are provided. Recently, these results have been surveyed and discussed in the paper [16]. In the present paper, series in Mittag-Leffler functions and their three-parametric Prabhakar generalizations are also studied and other results for them are discussed.

Practically, this survey paper is a natural continuation of the paper [16] and it contains propositions of Fatou type theorem and about their overconvergence as well. Finding such a kind of results is provoked by the fact that the solutions of some fractional order differential and integral equations can be written in terms of series (or series of integrals) of Mittag-Leffler type functions (as for example, in Kiryakova [3]). The functions (1) and series in them have recently been used to express solutions of the generalized Langevin equation by Sandev, Tomovski and Dubbeldam [19]. Other investigations, connected to various anomalous diffusion and relaxation processes, generalized diffusion and Fokker-Planck-Smoluchowski equations with the corresponding memory kernels, can be seen in the survey paper by Sandev, Chechkin, Kantz and Metzler [18].

2. Preliminary Results

Consider now the functions E_{α} with positive indices $\alpha = n \in \mathbb{N}$ and also generalized Mittag-Leffler functions (1) with integer indices of the kind $\beta = n$;

$n = 0, 1, 2, \dots$, namely:

$$\begin{aligned} E_n(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(nk+1)}, \quad n \in \mathbb{N}; \\ E_{\alpha,n}^{\gamma}(z) &= \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+n)} \frac{z^k}{k!}, \quad \alpha, \gamma \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \quad n \in \mathbb{N}_0. \end{aligned} \quad (2)$$

Since the generalized Mittag-Leffler functions reduce to two-parametric Mittag-Leffler functions for $\gamma = 1$, all the results connected with the three-parametric generalizations, discussed in this survey paper, hold true for the corresponding two-parametric Mittag-Leffler functions.

As it has been noted in [16], the coefficients in E_n are all different from zero, but the situation in $E_{\alpha,n}^{\gamma}$ is not the same. More detailed observation shows that some coefficients there can be zero, depending on γ and n . Namely, the following remark can be written.

Remark 1. *Given a number γ , suppose that some coefficients in $E_{\alpha,n}^{\gamma}$, defined by (2), equal zero. That is, there exists a number $p \in \mathbb{N}_0$, such that the second representation in (2) can be written as follows:*

$$E_{\alpha,n}^{\gamma}(z) = z^p \sum_{k=p}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+n)} \frac{z^{k-p}}{k!}. \quad (3)$$

More precisely, as it is given in [10, 16], if γ is different from zero, then $p = 0$ for each positive integer n and $p = 1$ for $n = 0$. If $\gamma = 0$, the second functions in (2) take the simplest form $E_{\alpha,n}^0(z) = \frac{1}{\Gamma(n)}$ for $n \in \mathbb{N}$, and $E_{\alpha,0}^0(z) = 0$.

Further, let us specify the families of Mittag-Leffler type functions

$$\left\{ \tilde{E}_n(z) \right\}_{n=0}^{\infty}, \quad \left\{ \tilde{E}_{\alpha,n}^{\gamma}(z) \right\}_{n=0}^{\infty}; \quad \alpha, \gamma \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \quad (4)$$

as follows below ($\tilde{E}_0(z) = \tilde{E}_{\alpha,0}^0(z) = 1$, just for completeness), namely:

$$\begin{aligned} \tilde{E}_0(z) &= 1, \quad \tilde{E}_n(z) = z^n E_n(z), \quad n \in \mathbb{N}, \\ \tilde{E}_{\alpha,0}^0(z) &= 1, \quad \tilde{E}_{\alpha,n}^0(z) = \Gamma(n) z^n E_{\alpha,n}^0(z); \quad n \in \mathbb{N}, \\ \tilde{E}_{\alpha,n}^{\gamma}(z) &= \frac{\Gamma(\alpha p+n)}{(\gamma)_p} z^{n-p} E_{\alpha,n}^{\gamma}(z), \quad \gamma \neq 0, \quad n \in \mathbb{N}_0. \end{aligned}$$

In this section we recall some results related to the asymptotic formula for ‘large’ values of indices n for $z, \alpha, \gamma \in \mathbb{C}, \gamma \neq 0$, and $\operatorname{Re}(\alpha) > 0$, applied in proving the main results. Namely (see e.g. in [16] and also [9, 10]) there exist entire functions θ_n and $\theta_{\alpha,n}^\gamma$ such that the functions (2), have the following asymptotic formulae:

$$\begin{aligned}\tilde{E}_n(z) &= z^n(1 + \theta_n(z)) \quad (n \in \mathbb{N}), \\ \tilde{E}_{\alpha,n}^\gamma(z) &= z^n(1 + \theta_{\alpha,n}^\gamma(z)) \quad (n \in \mathbb{N}_0), \\ \text{and } \theta_n(z) &\rightarrow 0, \quad \theta_{\alpha,n}^\gamma(z) \rightarrow 0 \quad \text{as } n \rightarrow \infty;\end{aligned}\tag{5}$$

with the corresponding p , depending on γ and n . Moreover, on the compact subsets of the complex plane \mathbb{C} , the convergence is uniform and

$$\theta_n(z) = O\left(\frac{1}{n!}\right), \quad \theta_{\alpha,n}^\gamma(z) = O\left(\frac{1}{n^{\operatorname{Re}(\alpha)}}\right) \quad (n \in \mathbb{N}).\tag{6}$$

Remark 2. According to the asymptotic formulae (5) and (6), it follows there exists a positive integer M such that the functions $E_n, E_{\alpha,n}^\gamma$ have no zeros for $n > M$, possibly except for the origin.

3. Series in Mittag-Leffler Type Functions and Previous Results about Them

Let us consider series in the functions of the families (4), namely:

$$\sum_{n=0}^{\infty} a_n \tilde{E}_n(z), \quad \sum_{n=0}^{\infty} a_n \tilde{E}_{\alpha,n}^\gamma(z),\tag{7}$$

with complex coefficients a_n ($n = 0, 1, 2, \dots$).

While studying the convergence of the series (7) in the complex plane, theorems corresponding to the classical results for the power series have been proposed. In this section we briefly recall the results given in the survey paper [16]. Let us start with the domain of convergence of the series (7), that it is the open disk $D(0; R) = \{z : |z| < R, z \in \mathbb{C}\}$ with a radius of convergence

$$R = \left(\limsup_{n \rightarrow \infty} (|a_n|)^{1/n} \right)^{-1}.\tag{8}$$

Both series are absolutely convergent in the disk $D(0; R)$ and they are divergent in the domain $|z| > R$. The cases $R = 0$ and $R = \infty$ fall in the general case. Farther, analogously to the classical Abel lemma, if any of the series (7) converges at the point $z_0 \neq 0$, then it is absolutely convergent in the disk $D(0; |z_0|)$. Moreover, inside the disk $D(0; R)$, i.e. on each closed disk $|z| \leq r < R$ (R defined by (8)), the series is uniformly convergent. Further, let $z_0 \in \mathbb{C}$, $0 < |z_0| = R < \infty$, and g_φ be an arbitrary angular domain with size of $2\varphi < \pi$ and with a vertex at the point $z = z_0$, which is symmetric with respect to the straight line passing through the origin and z_0 . Let d_φ be the part of the angular domain g_φ , situated between the angle arms and the arc of the circle centred at the point 0 and touching the arms of the angle. Another interesting result is the Abel type theorem, analogical to the classical Abel theorem for the power series. It refers to the uniform convergence of the series (7) in the set d_φ and the existence of the limits of their sums at the point z_0 from the boundary $C(0; R)$, provided $z \in D(0; R) \cap g_\varphi$, i.e. the limit of the sum of any of these series, convergent at the point z_0 , is equal to the corresponding series sum at the point z_0 . In general, the inverse proposition is not valid, i.e. the existence of the limit of the series at the point z_0 does not necessarily imply the convergence of the series at this point. However, as it is discussed e.g. in [16], under additional conditions, i.e. if $\lim_{n \rightarrow \infty} na_n = 0$, even more if $a_n = O(1/n)$, such a result holds true.

4. Behaviour on the Boundary of the Domains of Convergence

Let $\{a_n\}_{n=0}^\infty$ be a sequence of complex numbers with

$$\limsup_{n \rightarrow \infty} (|a_n|)^{1/n} = R^{-1}, \quad 0 < R < \infty,$$

and $f(z)$ be the sum of the power series $\sum_{n=0}^\infty a_n z^n$ in the open disk $D(0; R)$, i.e.

$$f(z) = \sum_{n=0}^\infty a_n z^n, \quad z \in D(0; R). \quad (9)$$

Definition 3. A point $z_0 \in \partial D(0; R)$ is called *regular* for the function f , if there exist a neighbourhood $U(z_0; \rho)$ and a function $f_{z_0}^* \in \mathcal{H}(U(z_0; \rho))$ (the space of complex-valued functions, holomorphic in the set $U(z_0; \rho)$), such that $f_{z_0}^*(z) = f(z)$ for $z \in U(z_0; \rho) \cap D(0; R)$.

By this definition it follows that the set of regular points of the power series is an open subset of the circle $C(0; R) = \partial D(0; R)$ with respect to the relative topology on $\partial D(0; R)$, i.e. the topology induced by that of \mathbb{C} .

In general, there is no relation between the convergence (divergence) of a power series at points on the boundary of its disk of convergence and the regularity (singularity) of its sum at such points. But under additional conditions on the sequence $\{a_n\}_{n=0}^{\infty}$, such a relation does exist (see for details [6], Vol.1, Ch. 3, §7, 7.3, p. 357), namely, if the coefficients of the power series with the unit disk of convergence tend to the zero, i.e. $\lim_{n \rightarrow \infty} a_n = 0$, then the power series converges, even uniformly, on each arc of the unit circle, all points of which (including the ends of the arc) are regular for the sum of the series.

A result, giving relation between the convergence (divergence) of the series (7) at points on the boundary of its disk of convergence and the regularity (singularity) of its sum at such points is formulated below. Analogical propositions have also been established for series in the Laguerre and Hermite polynomials by Rusev, as well as in Bessel type systems (see e.g. [11, 12]). Here we give such a type of theorem for the Mittag-Leffler type systems (for the line of proof, see [13]) as follows.

Theorem 4 (of Fatou type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the conditions*

$$\lim_{n \rightarrow \infty} a_n = 0, \quad \limsup_{n \rightarrow \infty} (|a_n|)^{1/n} = 1,$$

and $F(z)$ be the sum of any of the series (7) in the unit disk $D(0; 1)$. Let σ be an arbitrary arc of the unit circle $C(0; 1)$ with all its points (including the ends) regular to the function F . Then the corresponding series (7) converges, even uniformly, on the arc σ .

5. Overconvergence Theorem

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with

$$\limsup_{n \rightarrow \infty} (|a_n|)^{1/n} = R^{-1}, \quad 0 < R < \infty,$$

and $f(z)$ be given by (9), i.e. $f(z)$ be the sum of the power series $\sum_{n=0}^{\infty} a_n z^n$ in the open disk $D(0; R)$.

In order to introduce the next definition ([6, Vol. 2, p. 500]) and to expose the results in this section, we first set

$$\begin{aligned} s_p(z) &= \sum_{k=0}^p a_k z^k, & S_p(z) &= \sum_{k=0}^p a_k \tilde{E}_k(z), \\ \text{or resp. } S_p(z) &= \sum_{k=0}^p a_k \tilde{E}_{\alpha, k}^{\gamma}(z), \end{aligned} \quad (10)$$

for all the values $p = 0, 1, 2, \dots$.

Definition 5. A power series with a finite radius of convergence $0 < R < \infty$ is said to be *overconvergent*, if there exist a subsequence $\{s_{p_k}\}_{k=0}^{\infty}$ of the partial-sums sequence $\{s_p\}_{p=0}^{\infty}$ and a region G , containing the open disk $D(0; R)$ as a regular part ($G \cap \partial D(0; R) \neq \emptyset$), such that $\{s_{p_k}\}$ uniformly converges inside G . We say that the function f (or the series (9)), possesses *Hadamard gaps*, if there exist two sequences $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$, having the properties $q_{n-1} \leq p_n \leq q_n/(1 + \theta)$ ($\theta > 0$) and $a_k = 0$ for $p_n < k < q_n$ ($n = 0, 1, 2, \dots$).

Remark 6. To introduce the corresponding notions 'overconvergence' and 'gaps' for the series (7), the expression z^n has to be replaced by $\tilde{E}_n(z)$, respectively $\tilde{E}_{\alpha, n}^{\gamma}(z)$, and the sequence $\{s_{p_k}\}$ by the corresponding sequence $\{S_{p_k}\}$.

Thus, starting with the domain of convergence and series behaviour near its boundary, passing through the possible uniform convergence on an arbitrary closed arc of the boundary, we come to the natural question: "*What type of conditions should be imposed on the power series that ensure the existence of subsequence $\{s_{p_k}\}$, convergent outside the disk of convergence?*". The answer to this question is given in the early 20th century by Ostrowski [7, 8], see also [5]. Namely, one of his classical results states that a given power series with Hadamard gaps and existing regular points on the boundary of convergence disk is overconvergent. We draw the attention to the fact that merely the existence of Hadamard gaps does not imply overconvergence. For example, the power series $\sum_{n=0}^{\infty} a_{k_n} z^{k_n}$ with $k_{n+1} \geq (1 + \theta)k_n$ ($\theta > 0$) and $\limsup_{n \rightarrow \infty} (|a_{k_n}|)^{1/k_n} = 1$ possesses Hadamard gaps but nevertheless it is not overconvergent. Its natural boundary of analyticity is the unit circle $|z| = 1$ and that is nothing but the theorem about the gaps, belonging to Hadamard [2]. These assertions have recently been extended by the author for series in Bessel type functions (see

e.g. [14]). Overconvergence of the series (7) is discussed below. For example, the following statement holds true.

Theorem 7 (of overconvergence). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the condition $\limsup_{n \rightarrow \infty} (|a_n|)^{1/n} = 1$, $F(z)$ be the sum of any of the series (7) in the unit disk $D(0; 1)$, $F(z)$ have at least one regular point, belonging to the circle $C(0; 1)$, and let $F(z)$ possess Hadamard gaps. Then the corresponding series (7) is overconvergent.*

Proof. Here we expose the proof for the first of the series (7) and we only note that the other goes analogously (the details are in [15]). Without loss the generality we suppose that the point $z_0 = 1$ is regular to the function F . That means that F is analytically continuable in a neighbourhood U of the point 1. Denoting $\tilde{U} = U \cup D(0; 1)$, we define the function ψ in the region \tilde{U} by the equality

$$\psi(z) = F(z), \quad z \in D(0; 1),$$

i.e. ψ is a single valued analytical continuation of F in the domain \tilde{U} .

Letting $\theta > 0$ and taking $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ with the properties $q_n \geq (1 + \theta)p_n$ and $a_k = 0$ for $p_n < k < q_n$ ($n = 0, 1, 2, \dots$), we define the auxiliary function

$$\varphi_n(z) = \psi(z) - S_{p_n} = \psi(z) - \sum_{k=0}^{p_n} a_k \tilde{E}_k(z). \quad (11)$$

In order to prove that the sequence $\{S_{p_n}\}$ is uniformly convergent inside the region \tilde{U} , we are going to apply the Hadamard theorem for the three disks [6, Vol. 2].

To this end, taking $0 < \delta < 1/2$ and $0 < \omega < \delta$, we consider the three circles C_1, C_2, C_3 , centered at the point $1/2$ and having respectively radii $1/2 - \delta$, $1/2 + \omega$, $1/2 + \delta$, such that $C_3 \subset \tilde{U}$ and after that set

$$M_{n,j} = \max_{z \in C_j} |\varphi_n(z)| \quad j = 1, 2, 3; \quad M = \max_{z \in C_3} |\psi(z)|.$$

Before evaluating $|\varphi_n(z)|$ we come back to (6). Just mention that since $\lim_{n \rightarrow \infty} (1/n!) = 0$, there exists a number B such that $|1 + \theta_n(z)| \leq B$ for all the values of $n \in \mathbb{N}$ on an arbitrary compact subset of \mathbb{C} . Now, letting $0 < \eta < \delta/2$ implies the existence of $A = A(\eta)$ such that $|a_k| \leq AB^{-1}(1 - \eta)^{-k}$. To find an upper estimation of $|\varphi_n(z)|$ we intend to consider three different cases.

1. First, let $z \in C_1 \subset D(0; 1)$. In the unit disk, according to (11), we have

$$\varphi_n(z) = \sum_{k=q_n}^{\infty} a_k \tilde{E}_k(z).$$

Therefore,

$$\begin{aligned} |\varphi_n(z)| &\leq \sum_{k=q_n}^{\infty} |a_k \tilde{E}_k(z)| = \sum_{k=q_n}^{\infty} |a_k| |1 + \theta_k(z)| |z^k| \\ &\leq A \sum_{k=q_n}^{\infty} (1 - \eta)^{-k} (1 - \delta)^k = A \left(1 - \frac{1 - \delta}{1 - \eta}\right)^{-1} \left(\frac{1 - \delta}{1 - \eta}\right)^{q_n}, \end{aligned}$$

whence

$$M_{n,1} = O\left(\left(\frac{1 - \delta}{1 - \eta}\right)^{q_n}\right) = O\left(\left(\frac{1 - \delta}{1 - \eta}\right)^{(1+\theta)p_n}\right). \quad (12)$$

2. Now, let $z \in C_3$. In this case,

$$\begin{aligned} |\varphi_n(z)| &= |\psi(z) - S_{p_n}| = \left|\psi(z) - \sum_{k=0}^{p_n} a_k \tilde{E}_k(z)\right| \\ &\leq |\psi(z)| + \sum_{k=0}^{p_n} |a_k \tilde{E}_k(z)| \leq M + \sum_{k=0}^{p_n} |a_k| |1 + \theta_k(z)| |z^k| \\ &\leq M + A \sum_{k=0}^{p_n} \left(\frac{1 + \delta}{1 - \eta}\right)^k = O\left(\left(\frac{1 + \delta}{1 - \eta}\right)^{p_n}\right), \end{aligned}$$

and therefore,

$$M_{n,3} = O\left(\left(\frac{1 + \delta}{1 - \eta}\right)^{p_n}\right). \quad (13)$$

3. At last, let $z \in C_2$. Then, in view of (12) and (13) and according to the Hadamard theorem for the three disks (for details see [6, Vol. 2, formula (3.2:2)]), we can write

$$M_{n,2} = O\left(\left(\left(\frac{1 - \delta}{1 - \eta}\right)^{(1+\theta) \ln \frac{1+2\delta}{1+2\omega}} \left(\frac{1 + \delta}{1 - \eta}\right)^{\ln \frac{1+2\omega}{1-2\delta}}\right)^{p_n}\right). \quad (14)$$

Note that the limit of the inner part of (14) is equal to

$$a = (1 - \delta)^{(1+\theta) \ln(1+2\delta)} (1 + \delta)^{-\ln(1-2\delta)} \quad (15)$$

when ω and η tend to 0. Moreover, if δ also tends to 0 then $a < 1$. Indeed, taking the logarithm of a , we have

$$\begin{aligned} \ln a &= (1 + \theta) \ln(1 + 2\delta) \ln(1 - \delta) - \ln(1 - 2\delta) \ln(1 + \delta) \\ &= (1 + \theta)(2\delta + O(\delta^2))(-\delta + O(\delta^2)) - (-2\delta + O(\delta^2))(\delta + O(\delta^2)) \\ &= (1 + \theta)(-2\delta^2 + O(\delta^3)) + 2\delta^2 + O(\delta^3) = -2\theta\delta^2 + O(\delta^3). \end{aligned}$$

Therefore $\ln a < 0$ when $\delta \rightarrow 0$ and for this reason $a < 1$ if δ tends to 0.

That is why, $\lim_{n \rightarrow \infty} M_{n,2} = 0$. Additionally, the sequence $\{S_{p_n}\}_{n=0}^{\infty}$ uniformly converges inside the disk $D(0;1)$ (see Section 3). For these two reasons the sequence $\{S_{p_n}\}$ is uniformly convergent inside the whole region \tilde{U} . \square

After proving the Ostrowski type theorem we formulate the following result of Hadamard type, proved in [15].

Theorem 8 (of Hadamard about the gaps). *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of complex numbers satisfying the condition*

$$\limsup_{n \rightarrow \infty} (|a_{k_n}|)^{1/k_n} = 1,$$

for $k_{n+1} \geq (1 + \theta)k_n$ ($\theta > 0$) and $a_k = 0$ for $k_n < k < k_{n+1}$. Let $F(z)$ be the sum of any of the series (7) in the unit disk $D(0;1)$. Then all the points of the unit circle $C(0;1)$ are singular for the function F , i.e. the unit circle is a natural boundary of analyticity for the corresponding series.

6. Special Cases

In particular, as it has been mentioned in Introduction and Section 2, for $\gamma = 1$ the 3-parametric function $E_{\alpha,\beta}^{\gamma}$ defined by (1) coincides with the Mittag-Leffler function $E_{\alpha,\beta}$, i.e. $\tilde{E}_{\alpha,n}^1 = \tilde{E}_{\alpha,n}$. So, in this case the second of the series (7) takes the form

$$\sum_{n=0}^{\infty} a_n \tilde{E}_{\alpha,n}(z), \quad (16)$$

with the same complex coefficients. Thus, all the results, discussed in this survey and in [16], hold true for the family $\{\widehat{E}_{\alpha,n}\}$ and the series (16).

7. Conclusion

As a conclusion of this survey, all the discussed results can be briefly summarized in the following way. The basic properties of the series, objects of both surveys (this one and [16]), are sufficiently ‘close’ to the corresponding classical results for the power series with the same coefficients, i.e. their behaviour are quite similar to the one of the widely used power series.

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References

- [1] A. Erdélyi et al. (Ed-s), *Higher Transcendental Functions*, **1–3**, McGraw-Hill, New York-Toronto-London (1953–1955).
- [2] J. Hadamard, Essai sur l’étude des fonctions données par leur développement de Taylor, *J. Math. Pures Appl. (4)*, **8** (1892), 101–186.
- [3] V. Kiryakova, Fractional order differential and integral equations with Erdélyi-Kober operators: Explicit solutions by means of the transmutation method, In: *AIP Conf. Proc.* **1410** (2011), 247–258, doi:10.1063/1.3664376.
- [4] V. Kiryakova, The special functions of fractional calculus as generalized fractional calculus operators of some basic functions, *Computers and Mathematics with Appl.*, **59** (2010), 1128–1141, doi:10.1016/j.camwa.2009.05.014.
- [5] R. Kovacheva, Zeros of sequences of partial sums and overconvergence, *Serdica Math. J.*, **34** (2008), 467–482.

- [6] A. Markushevich, *A Theory of Analytic Functions*, Vols. **1, 2** (In Russian), Nauka, Moscow (1967).
- [7] A. Ostrowski, Über eine Eigenschaft gewisser Potenzreihen mit unendlichvielen verschwindenden Koeffizienten, *Berl. Ber.* (1921), 557–565.
- [8] A. Ostrowski, Über Potenzreihen, die überkonvergente Abschnittsfolgen besitzen, *Deutsche Math.-Ver.*, **32** (1923), 185–194.
- [9] J. Paneva-Konovska, Series in Mittag-Leffler functions: inequalities and convergent theorems, *Fract. Calc. Appl. Anal.*, **13** (2010), 403–414, at <http://www.math.bas.bg/~fcaa>.
- [10] J. Paneva-Konovska, Convergence of series in three-parametric Mittag-Leffler functions, *Mathematica Slovaca*, **64**, No 1 (2014), 73–84, DOI: 10.2478/s12175-013-0188-0.
- [11] J. Paneva-Konovska, Fatou theorems for multi-index Bessel series, In: *AIP Conf. Proc.*, **1631** (2014), 303–312, doi: 10.1063/1.4902491.
- [12] J. Paneva-Konovska, A family of hyper-Bessel functions and convergent series in them, *Fract. Calc. Appl. Anal.*, **17**, No 4 (2014), 1001–1015, DOI:10.2478/s13540-014-0211-3.
- [13] J. Paneva-Konovska, On the multi-index Mittag-Leffler series and their boundary behaviour, *Integral Transforms and Special Functions*, **26**, No 3 (2015), 152–164, DOI: 10.1080/10652469.2014.975129.
- [14] J. Paneva-Konovska, Overconvergence of Bessel series, *Compt. rend. Acad. bulg. Sci.*, **68**, No 9 (2015), 1091–1098.
- [15] J. Paneva-Konovska, On the convergence and overconvergence of Mittag-Leffler series, *Adv. Math. Sci. J.*, **4**, No 2 (2015), 175–181.
- [16] J. Paneva-Konovska, Periphery behaviour of series in Mittag-Leffler type functions, I, *Intern. J. Appl. Math.*, **29**, No 1 (2016), 69–78, doi:10.12732/ijam.v29i1.6.
- [17] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, *Yokohama Math. J.*, **19** (1971), 7–15.
- [18] T. Sandev, A. Chechkin, H. Kantz, R. Metzler, Diffusion and Fokker-Planck-Smoluchowski equations with generalized memory kernel, *Fract. Calc. Appl. Anal.*, **18**, No 4 (2015), 1006–1038, DOI: 10.1515/fca-2015-0059.

- [19] T. Sandev, Ž. Tomovski, J. Dubbeldam, Generalized Langevin equation with a three parameter Mittag-Leffler noise, *Physica A*, **390** (2011), 3627–3636, doi: 10.1016/j.physa.2011.05.039.

