

ON THE FOUR-PARAMETER BOND PRICING MODEL

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Abstract: A four-parameter random walk model for the short rate of interest is described in Wilmott et al. [15]. For pricing zero-coupon bonds from the resulting partial differential equation based on this short rate model, a certain form of solution requires the solution of two first-order *nonlinear* ordinary differential equations. In the present paper we show the interesting result that, for obtaining solutions of the bond pricing equation, neither of these two equations requires any differential equation solving techniques; in fact, both these first-order *nonlinear* differential equations can be solved simply by elementary integration. We include the corresponding yield curve and its asymptotic behavior. We identify our results obtained here for the general four-parameter model in the two special cases of Vasicek [14] and Cox, Ingersoll and Ross [4] with those given by these authors.

AMS Subject Classification: 91B24, 91B28, 91B30

Key Words: four-parameter short rate model, bond pricing equation, general solution, yield curve, Vasicek model, Cox-Ingersoll-Ross model

1. Introduction

The first short rate model for the evolution of interest rates was proposed by Vasicek [14], and since then various short rate models have been suggested with various degrees of generalizations.

Vasicek model [14] is a three constant-parameter short rate model described by

$$dr = a(b - r)dt + \sigma dX, \quad (1.1)$$

where a , b and σ are constants, σ is volatility of interest rate and dX is a Wiener process drawn from a normal distribution with mean zero and variance dt . While the drift term indicates Vasicek model incorporates *mean reversion*, however under Vasicek model it is possible for interest rates to become negative. To fix this shortcoming of Vasicek model, Cox, Ingersoll and Ross [4] extended Vasicek model and proposed for the short rate the following stochastic differential equation:

$$dr = a(b - r)dt + \sigma\sqrt{r}dX. \quad (1.2)$$

While, like Vasicek model, Cox-Ingersoll-Ross model has mean reversion, however $\sigma\sqrt{r}$ in volatility term helps prevent interest rates becoming negative or zero. A general treatment is given by Maghsoodi [12] and consistency of the model with an input term structure of interest rates is given by Brigo and Mercurio [2].

Another weakness of the Vasicek model is that while the model produces a term structure as an output but it does not accept today's term structure as input. In the solution of initial-value problems for differential equations, of the many solutions possible, the one that is relevant and useful is the one that also satisfies the initial condition. Likewise in financial mathematics of interest rates and bond pricing, the one solution of the bond pricing equation that is relevant and useful is the one that incorporates today's term structure into the bond pricing model. The first such model was proposed by Ho and Lee [6] with the short rate model:

$$dr = \theta(t)dt + \sigma dX, \quad (1.3)$$

with σ a constant and $\theta(t)$ is a time-dependent parameter which is utilized to fit exactly today's term structure into the Ho-Lee model of pricing zero-coupon bonds. Later, Hull and White [9], by combining the ideas of Vasicek and Ho and Lee, considered an *extended Vasicek model* with the short rate model:

$$dr = [\theta(t) - ar]dt + \sigma dX. \quad (1.4)$$

Again, as in the Ho-Lee model, the time dependent parameter $\theta(t)$ is utilized to fit today's term structure of interest rates in the bond pricing model. For more discussion of interest rate models and pricing of interest rate derivative securities, see Black, Derman and Toy [1]. Duffie and Kan [5], Hughston [7], Hull [8], Klugman [10] and Klugman and Wilmott [11].

We consider a four-parameter random walk model for the short rate of interest as described, for example, in Wilmott et al. [15]. For pricing zero-coupon bonds from the resulting partial differential equation based on this short rate model, a certain form of solution requires the solution of two first-order *nonlinear* ordinary differential equations. In the present paper we show the interesting result that, for obtaining solutions of the bond pricing equation, neither of these two equations requires any differential equation solving techniques; in fact, both these first-order *nonlinear* differential equations can be solved simply by elementary integration. We include the corresponding yield curve and its asymptotic behavior. We identify our results obtained here for the general four-parameter model in the two special cases of Vasicek [14] and Cox, Ingersoll and Ross [4] with those given by these authors.

2. The Four-Parameter Model

We consider the four-parameter random walk model for the short term rate of interest described by the stochastic differential equation:

$$dr = u(r, t) dt + w(r, t) dX, \quad (2.1)$$

where

$$w(r, t) = \sqrt{\alpha r - \beta}, \quad u(r, t) = (\eta - \gamma r) + \lambda w(r, t). \quad (2.2)$$

We are concerned with the pricing of zero-coupon bonds with this four-parameter short rate model. Let $B(t, T)$ denote the value of a zero-coupon bond at time t with maturity T , $t < T$, and value on maturity $B(T, T) = Z$. Though interest rates are random, for a known interest rate,

$$B(t, T) = B(T, T) e^{-\int_t^T r(s) ds}. \quad (2.3)$$

As a measure of future interest rates, the *yield curve* is defined by

$$Y(t, T) = -\frac{1}{T-t} \ln \left(\frac{B(t, T)}{B(T, T)} \right), \quad (2.4)$$

and then the *interest rate implied by the yield curve* is given by

$$r(t, T) = \frac{d}{dT} [Y(t, T) (T - t)]. \quad (2.5)$$

The bond pricing equation providing values of zero-coupon bonds $B(t, T)$, at time $t < T$, is

$$\frac{\partial B}{\partial t} + \frac{1}{2}(\alpha r - \beta) \frac{\partial^2 B}{\partial r^2} + (\eta - \gamma r) \frac{\partial B}{\partial r} - rB = 0. \quad (2.6)$$

Note that λ does not appear in the bond pricing equation (2.6). It will be helpful to introduce *time to expiry* $\tau = T - t$ and set an $f(t, T) = f(T - t) = f(\tau)$. We seek a solution of the bond pricing equation (2.6) in the form:

$$B(t, T) = Ze^{A(t, T) - rC(t, T)}. \quad (2.7)$$

This leads to two first-order *nonlinear* ordinary differential equations for the determination of the functions $A(\tau)$ and $B(\tau)$:

$$\frac{dA(\tau)}{d\tau} = -\eta C(\tau) - \frac{1}{2}\beta C^2(\tau), \quad (2.8)$$

and

$$\frac{dC(\tau)}{d\tau} = -\frac{1}{2}\alpha C^2(\tau) - \gamma C(\tau) + 1, \quad (2.9)$$

with now the initial conditions

$$A(0) = 0 \quad \text{and} \quad C(0) = 0.$$

We note here that Chawla [3] solved (2.9) by first homogenizing the equation and then solving it as a Bernoulli equation with index two. Shreve [13], page 285, first transforms the first-order nonlinear equation (2.9), using an exponential transformation, into a second order linear ordinary differential equation from whose solution is recovered the solution of (2.9). Even though both (2.8) and (2.9) are nonlinear differential equations, no special differential equation solving techniques are needed; in fact, both these equations can be solved simply by elementary integration as we show in the following.

3. Solution of the Bond Pricing Equation

We first consider solution of the nonlinear differential equation (2.9). For $\alpha > 0$, we can write (2.9) as

$$\frac{dC}{C^2 + \frac{2\gamma}{\alpha}C - \frac{2}{\alpha}} = -\frac{1}{2}\alpha d\tau.$$

Factorizing the quadratic expression in the denominator, we get

$$\frac{dC}{(C - a)(C + b)} = -\frac{1}{2}\alpha d\tau,$$

where we have set

$$\psi = \sqrt{\gamma^2 + 2\alpha}, \quad a = \frac{-\gamma + \psi}{\alpha}, \quad b = \frac{\gamma + \psi}{\alpha}.$$

Partial fractioning gives

$$\left(\frac{1}{C-a} - \frac{1}{C+b} \right) dC = -\frac{1}{2}\alpha(a+b) d\tau = -\psi d\tau,$$

since $a+b = \frac{2\psi}{\alpha}$. Integrating we have

$$\frac{C-a}{C+b} = k_1 e^{-\psi\tau},$$

for a constant k_1 . Applying the initial condition $C(0) = 0$ we have $k_1 = \frac{-a}{b}$, therefore

$$\frac{C-a}{C+b} = \left(\frac{-a}{b} \right) e^{-\psi\tau}.$$

Solving for C we have

$$C(b + ae^{-\psi\tau}) = ab(1 - e^{-\psi\tau}).$$

Since $ab = \frac{2}{\alpha}$, we obtain the solution of (2.9) as

$$C(\tau) = \frac{2}{\alpha} \frac{1 - e^{-\psi\tau}}{b + ae^{-\psi\tau}}. \quad (3.1)$$

We next consider the solution of (2.8). With the initial condition $A(0) = 0$, integrating (2.8) from 0 to τ we have

$$A(\tau) = -\eta I(C) - \frac{1}{2}\beta I(C^2), \quad (3.2)$$

where we have set

$$I(C) = \int_0^\tau C(u) du, \quad I(C^2) = \int_0^\tau C^2(u) du.$$

First consider evaluation of $I(C)$. With (3.1) this can be written as

$$\begin{aligned} I(C) &= \frac{2}{\alpha} \int_0^\tau \frac{1 - e^{-\psi u}}{b + ae^{-\psi u}} du \\ &= \frac{2}{\alpha} \int_0^\tau \frac{e^{\psi u}}{be^{\psi u} + a} du - \frac{2}{\alpha} \int_0^\tau \frac{e^{-\psi u}}{b + ae^{-\psi u}} du. \end{aligned}$$

Performing the two integrations we get

$$I(C) = \frac{2}{\alpha b \psi} \ln \left(\frac{be^{\psi\tau} + a}{b + a} \right) + \frac{2}{\alpha a \psi} \ln \left(\frac{b + ae^{-\psi\tau}}{b + a} \right).$$

This can be written as

$$\begin{aligned} I(C) &= \frac{2}{\alpha b \psi} \left[\psi \tau + \ln \left(\frac{b + ae^{-\psi \tau}}{b + a} \right) \right] + \frac{2}{\alpha a \psi} \ln \left(\frac{b + ae^{-\psi \tau}}{b + a} \right) \\ &= \frac{2}{\alpha b} \tau + \frac{2}{\alpha \psi} \left(\frac{1}{b} + \frac{1}{a} \right) \ln \left(\frac{b + ae^{-\psi \tau}}{b + a} \right). \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{\alpha b} &= \frac{1}{\psi + \gamma} = \frac{\psi - \gamma}{2\alpha} = \frac{a}{2}, \\ \frac{1}{b} + \frac{1}{a} &= \frac{a + b}{ab} = \frac{2\psi/\alpha}{2/\alpha} = \psi, \end{aligned}$$

therefore

$$I(C) = a\tau + \frac{2}{\alpha} \ln \left(\frac{b + ae^{-\psi \tau}}{b + a} \right). \quad (3.3)$$

For the evaluation of $I(C^2)$, substituting for $C^2(u)$ from the differential equation in (2.9) we have

$$I(C^2) = -\frac{2}{\alpha} \int_0^\tau \left[\frac{dC(u)}{du} + \gamma C(u) - 1 \right].$$

With the initial condition $C(0) = 0$, we get

$$I(C^2) = -\frac{2}{\alpha} [C(\tau) + \gamma I(C) - \tau]. \quad (3.4)$$

Substituting from (3.3) and (3.4) into (3.2) we have

$$\begin{aligned} A(\tau) &= -\eta I(C) + \frac{\beta}{\alpha} [C(\tau) + \gamma I(C) - \tau] \\ &= \left(-\eta + \frac{\beta\gamma}{\alpha} \right) I(C) + \frac{\beta}{\alpha} (C(\tau) - \tau) \\ &= \left(-\eta + \frac{\beta\gamma}{\alpha} \right) \left[a\tau + \frac{2}{\alpha} \ln \left(\frac{b + ae^{-\psi \tau}}{b + a} \right) \right] + \frac{\beta}{\alpha} (C(\tau) - \tau), \end{aligned}$$

from which we finally obtain

$$A(\tau) = \left(\frac{\delta a - \beta}{\alpha} \right) \tau + \frac{\beta}{\alpha} C(\tau) + \frac{2\delta}{\alpha^2} \ln \left(\frac{b + ae^{-\psi \tau}}{b + a} \right), \quad (3.5)$$

$$\delta = \beta\gamma - \alpha\eta.$$

Thus, for the four-parameter model price of a zero-coupon bond is given by (2.7) with $C(\tau)$ and $A(\tau)$ given by (3.1) and (3.5).

With the values of $A(\tau)$ and $C(\tau)$ given by (3.5) and (3.1), from (2.4) the yield curve for the four-parameter bond pricing model is given by

$$\begin{aligned} Y(t, T) &= -\frac{1}{\tau} [A(\tau) - rC(\tau)] \\ &= \left(\frac{\beta - \delta a}{\alpha} \right) - \frac{1}{\tau} \left[\left(\frac{\beta}{\alpha} - r \right) C(\tau) + \frac{2\delta}{\alpha^2} \ln \left(\frac{b + ae^{-\psi\tau}}{b + a} \right) \right]. \end{aligned} \quad (3.6)$$

Since

$$\lim_{\tau \rightarrow \infty} C(\tau) = \frac{2}{\alpha b} \text{ and } \lim_{\tau \rightarrow \infty} \ln \left(\frac{b + ae^{-\psi\tau}}{b + a} \right) = \ln \left(\frac{b}{b + a} \right),$$

it is clear that asymptotic ($\tau \rightarrow \infty$) behavior of the yield curve for the four-parameter model is

$$Y(t, T) \sim \left(\frac{\beta - \delta a}{\alpha} \right). \quad (3.7)$$

This is positive if $\beta > \delta a$.

3.1. Solution for the Vasicek Case

We next consider the special case of Vasicek model [14] which corresponds to random walk for the short rate (2.1)-(2.2) with $\alpha = 0$. For $\alpha = 0$ equation (2.9) simplifies to

$$\frac{dC}{\gamma C - 1} = -d\tau.$$

Integrating we get

$$\gamma C - 1 = k_2 e^{-\gamma\tau}.$$

The initial condition $C(0) = 0$ gives $k_2 = -1$, and the solution now called $C_V(\tau)$ is, for $\gamma > 0$,

$$C_V(\tau) = \frac{1 - e^{-\gamma\tau}}{\gamma}. \quad (3.8)$$

Next, for the solution of (2.8) with the initial condition $A(0) = 0$, integrating from 0 to τ the solution now called $A_V(\tau)$ is given as

$$A_V(\tau) = -\eta I(C_V) - \frac{1}{2} \beta I(C_V^2). \quad (3.9)$$

Note that with $\alpha = 0$ from (2.9) we have

$$C_V = \frac{1}{\gamma} \left(1 - \frac{dC_V}{d\tau} \right). \quad (3.10)$$

With (3.10) we immediately have

$$\begin{aligned} I(C_V) &= \int_0^\tau C_V(u) du = \frac{1}{\gamma} \int_0^\tau \left(1 - \frac{dC_V(u)}{du} \right) du \\ &= \frac{1}{\gamma} (\tau - C_V(\tau)). \end{aligned} \quad (3.11)$$

Again, with (3.10) we obtain

$$\begin{aligned} I(C_V^2) &= \int_0^\tau C_V^2(u) du = \frac{1}{\gamma} \int_0^\tau C_V(u) \left(1 - \frac{dC_V(u)}{du} \right) du \\ &= \frac{1}{\gamma} \left[I(C_V) - \frac{1}{2} C_V^2(\tau) \right]. \end{aligned} \quad (3.12)$$

With (3.12) from (3.9) we get

$$\begin{aligned} A_V(\tau) &= -\eta I(C_V) - \frac{\beta}{2\gamma} \left[I(C_V) - \frac{1}{2} C_V^2(\tau) \right] \\ &= -\left(\eta + \frac{\beta}{2\gamma} \right) I(C_V) + \frac{\beta}{4\gamma} C_V^2(\tau). \end{aligned}$$

Substituting for $I(C_V)$ from (3.11) we finally get

$$A_V(\tau) = \frac{1}{\gamma} \left(\eta + \frac{\beta}{2\gamma} \right) [C_V(\tau) - \tau] + \frac{\beta}{4\gamma} C_V^2(\tau). \quad (3.13)$$

Thus, for the Vasicek model the price of a zero-coupon bond is given by (2.7) where $C_V(\tau)$ and $A_V(\tau)$ are given by (3.8) and (3.13). The yield curve for the Vasicek model is

$$\begin{aligned} Y_V(t, T) &= -\frac{1}{\tau} [A_V(\tau) - rC_V(\tau)] \\ &= \frac{1}{\gamma} \left(\eta + \frac{\beta}{2\gamma} \right) \left[1 - \frac{C_V(\tau)}{\tau} \right] - \frac{\beta}{4\gamma} \frac{C_V^2(\tau)}{\tau} + \frac{r}{\tau} C_V(\tau). \end{aligned} \quad (3.14)$$

Since

$$\lim_{\tau \rightarrow \infty} C_V(\tau) = \frac{1}{\gamma},$$

asymptotic behavior of the Vasicek yield curve is

$$Y_V(t, T) \sim \frac{1}{\gamma} \left(\eta + \frac{\beta}{2\gamma} \right). \quad (3.15)$$

If, in addition to $\alpha = 0$, we set $\gamma = 0$ we have the Ho and Lee [6] model of short rate (1.3) with a constant θ . We denote the corresponding results by a subscript *HLL*. Now, with $\alpha = \gamma = 0$, integrating (2.9) from 0 to τ with the initial condition $C(0) = 0$ we have

$$C_{HL}(\tau) = \int_0^\tau du = \tau,$$

while integration of (2.8) with the initial condition $A(0) = 0$ gives

$$\begin{aligned} A_{HL}(\tau) &= -\eta \int_0^\tau u du - \frac{1}{2}\beta \int_0^\tau u^2 du \\ &= -\frac{1}{2}\eta\tau^2 - \frac{1}{6}\beta\tau^3, \end{aligned}$$

and the yield curve for the Ho-Lee model is

$$\begin{aligned} Y_{HL}(t, T) &= -\frac{1}{\tau} [A_{HL}(\tau) - rC_{HL}(\tau)] \\ &= r + \frac{1}{2}\eta\tau + \frac{1}{6}\beta\tau^2. \end{aligned}$$

In order that the yield remains finite for $\tau \rightarrow \infty$ we must have, in addition, $\eta = 0$ and $\beta = 0$, implying an asymptotic yield with constant rate of interest:

$$Y_{HL}(t, T) \sim r.$$

3.2. Solution for the Cox-Ingersoll-Ross Case

The special case of Cox-Ingersoll-Ross model [4] corresponds to random walk for the short rate (2.1)-(2.2) with $\beta = 0$. Now, we need not perform any new calculations and the results for this case can simply be obtained by substituting $\beta = 0$ in our general four-parameter model. We denote the corresponding results by putting a subscript *CIR*. Note that solution of (2.9) remains the same as obtained in (3.1), thus

$$C_{CIR}(\tau) = \frac{2}{\alpha} \frac{1 - e^{-\psi\tau}}{b + ae^{-\psi\tau}}. \quad (3.16)$$

With $\beta = 0$, from (3.5) we have

$$A_{CIR}(\tau) = -\eta \left[a\tau + \frac{2}{\alpha} \ln \left(\frac{b + ae^{-\psi\tau}}{b + a} \right) \right]. \quad (3.17)$$

From (3.6), with $\beta = 0$, the yield curve for the Cox-Ingersoll-Ross model is

$$Y_{CIR}(t, T) = \eta a + \frac{1}{\tau} \left[rC_{CIR}(\tau) + \frac{2\eta}{\alpha} \ln \left(\frac{b + ae^{-\psi\tau}}{b + a} \right) \right], \quad (3.18)$$

with asymptotic value

$$Y_{CIR}(t, T) \sim \eta a. \quad (3.19)$$

3.3. Behavior of the Price of a Zero-Coupon Bond

We show here analytically that the value of a zero-coupon bond $B(t, T)$ decreases steadily, subject to variation in the value of $r(t)$, from its value Z at maturity T down to a value at time t .

For $\alpha > 0$, from (3.1) $C(\tau) > 0$. If $\alpha = 0$, from (3.8) $C_V(\tau) > 0$ for $\gamma > 0$; if in addition $\gamma = 0$, then $C_{HL}(\tau) > 0$. So, $C(\tau)$ is always positive. Now, from (2.8) for $\eta > 0$ and $\beta \geq 0$, $\frac{dA}{d\tau} < 0$. Since $A(0) = 0$ it follows that $A(\tau)$ is negative for $\tau > 0$ and that $A(\tau)$ monotonically increases negatively with τ increasing.

As for $C(\tau)$, we may write (2.9), as in Section 3, as

$$\frac{dC(\tau)}{d\tau} = \frac{1}{2}\alpha(a - C)(C + b).$$

For $\alpha > 0$, clearly $C + b > 0$. For $a - C$, with (3.1) we can write it as

$$\begin{aligned} a - C &= a - \frac{2}{\alpha} \frac{1 - e^{-\psi\tau}}{b + ae^{-\psi\tau}} \\ &= \frac{Num}{b + ae^{-\psi\tau}}, \end{aligned}$$

where we have set

$$Num = ab + a^2e^{-\psi\tau} - \frac{2}{\alpha} (1 - e^{-\psi\tau}).$$

Since $ab = \frac{2}{\alpha}$,

$$Num = \left(a^2 + \frac{2}{\alpha} \right) e^{-\psi\tau}.$$

Now,

$$a^2 = \frac{1}{\alpha^2} (\psi^2 + \gamma^2 - 2\gamma\psi),$$

and substituting for ψ^2 ,

$$\begin{aligned} a^2 &= \frac{2}{\alpha^2} (\alpha - \gamma (\psi - \gamma)) \\ &= \frac{2}{\alpha} (1 - \gamma a). \end{aligned}$$

Therefore,

$$Num = \frac{2}{\alpha} (2 - \gamma a) e^{-\psi\tau}.$$

Again, since

$$\begin{aligned} 2 - \gamma a &= \frac{2\alpha - \gamma (\psi - \gamma)}{\alpha} \\ &= \frac{\psi^2 - \gamma\psi}{\alpha} = \psi a, \end{aligned}$$

we get

$$Num = \frac{2}{\alpha} \psi a e^{-\psi\tau}.$$

We thus obtain

$$a - C = \frac{2}{\alpha} \psi a \frac{e^{-\psi\tau}}{b + a e^{-\psi\tau}}.$$

This shows that $a - C > 0$ for $\alpha > 0$. So, for $\alpha > 0$, $\frac{dC}{d\tau} > 0$ implying that $C(\tau)$ monotonically increases with τ increasing. For $\alpha = 0$, for $\gamma > 0$ from (3.8) we have $dC_V/d\tau = e^{-\gamma\tau}$; if in addition $\gamma = 0$, then $dC_{HL}/d\tau = 1$, implying that in both these cases also $C(\tau) \uparrow$ with $\tau \uparrow$.

We have thus shown that in all cases $A(\tau)$ steadily increases negatively and $C(\tau)$ steadily increases positively with τ increasing. It follows that the price of a zero-coupon bond $B(t, T)$ in the four-parameter model given by (2.7) decreases steadily, subject to variation in the value of $r(t)$, from its value Z at maturity to a value at time t .

4. Identification of Results in Two Special Cases

For special cases of the four-parameter random walk (2.1)-(2.2), solutions of the bond pricing equation have been given using different notations with different

forms of solution. In this section we identify our results obtained here for the general four-parameter model in the two special cases of Vasicek [14] and Cox-Ingersoll-Ross [4] with those given by these authors. We note that alternatively bond price is written as

$$P(t, T) = A(t, T) e^{-rB(t, T)}.$$

So, in our notation, with $Z = 1$, this corresponds to our

$$B(t, T) \rightarrow P(t, T), \quad A(t, T) \rightarrow \ln A(t, T), \quad C(t, T) \rightarrow B(t, T).$$

Now, the Vasicek short rate model (1.1), in our notation corresponds to

$$\alpha = 0, \quad \beta = -\sigma^2, \quad \gamma = a, \quad \eta = ab.$$

From equation (3.8) with $\gamma = a$ we have

$$C_V(\tau) = \frac{1 - e^{-a\tau}}{a}.$$

Again, from (3.13), switching to the above notation, we get

$$A_V(\tau) = \frac{1}{a} \left(ab - \frac{\sigma^2}{2a} \right) [C_V(\tau) - \tau] - \frac{\sigma^2}{4a} C_V^2(\tau).$$

These results agree with those given for the Vasicek model in Hull [8].

If in addition $a = 0$, then from the results following equation (3.15), with $\eta = ab = 0$ we have

$$C_{HL}(\tau) = \tau, \quad A_{HL}(\tau) = \frac{\sigma^2}{6} \tau^3,$$

which agree with the results given in Hull [8].

Next, the Cox-Ingersoll-Ross short rate model (1.2), in our notation corresponds to

$$\begin{aligned} \beta &= 0, \quad \alpha = \sigma^2, \quad \eta = ab, \quad \gamma = a, \\ \psi &\rightarrow \gamma = \sqrt{a^2 + 2\sigma^2}, \quad b = \frac{\gamma + a}{\sigma^2}, \quad a = \frac{\gamma - a}{\sigma^2}. \end{aligned}$$

Switching to the above notation, from (3.16) we have

$$\begin{aligned} C_{CIR}(\tau) &= \frac{2}{\sigma^2} \frac{1 - e^{-\gamma\tau}}{b + ae^{-\gamma\tau}} \\ &= \frac{2(e^{\gamma\tau} - 1)}{(\gamma + a)(e^{\gamma\tau} - 1) + 2\gamma}. \end{aligned}$$

From (3.17) we have

$$A_{CIR}(\tau) = -\eta \left[a\tau + \frac{2}{\alpha} \ln \left(\frac{b + ae^{-\psi\tau}}{b + a} \right) \right].$$

Combining the two terms in square brackets, this can be written as

$$A_{CIR}(\tau) = -\frac{2\eta}{\alpha} \ln \left(\frac{be^{\psi\tau} + a}{(b + a)e^{(\psi - \alpha a/2)\tau}} \right).$$

Since $\psi - \frac{\alpha a}{2} = \frac{\psi + \gamma}{2}$, and simplifying we get

$$\begin{aligned} A_{CIR}(\tau) &= -\frac{2\eta}{\alpha} \ln \left(\frac{(\psi + \gamma)(e^{\psi\tau} - 1) + 2\psi}{2\psi e^{((\psi + \gamma)/2)\tau}} \right) \\ &= \left(\frac{2\psi e^{((\psi + \gamma)/2)\tau}}{(\psi + \gamma)(e^{\psi\tau} - 1) + 2\psi} \right)^{\frac{2\eta}{\alpha}}. \end{aligned}$$

Finally switching to the above notation we have

$$A_{CIR}(\tau) = \left(\frac{2\gamma e^{(\gamma + a)\tau/2}}{(\gamma + a)(e^{\gamma\tau} - 1) + 2\gamma} \right)^{2ab/\sigma^2}.$$

These results for the Cox-Ingersoll-Ross model agree with those given in Hull [8].

4.1. The Case of Fitting Initial Yield

We also include identification of results obtained in Chawla [3] with those of Ho and Lee model [6] and the extended Vasicek model of Hull and White [9] in the case of fitting today's yield to the four-parameter model with short rate (2.1)-(2.2) for the case $\alpha = 0$.

The idea is to treat η as a function of time and utilize it to fit today's (at $t^* = 0$) term structure of interest rates into the bond pricing model. For the purpose, write equation (3.2) as

$$A(t, T) = - \int_0^T \eta(s) C_V(s, T) ds - \frac{1}{2} \beta I(C_V^2), \quad (4.1)$$

where from (3.11) and (3.12),

$$I(C_V^2) = \frac{1}{\gamma} \left[\frac{1}{\gamma} (\tau - C_V(\tau)) - \frac{1}{2} C_V^2(\tau) \right].$$

Fitting today's yield from (2.4):

$$Y(0, T) = -\frac{1}{T} (A(0, T) - r(0) C(0, T)),$$

to (4.1) we can write

$$\int_0^T \eta(s) C_V(s, T) ds = F^*(T), \quad (4.2)$$

where we have set

$$F^*(T) = TY(0, T) - r(0) C_V(0, T) - \frac{1}{2} \beta I(C_V^2).$$

We solve (4.2) for $\eta^* = \eta(T)$ and get the corresponding $A = A^*(t, T)$ from (4.1).

From Chawla [3] we have (with minor correction):

$$C_V(\tau) = \frac{1 - e^{-\gamma\tau}}{\gamma},$$

$$\eta^*(t) = \frac{d}{dt} r(0, t) + \gamma r(0, t) - \frac{1}{2} \beta C_V(0, t) (1 + e^{-\gamma t}), \quad (4.3)$$

and, with the simplification:

$$C_V^2(\tau) - \{C_V(0, T) - C_V(0, t)\}^2 = C_V^2(\tau) (1 - e^{-2\gamma\tau}),$$

that

$$A^*(t, T) = -f(0, t, T) \tau + r(0, t) C_V(\tau) + \frac{\beta}{4\gamma} C_V^2(\tau) (1 - e^{-2\gamma\tau}). \quad (4.4)$$

Note that $r(0, t) = F(0, t)$ is *forward rate* at time t and $f(0, t, T)$ is the *forward yield* which with (2.4) can be written as

$$\begin{aligned} f(0, t, T) &= \frac{Y(0, T) T - Y(0, t) t}{T - t} \\ &= -\frac{1}{\tau} \ln \left(\frac{B(0, T)}{B(0, t)} \right). \end{aligned}$$

For the Ho and Lee model [4], since $C_{HL}(\tau) = \tau$ and

$$\lim_{\gamma \rightarrow 0} \left(\frac{1 - e^{-2\gamma t}}{\gamma} \right) = 2t,$$

from (4.3) and (4.4) we get

$$\theta^*(t) = \frac{d}{dt}r(0, t) + \sigma^2 t,$$

and

$$A^*(t, T) = -f(0, t, T)\tau + r(0, t)C_{HL}(\tau) - \frac{\sigma^2}{2}t\tau^2.$$

These results agree with those given for the Ho and Lee model in Hull [8].

For the extended Vasicek model of Hull and White [9], from (4.3) and (4.4), with $C_V(\tau) = \frac{1-e^{-a\tau}}{a}$, we have

$$\theta^*(t) = \frac{d}{dt}r(0, t) + ar(0, t) + \frac{\sigma^2}{2}C_V(0, t)(1 + e^{-at}),$$

and

$$A^*(t, T) = -f(0, t, T)\tau + r(0, t)C_V(\tau) - \frac{\sigma^2}{4a}C_V^2(\tau)(1 - e^{-2at}).$$

These results agree with those given for the extended Vasicek model of Hull and White in Hull [8].

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