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BLOW-UP TIME OF SOLUTIONS FOR SOME NONLINEAR PARABOLIC EQUATIONS

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Abstract: In this paper, we consider the following initial-boundary value problem

$$\begin{cases} u_t = \varepsilon\Delta u + b(t)f(u) \quad \text{in} \quad \Omega\times(0,T), \\ u = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \end{cases}$$

where $b \in C^1(\mathbb{R}_+)$, $b(t) \geq b_0 > 0$, $b^{'}(t) \geq 0$ for $t \geq 0$, ε is a positive parameter, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $f(s)$ is positive, nondecreasing, convex function for positive values of s and $\int^{\infty}\frac{ds}{f(s)} < \infty$. We show that if ε is small enough, the solution u of the above problem blows up in a finite time and its blow-up time tends to the one of the solution of the following differential equation

$$\begin{cases} \alpha^{'}(t) = b(t)f(\alpha(t)), \\ \alpha(0) = M, \end{cases}$$

as ε goes to zero, where $M = \sup_{x\in\Omega} u_0(x)$.

Finally, we give some numerical results to illustrate our analysis.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem for a nonlinear parabolic equation of the form

$$u_t = \varepsilon \Delta u + b(t) f(u) \quad \text{in} \quad \Omega \times (0, T), \tag{1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{2}$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \Omega, \tag{3}$$

which models the temperature distribution of a large number of physical phenomena from physics, chemistry and biology. The term $b(t)f(u)$ represents the nonlinear heat generation and $f(s)$ is a positive, increasing, convex function for the positive values of s, $\int^{+\infty} \frac{ds}{f(s)} < +\infty$, $b \in C^1(\mathbb{R}_+)$, $b(t) \geq b_0 > 0$, $b'(t) \geq 0$ for $t \geq 0$.

The initial data $u_0 \in C^1(\overline{\Omega})$, $u_0(x) = 0$ on Ω. Here $(0, T)$ is the maximal time interval on which the solution u exists. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite the solution u develops a singularity in a finite time, namely

$$\lim_{t \to T} \|u(., t)\|_\infty = +\infty$$

where $\|u(., t)\|_\infty = \sup_{x \in \Omega} |u(x, t)|$. In this last case, we say that the solution u blows up in a finite time and the time T is called the blow-up time of the solution u. Using standard methods based on the maximum principle, it is not hard to prove the local existence and the uniqueness of the solution (see for instance [11]). Solutions of nonlinear parabolic equations which blow up in a finite time have been the subject of investigations of many authors (see [4], [6]–[10], [12], [13], [15], [17]–[21] and the references cited therein). In particular in [8], Friedman and Lacey have considered the problem (1)–(3) in the case where $b(t) = 1$ and $f(0) > 0$. Under some additional conditions on the initial data, they have shown that the solution u of (1)–(3) blows up in a finite time and its blow-up time goes to the one of the solution of the following differential equation

$$\alpha^{'}(t) = f(\alpha(t)), \quad \alpha(0) = M, \tag{4}$$

as ε tends to zero, where $M = \sup_{x\in\Omega} u_0(x)$.

Let us notice that when $u_0(x) = 0$, the result in [8] is not valid. On the other hand, the case where $f(0) = 0$ has not been treated but Friedman and Lacey have noticed that it is possible to extend their result if the solution is increasing in t. The proof developed in [8] are based on the construction of upper and lower solutions. In this paper, we obtain the same result using both a modification of Kaplan's method (see [10]) and a method based on the construction of upper solutions. These methods are simple and may be generalized to other classes of parabolic equations. We have also handled the case where $u_0(x) = 0$ and the one where $f(0) = 0$.

Our paper is written in the following manner. In the next Section 2, we show that when ε is sufficiently small, the solution u of (1)–(3) blows up in a finite time and its blow-up time goes to the one of the solution of the differential equation in (4) when ε tends to zero. Finally, in the last Section 3 we give some numerical results to illustrate our analysis.

2. Blow-up solutions

In this section, under some assumptions, we show that the solution u of the problem (1)–(3) blows up in a finite time for ε sufficiently small. In addition, we prove that its blow-up time tends to the one of the solution of a certain differential equation as ε goes to zero.

Before starting, let us recall a well known result.

Consider the eigenvalue problem

$$-\Delta\varphi(x) = \lambda\varphi(x) \quad \text{in} \quad \Omega, \tag{5}$$

$$\varphi(x) = 0 \quad \text{on} \quad \partial\Omega, \tag{6}$$

$$\varphi(x) > 0 \quad \text{in} \quad \Omega. \tag{7}$$

We know that the above problem has a solution (φ, λ) such that $\lambda > 0$. Without loss of generality, we may suppose that $\int_\Omega \varphi(x)dx = 1$.

Our first result on blow-up is the following.

Theorem 1. *Assume that $u_0(x) = 0$ and $f(0) > 0$. Suppose that $\varepsilon < \frac{1}{A}$, where $A = \frac{\lambda}{b_0}\int_0^\infty \frac{ds}{f(s)}$. Then the solution u of (1)–(3) blows up in a finite time and its blow-up time T satisfies the following estimates*

$$0 \leq T - T_0 \leq \varepsilon A T_0 + o(\varepsilon),$$

where $T_0 = \int_0^\infty \frac{ds}{f(s)}$ is the blow-up time of the solution $\alpha(t)$ of the differential equation defined as follows

$$\begin{cases} \alpha'(t) = b(t)f(\alpha(t)), & t > 0, \\ \alpha(0) = 0. \end{cases}$$

Proof. Since $(0, T)$ is the maximal time interval on which $\|u(.,t)\|_\infty$ is finite. Our aim is to show that T is finite and satisfies the above estimates. Since the initial data $u_0(x)$ is nonnegative in Ω. From the maximum principle, u is also nonnegative in $\Omega \times (0, T)$. Introduce the function $v(t)$ defined as follows

$$v(t) = \int_\Omega u(x,t)\varphi(x)dx \quad \text{for} \quad t \in (0, T).$$

Differentiating v in t and using (1), we have

$$v'(t) = \varepsilon \int_\Omega \varphi(x)\Delta u(x,t)dx + b(t)\int_\Omega f(u(x,t))\varphi(x)dx.$$

Apply Green's formula to obtain

$$v'(t) = \varepsilon \int_\Omega u(x,t)\Delta\varphi(x)dx + b(t)\int_\Omega f(u(x,t))\varphi(x)dx.$$

It follows from (5) and Jensen's inequality that

$$v'(t) \geq -\lambda\varepsilon v(t) + b(t)f(v(t)),$$

which implies that

$$v'(t) \geq b(t)f(v(t))\left(1 - \frac{\lambda\varepsilon v(t)}{b(t)f(v(t))}\right).$$

We observe that $b(t) \geq b_0 > 0$ and

$$\int_0^\infty \frac{dt}{f(t)} \geq \sup_{s\geq 0}\int_0^s \frac{dt}{f(t)} \geq \sup_{t\geq 0}\frac{t}{f(t)}$$

because $f(s)$ is a nondecreasing function for $s \geq 0$. We deduce that

$$v^{'}(t) \geq b(t)f(v(t))(1-A\varepsilon) \quad \text{for} \quad t \in (0,T).$$

Set

$$w(t) = v(\frac{t}{1-A\varepsilon}) \quad \text{for} \quad t \in (0,(1-A\varepsilon)T).$$

A straightforward computation reveals that

$$w^{'}(t) \geq b(\frac{t}{1-A\varepsilon})f(w(t)) \quad \text{for} \quad t \in (0,(1-A\varepsilon)T),$$

$$w(0) = 0.$$

Since $b(s)$ is nondecreasing for $s \geq 0$, we arrive at

$$w^{'}(t) \geq b(t)f(w(t)) \quad \text{for} \quad t \in (0,(1-A\varepsilon)T),$$

$$w(0) = 0.$$

Apply the maximum principle to obtain

$$w(t) \geq \alpha(t) \quad \text{for} \quad t \in (0,T_*)$$

where $T_* = \min\{T_0,(1-\varepsilon A)T\}$. We deduce that

$$T \leq \frac{T_0}{1-\varepsilon A}. \tag{8}$$

Indeed, suppose that $T > \frac{T_0}{1-A\varepsilon} \geq T^{'}$. We get

$$\|u(.,T^{'})\|_\infty \geq v(T^{'}) = w(T_0) = \alpha(T_0) = +\infty$$

which contradicts the fact that $(0,T)$ is the maximal time interval of existence of the solution u. On the other hand, consider the function $z(x,t)$ defined as follows

$$z(x,t) = \alpha(t) \quad \text{in} \quad \overline{\Omega} \times (0,T_0).$$

It is easy to check that

$$z_t(x,t) = \varepsilon\Delta z(x,t) + b(t)f(z(x,t)) \geq 0 \quad \text{in} \quad \Omega \times (0,T_0),$$

$$z(x,t) \geq 0 \quad \text{on} \quad \partial\Omega \times (0, T_0),$$

$$z(x,0) \geq u(x,0) \quad \text{in} \quad \Omega.$$

We deduce from the maximum principle that

$$0 \leq u(x,t) \leq z(x,t) = \alpha(t) \quad \text{in} \quad \Omega \times (0, T^0),$$

where $T^0 = \min\{T, T_0\}$. It follows that

$$T \geq T_0. \tag{9}$$

Indeed, suppose that $T < T_0$. We obtain $\|u(.,T)\|_\infty \leq \alpha(T) < +\infty$. But this contradicts the fact that $(0,T)$ is the maximal time interval of existence of $\alpha(t)$. We conclude

$$T_0 \leq T \leq \frac{T_0}{1 - \varepsilon A}. \tag{10}$$

Apply Taylor's expansion to obtain

$$\frac{1}{1 - \varepsilon A} = 1 + \varepsilon A + o(\varepsilon).$$

Use (10) and the above relation to complete the rest of the proof. □

Now, let us consider the case where the initial data is not null. Let $a \in \Omega$ be such that $u_0(x) = M > 0$ and consider the following eigenvalue value problem

$$-\Delta\psi(x) = \lambda_\delta \psi(x) \quad \text{in} \quad B(a, \delta), \tag{11}$$

$$\psi(x) = 0 \quad \text{on} \quad \partial B(a, \delta), \tag{12}$$

$$\psi(x) > 0 \quad \text{in} \quad B(a, \delta), \tag{13}$$

where $\delta > 0$, such that, $B(a,\delta) = \{x \in \mathbb{R}^N;\ \| x - a \| < \delta\} \subset \Omega$. It is well known that the above eigenvalue problem admits a solution (ψ, λ_δ) such that $\lambda_\delta = \frac{\lambda_1}{\delta^2}$, where λ_1 is the eigenvalue for the above eigenvalue problem for $\delta = 1$.

Theorem 2. *Suppose that $f(0) = 0$ and $\sup_{x \in \Omega} u_0(x) = M > 0$. Let K be an upper bound of the first derivative of u_0 and let $A = \frac{\lambda_1 K^2 M}{2 b_0 f(\frac{M}{2})}$. If*

$\varepsilon < \min\{(\frac{M}{2})^3, (2A)^{-3}\}$ *then the solution u of (1)–(3) blows up in a finite time and its blow-up time obeys the following estimates*

$$0 \le T - T_0 \le (T_0 A + C)\varepsilon^{1/3} + o(\varepsilon^{1/3}),$$

where $C = \frac{2}{b_0 f(\frac{M}{2})}$ *and* T_0 *is the blow-up time of the solution* $\alpha(t)$ *of the differential equation defined as follows*

$$\begin{cases} \alpha'(t) = b(t)f(\alpha(t)), \quad t > 0, \\ \alpha(0) = M. \end{cases}$$

Proof. Since $u_0 \in C^1(\overline{\Omega})$, using the mean value theorem, we get

$$u_0(x) \ge u_0(a) - \varepsilon^{1/3} \quad \text{for} \quad x \in B(a,\delta) \subset \Omega,$$

where $\delta = \frac{\varepsilon^{1/3}}{K}$. Due to the fact that the initial data u_0 is nonnegative in Ω, from the maximum principle, u is also nonnegative in $\Omega \times (0,T)$. Introduce the function $v(t)$ defined as follows

$$v(t) = \int_{B(a,\delta)} u(x,t)\varphi(x)dx.$$

Differentiating v in t and using (1), we have

$$v'(t) = \varepsilon \int_{B(a,\delta)} \psi(x)\Delta u(x,t)dx + b(t)\int_{B(a,\delta)} \psi(x)f(u(x,t))dx.$$

Apply Green's formula to obtain

$$\begin{aligned} v'(t) = \varepsilon \int_{B(a,\delta)} u(x,t)\Delta\psi(x)dx + \varepsilon \int_{\partial B(a,\delta)} \psi(x)\frac{\partial u(x,t)}{\partial \nu}ds \\ -\varepsilon \int_{\partial B(a,\delta)} u(x,t)\frac{\partial \psi(x)}{\partial \nu}ds + b(t)\int_{B(a,\delta)} \psi(x)f(u(x,t))dx. \end{aligned}$$

We know that $\frac{\partial\psi(x)}{\partial\nu} \le 0$ on $\partial B(a,\delta)$. Taking into account (11), we arrive at

$$v'(t) \ge -\varepsilon\lambda_\delta v(t) + b(t)\int_{B(a,\delta)} \psi(x)f(u(x,t))dx.$$

It follows from Jensen's inequality that

$$v'(t) \ge -\varepsilon\lambda_\delta v(t) + b(t)f(v(t))$$

which implies that

$$v^{'}(t) \geq b(t)f(v(t))(1-\frac{\varepsilon\lambda_\delta v(t)}{b(t)f(v(t))}).$$

Since $\lambda_\delta = \frac{\lambda_1}{\delta^2} = \frac{\lambda_1 K^2}{\varepsilon^{2/3}}$ and $b(t) \geq b_0$, we discover that

$$v^{'}(t) \geq b(t)f(v(t))(1-\varepsilon^{1/3}\frac{\lambda_1 K^2}{b_0}\frac{v(t)}{f(v(t))}).$$

We observe that $v^{'}(0) \geq b_0 f(v(0))(1-\frac{\varepsilon^{1/3}\lambda_1 K^2 v(0)}{b_0 f(v(0))})$. Since $f(0)=0$, we see that $\frac{f(s)}{s}$ is an increasing function for the positive values of s. Due to the fact that $v(0) \geq M-\varepsilon^{1/3} \geq M/2$, we see that

$$1-\frac{\varepsilon^{1/3}\lambda_1 K^2 v(0)}{b_0 f(v(0))} \geq 1-\frac{\varepsilon^{1/3}\lambda_1 K^2 M}{2b_0 f(\frac{M}{2})} > 0$$

which implies that $v^{'}(0) > 0$. We deduce that $v^{'}(t) > 0$ for $t \in (0,T)$. Indeed, let t_0 be the first $t>0$ such that $v^{'}(t) > 0$ for $t \in [0,T_0)$ but $v^{'}(t_0)=0$. Since $\frac{f(s)}{s}$ is an increasing function for the positive values of s, we get $\frac{f(v(t_0))}{v(t_0)} \geq \frac{f(v(0))}{v(0)}$ because $v(t_0) \geq v(0)$. Therefore, we have

$$0 = v^{'}(t_0) \geq b(t_0)f(v(t_0))(1-\varepsilon^{1/3}\frac{\lambda_1 K^2}{b_0}\frac{v(0)}{f(v(0))}) > 0$$

which is a contradiction. Consequently, we have

$$v^{'}(t) \geq b(t)f(v(t))(1-\varepsilon^{1/3}\frac{\lambda_1 K^2}{b_0}\frac{v(0)}{f(v(0))}).$$

Since $v(0) \geq M-\varepsilon^{1/3} \geq \frac{M}{2}$, we find that

$$v^{'}(t) \geq b(t)f(v(t))(1-\varepsilon^{1/3}\frac{\lambda_1 K^2}{b_0}\frac{M}{2f(\frac{M}{2})}) \quad for \quad t \in (0,T),$$

$$v(0) \geq M-\varepsilon^{1/3}.$$

Hence, it is not hard to see that

$$\begin{cases} v^{'}(t) \geq b(t)f(v(t))(1-\varepsilon^{1/3}A), \\ v(0) \geq M-\varepsilon^{1/3}. \end{cases}$$

We have $v^{'}(t) \geq \frac{1}{2}b_0 f(\frac{M}{2})$ for $t \in (0, T)$. Use the mean value theorem to obtain

$$v(C\varepsilon^{1/3}) = v(0) + C\varepsilon^{1/3}v^{'}(\xi),$$

where $\xi \in (0, C\varepsilon^{1/3})$, which implies that

$$v(C\varepsilon^{1/3}) \geq v(0) + \varepsilon^{1/3} \geq M.$$

Set

$$w(t) = v(\frac{t}{1-\varepsilon^{1/3}A} + C\varepsilon^{1/3}) \quad \text{for} \quad t \in (0, (1-\varepsilon^{1/3}A)(T-\varepsilon^{1/3})).$$

A straightforward computation reveals that

$$w(t) \geq b(\frac{t}{1-\varepsilon^{1/3}A} + C\varepsilon^{1/3})f(w(t))$$

$$w(0) \geq M,$$

which implies that

$$\begin{cases} w^{'}(t) \geq b(t)f(v(t)) \quad \text{for} \quad t \in (0, (1-\varepsilon^{1/3}A)(T-\varepsilon^{1/3})) \ , \\ w(0) \geq M, \end{cases}$$

because $b(t)$ is a nondecreasing function for the nonnegative values of t.

The maximum principle implies that

$$w(t) \geq \alpha(t) \quad \text{for} \quad t \in (0, T_*),$$

where $T_* = \min\{T_0, (1-\varepsilon^{1/3}A)(T-\varepsilon^{1/3})\}$. We deduce that

$$T < \frac{T_0}{1-\varepsilon^{1/3}A} + C\varepsilon^{1/3}. \tag{14}$$

Indeed, suppose that

$$T > \frac{T_0}{1-\varepsilon^{1/3}A} + C\varepsilon^{1/3} = T^{'}.$$

We get $\|u(.,T^{'})\|_\infty \geq v(T^{'}) = w(T_0) \geq \alpha(T_0) = +\infty$ which contradicts the fact that $(0, T)$ is the maximal time interval of existence of the solution u. On the other hand, setting

$$z(x,t) = \alpha(t) \quad \text{in} \quad \overline{\Omega} \times (0, T_0),$$

a direct calculation yields

$$z_t(x,t) = \varepsilon\Delta z(x,t) + b(t)f(z(x,t)) \quad \text{in} \quad \Omega \times (0,T_0),$$

$$z(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T_0),$$

$$z(x,0) = u_0(x) \quad \text{in} \quad \Omega.$$

The maximum principle implies that

$$z(x,t) \geq u(x,t) \quad \text{in} \quad \Omega \times (0,T_*^0),$$

where $T_*^0 = \min\{T, T_0\}$. Reasoning as in the proof of Theorem 1, we get

$$T \geq T_0. \tag{15}$$

Apply Taylor's expansion to obtain

$$\frac{1}{1-\varepsilon^{1/3}A} = 1 + \varepsilon^{1/3}A + o(\varepsilon^{1/3}A).$$

Use (14), (15) and the above relation to complete the rest of the proof. □

3. Numerical results

In this section, we consider the radial symmetric solution of the following initial-boundary value problem:

$$u_t = \varepsilon\Delta u + e^t e^u \quad \text{in} \quad B \times (0,T),$$

$$u(x,t) = 0 \quad \text{on} \quad S \times (0,T),$$

$$u(x,0) = u_0(x) \quad \text{in} \quad B,$$

where $B = \{x \in \mathbb{R}^N \ ; \ \|x\| < 1\}$, $S = \{x \in \mathbb{R}^N \ ; \ \|x\| = 1\}$. The above problem may be rewritten in the following form:

$$u_t = \varepsilon(u_{rr} + \frac{N-1}{r}u_r) + e^t e^u, \quad r \in (0,1), \quad t \in (0,T), \tag{16}$$

$$u_r(0,t)=0, \quad u(1,t)=0, \quad t\in(0,T), \tag{17}$$

$$u(r,0)=\varphi(r), \quad r\in(0,1). \tag{18}$$

Here, we take $\varphi(r)=a\sin(\pi r)$ with $a\geq 0$.

Let I be a positive integer and let $h=1/I$. Define the grid $x_i=ih$, $0\leq i\leq I$ and approximate the solution u of (16)–(18) by the solution $U_h^{(n)}=(U_0^{(n)},U_1^{(n)},...,U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta t_n}=\varepsilon N\frac{2U_1^{(n)}-2U_0^{(n)}}{h^2}+e^{t_n}e^{U_0^{(n)}},$$

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}=\varepsilon(\frac{U_{i+1}^{(n)}-2U_i^{(n)}+U_{i-1}^{(n)}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n)}-U_{i-1}^{(n)}}{2h})$$
$$+e^{t_n}e^{U_i^{(n)}}, 1\leq i\leq I-1,$$

$$U_I^{(n)}=0,$$

$$U_i^{(0)}=\varphi_i, \quad 0\leq i\leq I,$$

and also by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta t_n}=\varepsilon N\frac{2U_1^{(n+1)}-2U_0^{(n+1)}}{h^2}+e^{t_n}e^{U_0^{(n)}},$$

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}=\varepsilon(\frac{U_{i+1}^{(n+1)}-2U_i^{(n+1)}+U_{i-1}^{(n+1)}}{h^2}$$
$$+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n+1)}-U_{i-1}^{(n+1)}}{2h})+e^{t_n}e^{U_i^{(n)}}, 1\leq i\leq I-1,$$

$$U_I^{(n+1)}=0,$$

$$U_i^{(0)}=\varphi_i, \quad 0\leq i\leq I,$$

where $t_n = \sum_{j=0}^{n-1} \Delta t_j$.

We take $\Delta t_n = \min\{\frac{h^2}{2N\varepsilon}, h^2 e^{-\|U_h^{(n)}\|_\infty}\}$ for the explicit scheme and $\Delta t_n = h^2 e^{-\|U_h^{(n)}\|_\infty}$ for the implicit scheme where $\|U_h^{(n)}\|_\infty = \sup_{0\leq i\leq I} |U_i^{(n)}|$. Let us notice that in the case of the explicit scheme, the restriction on the time step ensures the nonnegativity of the discrete solution. For the implicit scheme, the existence and the nonnegativity of the discrete solution is also guaranteed using standard method (see for instance [3]). We remark that $\lim_{r\to 0} \frac{u_r}{r}(r,t) = u_{rr}(0,t)$ which implies that $u_t(0,t) = \varepsilon N u_{rr}(0,t) + e^t e^{u(0,t)}$. This remark has been taken into account in the construction of the schemes for $i = 0$. We need the following definition.

Definition 3. We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme blows up in a finite time if $\lim_{n\to+\infty} \|U_h^{(n)}\|_\infty = +\infty$ and the series $\sum_{n=0}^{+\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{+\infty} \Delta t_n$ is called the numerical blow-up time of the solution $U_h^{(n)}$.

In the following tables, in rows, we present the numerical blow-up times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512. We take for the numerical blow-up time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$|T^{n+1} - T^n| \leq 10^{-16}.$$

The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $a = 0$**,** $N = 2$.

First case: $\varepsilon = \frac{1}{100}$**.**

I	T^n	n	$CPU\ time$	s
16	0.694125	4166	-	-
32	0.693391	15966	-	-
64	0.693208	61038	1	2.00
128	0.693162	232803	5	2.00
256	0.693151	885790	36	2.00
512	0.693148	3361464	268	2.00

Table 1: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.694125	4166	-	-
32	0.693391	15966	1	-
64	0.693208	61038	1	2.00
128	0.693162	232803	6	2.00
256	0.693151	885790	42	2.00
512	0.693148	3361464	315	2.00

Table 2: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

Second case: $\varepsilon = \frac{1}{1000}$.

I	T^n	n	$CPU\ time$	s
16	0.694125	4166	-	-
32	0.693391	15966	-	-
64	0.693208	61038	1	2.00
128	0.693162	232803	5	2.00
256	0.693151	885790	36	2.00
512	0.693148	3361464	269	2.00

Table 3: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.694125	4166	-	-
32	0.693391	15966	-	-
64	0.693208	61038	1	2.00
128	0.693162	232803	5	2.00
256	0.693151	885790	41	2.00
512	0.693148	3361464	313	2.00

Table 4: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

Third case: $\varepsilon = \frac{1}{10000}$.

I	T^n	n	$CPU\ time$	s
16	0.694125	4166	-	-
32	0.693391	15966	-	-
64	0.693208	61038	1	2.00
128	0.693162	232803	5	2.00
256	0.693151	885790	36	2.00
512	0.693148	3361464	264	2.00

Table 5: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.694125	4166	-	-
32	0.693391	15966	-	-
64	0.693208	61038	0	2.00
128	0.693162	232803	5	2.00
256	0.693151	885790	40	2.00
512	0.693148	3361464	308	2.00

Table 6: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

Numerical experiments for $a = 20$, $N = 2$ when the reaction term $e^{t_n} e^{U_h^{(n)}}$ is replaced by $e^{t_n} (U_h^{(n)})^2$.

In this case we take $\Delta t_n = \min\{\frac{h^2}{2N\varepsilon}, \frac{h^2}{\|U_h^{(n)}\|_\infty}\}$ for the explicit scheme and $\Delta t_n = \frac{h^2}{\|U_h^{(n)}\|_\infty}$ for the implicit scheme.

First case: $\varepsilon = \frac{1}{100}$.

I	T^n	n	$CPU\ time$	s
16	0.049193	6927	-	-
32	0.049061	26321	-	-
64	0.049029	99862	1	2.04
128	0.049021	377874	8	2.04
256	0.049019	1425289	61	2.03
512	0.049019	5355953	461	2.00

Table 7: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.049195	6927	-	-
32	0.049062	26321	-	-
64	0.049029	99862	2	2.04
128	0.049021	377874	13	2.04
256	0.049019	1425289	101	2.03
512	0.049019	5355953	758	2.00

Table 8: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

Second case: $\varepsilon = \frac{1}{1000}$.

I	T^n	n	$CPU\ time$	s
16	0.049000	6924	-	-
32	0.048860	26304	-	-
64	0.048825	99772	1	2.00
128	0.048816	377430	8	2.00
256	0.048814	1423199	61	2.00
512	0.048813	5346822	457	2.00

Table 9: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.049000	6924	-	-
32	0.048860	26304	1	-
64	0.048825	99772	2	2.00
128	0.048816	377430	14	2.00
256	0.048814	1423199	100	2.00
512	0.048813	5346822	747	2.00

Table 10: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

Third case: $\varepsilon = \frac{1}{10000}$.

I	T^n	n	$CPU\ time$	s
16	0.048981	6924	-	-
32	0.048839	26302	1	-
64	0.048804	99763	2	2.00
128	0.048795	377383	9	2.00
256	0.048793	1422975	61	2.00
512	0.048793	5345776	459	2.00

Table 11: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.048981	6924	-	-
32	0.048839	26302	-	-
64	0.048804	99763	2	2.00
128	0.048795	377383	13	2.00
256	0.048793	1422975	101	2.00
512	0.048793	5345776	759	2.00

Table 12: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

Remark 4. If we consider the problem (16)–(18) in the case where the initial data is null and the reaction term is $e^t e^u$, it is not hard to see that the blow-up time of the solution of the differential equation defined in Theorem 1 equals $\ln(2) \simeq 0.693$. We observe from Tables 1-6 that when ε diminishes, the numerical blow-up time decays to $\ln(2)$. This result has been proved in Theorem 1. When the initial data $\varphi(r) = 20\sin(x\pi)$ and the reaction term is $e^t u^2$, we find that the blow-up time of the solution of the differential equation defined in Theorem 2 equals $\ln(1.05) \simeq 0.04879$. We discover from Tables 7-12 that when ε diminishes, the numerical blow-up time decays to $\ln(1.05)$ which is a result proved in Theorem 2.

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where $I = 16$.

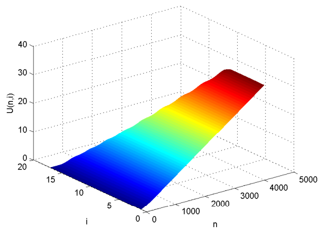

Figure 1: Evolution of the discrete solution for $\varepsilon = 10^{-3}$ with a reaction term $e^{t_n} e^{U_h^{(n)}}$ (Explicit scheme).

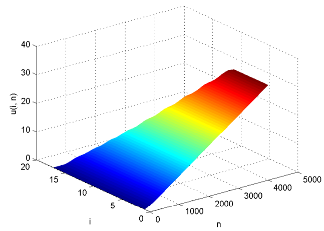

Figure 2: Evolution of the discrete solution for $\varepsilon = 10^{-3}$ with a reaction term $e^{t_n} e^{U_h^{(n)}}$ (Implicit scheme).

References

[1] L.M. Abia, J.C. López-Marcos and J. Martinez, On the blow-up time convergence of semidiscretizations of reaction-diffusion equations, *Appl. Numer. Math.*, **26** (1998), 399-414.

[2] L.M. Abia, J.C. López-Marcos and J. Martinez, Blow-up time for semidiscretizations of reaction-diffusion equations, *Appl. Numer. Math.*, **20** (1996), 145-156.

[3] T.K. Boni, Extinction for discretizations of some semilinear parabolic equations, *C.R.A.S*, Serie I, **333** (2001), 795-800.

[4] T.K. Boni, On the blow-up and asymptotic behavior of solutions to a nonlinear parabolic equations of second order, *Asympt. Anal.*, **21** (1999), 187-208.

[5] H. Brezis, *Analyse Fonctionnelle, Théorie et Applications*, Collection mathématiques appliquées pour la matrise, Masson, Paris (2005).

[6] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, Blow-up of $u_t = \triangle u + g(u)$ revised, *Adv. Diff. Equat.*, **1** (1996), 73-90.

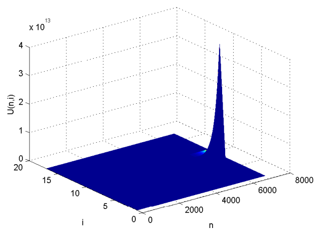

Figure 3: Evolution of the discrete solution for $\varepsilon = 10^{-3}$ with a reaction term $e^{t_n}(U_h^{(n)})^2$ (Explicit scheme)

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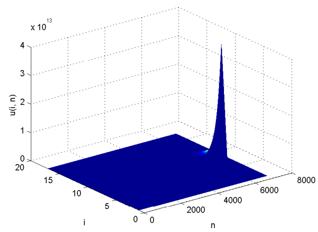

Figure 4: Evolution of the discrete solution for $\varepsilon = 10^{-3}$ with a reaction term $e^{t_n}(U_h^{(n)})^2$(Implicit scheme).

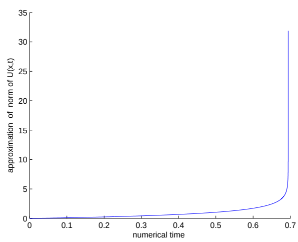

Figure 5: Profil of the approximation of $u(x,t)$ for $\varepsilon = 10^{-3}$ with a reaction term $e^t e^u$(Explicit scheme).

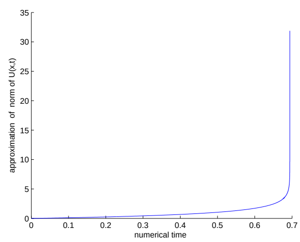

Figure 6: Profil of the approximation of $u(x,t)$ for $\varepsilon = 10^{-3}$ with a reaction term $e^t e^u$ (Implicit scheme).

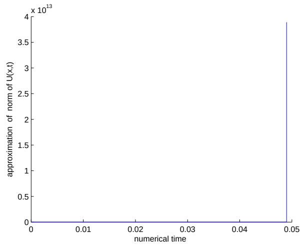

Figure 7: Profil of the approximation of $u(x,t)$ for $\varepsilon = 10^{-3}$ with a reaction term $e^t u^2$ (Explicit scheme).

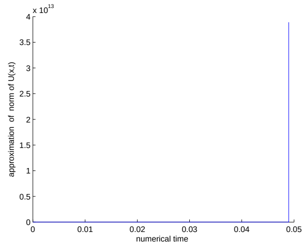

Figure 8: Profil of the approximation of $u(x,t)$ for $\varepsilon = 10^{-3}$ with a reaction term $e^t u^2$(Implicit scheme).

[7] J. Ding, Global and blow-up solutions for nonlinear parabolic equations with robin boundary conditions, *Computers Math. Appl.*, **65** (2013), 1806-1822.

[8] A. Friedman and A.A. Lacey, The blow-up time for solutions of nonlinear heat equations with small diffusion, *SIAM J. Math. Anal.*, **18** (1987), 711-721.

[9] V. Galaktionov and J.L. Vasquez, The problem of blow-up in nonlinear parabolic equation, current developments in PDE (Temuco, 1999), *Discrete Contin. Dyn. Syst.*, **8** (2002), 399-433.

[10] S. Kaplan, On the growth of the solutions of quasi-linear parabolic equation, *Comm. Pure. Appl. Math.*, **16** (1963), 305-330.

[11] O.A. Ladyzenskaya, V.A. Solonnikov and N.N. Ural'ceva, Linear and quasilinear equations of parabolic type, *Transl. Math. Monogr.*, **23**, AMS, Providence, RI (1968).

[12] H. Li, W. Gao and Y. Han, Lower bounds for the blow-up time of solutions to a nonlinear parabolic problem, *EJDE*, **2014** (2014), No 20, 1-6.

[13] I. Mai and K. Mochizuki, On blow-up of solutions for quasilinear degenerate parabolic equation, *Publ. RIMS Kyoto Univ.*, **27** (1991), 695-709.

[14] T. Nakagawa, Blowing up on the finite difference solution to $u_t = u_{xx} + u^2$, *Appl. Math. Optim.*, **2** (1976), 337-350.

[15] L.E. Payne, G.A. Philippin and P.W. Schaefer, Blow-up phenomenon for some nonlinear parabolic problems, *Nonlinear Anal. TMA*, **69** (2008), 3495-3502.

[16] M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Prentice Hall, Englewood Cliffs, NJ (1967).

[17] P. Quittner and P. Souplet, *Superlinear parabolic problems blow-up, Global Existence and Steady States Series*, Birkhuser Advanced Texts, Basler Lehbcher (2007).

[18] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov and A.P. Mikhailov, *Blow-up in Problems for Quasilinear Parabolic Equations*, Nauka, Moscow (1987) (in Russian); English transl.: Walter de Gruyter, Berlin (1995).

[19] R. Suzuki, On blow-up sets and asymptotic behavior of interface of one dimensional quasilinear degenerate parabolic equation, *Publ. RIMS., Kyoto Univ.*, **27** (1991), 375-398.

[20] W. Walter, *Differential-und Integral-Ungleichungen*, Springer, Berlin (1964).

[21] F.B. Weissler, An L^∞ blow-up estimate for a nonlinear heat equation, *Comm. Pure. Appl. Math.*, **38** (1985), 291-295.