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GENERALIZATION OF THE WIMAN-VALIRON METHOD FOR FRACTIONAL DERIVATIVES

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Abstract: We generalize the Wiman-Valiron method for fractional derivatives proving that

$$|z|^q D^q f(z) \sim (\nu(r,f))^q f(z)$$

holds in a neighborhood of a maximum modulus point outside an exceptional set of values of $|z|$ as $|z| \to \infty$, where D^q is the Riemann-Liouville fractional derivative of order $q > 0$, $\nu(r,f)$ is the central index of the Taylor representation of f. We use this result to find the precise value for the order of growth of solutions of a fractional differential equation.

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Key Words: transcendental entire function, Wiman-Valiron method, Riemann-Liouville fractional derivative, Riemann-Liouville fractional integral, fractional differential equation

1. Introduction

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z = re^{i\theta} \tag{1}$$

be a transcendental entire function. For $r \in [0, +\infty)$ we denote $M(r,f) =$

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$\max\{|f(z)| : |z| = r\}$, and let $\mu(r, f) = \max\{|a_n|r^n : n \geq 0\}$ be the maximum term and $\nu(r, f) = \max\{n \geq 0 : |a_n|r^n = \mu(r, f)\}$ be the central index of the series (1).

The theory, initiated by A. Wiman ([14, 15]) and developed by many other mathematicians such as G. Valiron, J. Clunie, T. Kővari, describes the local behavior of f near a point z_r, $|z_r| = r$, satisfying $|f(z_r)| = M(r, f)$ in terms of the power series (1). A nice exposition is due to W.K. Hayman [4], where bibliographical references are given. The seminal result of the theory states that given $q \in \mathbf{N}$ in a neighborhood of z_r one has

$$f(z) \sim \Big(\frac{z}{z_r}\Big)^{\nu(r,f)} f(z_r), \quad f^{(q)}(z) \sim \Big(\frac{\nu(r, f)}{z}\Big)^q f(z) \tag{2}$$

for $r \in [1, \infty)\backslash E$ where E is a set of finite logarithmic measure, i.e. $\int_{E\cap[1,\infty)} \frac{dx}{x} < \infty$. Another elegant approach not involving power series was proposed by A. Macintyre ([7]), who used $K(r, f) := z_r f'(z_r)/f(z_r) = r(\log M(r, f))'$ instead of the central index. This approach was developed by Sh. Strelitz in his book [13], who proved counterparts of (2) with $K(r, f)$ instead of $\nu(r, f)$ for functions analytic in a strip or in the unit disc, and for Dirichlet series. On the other hand, the theory has been developed for Dirichlet series by M. Sheremeta and O. Skaskiv (see e.g. [9, 10, 12]). Recently, W. Bergweiler and others developed Macintyre's approach for meromorphic functions having a direct tract in $\mathbf{C}$ ([1]). Correlations (2) are very useful in studying differential equations. They allow to obtain sharp asymptotic estimates for the growth of solutions (see [16, 6, 1]). Counterparts of (2) for fractional values of q is unknown.

The aim of the paper is to obtain an analogue of the second relation of (2) for the Riemann-Liouville fractional derivatives.

2. Generalization of the Wiman-Valiron Method for Fractional Derivatives

We start with the settings of the Wiman-Valiron theory.

Let $(\alpha_n)_{n=0}^{\infty}$ be a sequence of positive numbers such that α_{n+1}/α_n decreases with increasing n. Let (ϱ_n) be a sequence of numbers such that

$$0 < \varrho_0 < \frac{\alpha_0}{\alpha_1}, \quad \frac{\alpha_{n-1}}{\alpha_n} < \varrho_n < \frac{\alpha_n}{\alpha_{n+1}} \quad (n \geq 1),$$

so that (ϱ_n) increases with increasing n. We shall say that a value r is normal (for the sequence (a_n),(α_n) and (ϱ_n)), if we have for some ν

$$|a_n|r^n \leq |a_\nu|r^\nu \frac{\alpha_n \varrho_\nu^n}{\alpha_\nu \varrho_\nu^\nu} \quad (n \geq 0).$$

Let V be the class of positive continuous nondecreasing functions v on $[0, +\infty)$ such that $\frac{x^2}{v(x)\ln v(x)}$ increases to $+\infty$ on $x \in [x_0; +\infty)$, $x_0 > 0$, and $\int\limits_0^{+\infty} \frac{dx}{v(x)} < +\infty$. For example, the functions $v(x) = x\ln^{\alpha+1} x$, $(x \geq e)$, $\alpha > 0$, and $v(x) = x^{\delta+1}$, $(x \geq 1)$, $\delta \in (0,1)$ belong to V.

The main result of the Wiman-Valiron theory is formulated as follows.

Theorem 1. *Let $v \in V$ and $\kappa(t) = 4\sqrt{v(t)\ln v(t)}$. Suppose that f is an entire function, a value r is normal and large enough, $|z_0| = r$,*

$$|f(z_0)| \geq \eta M(r,f), \quad v^{-2}(\nu(r,f)) \leq \eta \leq 1,$$

holds, and

$$r\left(1 - \frac{1}{40\kappa(\nu)}\right) < \rho < r\left(1 + \frac{1}{40\kappa(\nu)}\right), \quad \nu = \nu(r,f).$$

Then if $q \in \mathbf{Z}_+$ we have for $|z| = \rho$

$$\left(\frac{z}{\nu}\right)^q f^{(q)}(z) = f(z) + O\left(\frac{\kappa(\nu)}{\nu}\right) M(\rho, f).$$

In particular, if $\ln\rho - \ln r = o\left(\frac{1}{\kappa(\nu)}\right)$, then

$$\begin{aligned} M(\rho, f^{(q)}) &= \left(\frac{\nu}{\rho}\right)^q \left\{1 + O\left(\frac{\kappa(\nu)}{\nu}\right)\right\} M(\rho, f) \\ &= (1 + o(1))\left(\frac{\nu}{r}\right)^q M(r,f) \end{aligned}$$

as $r \to +\infty$ outside a set of finite logarithmic measure.

We generalize the Wiman-Valiron method for fractional derivatives.

Let $f \in L(0,a)$, $a > 0$. The Riemann-Liouville fractional derivative of order $\alpha > 0$ for f is defined as

$$D^\alpha f(x) = \frac{d^n}{dx^n}\{I^{n-\alpha} f(x)\}, \quad \alpha \in (n-1, n], \quad n \in \mathbf{N},$$

where

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^x \frac{f(t)dt}{(x-t)^{1-\alpha}}$$

is the Riemann-Liouville fractional integral of order $\alpha > 0$ for f, $\Gamma(\alpha)$ is the Gamma function. In particular, if $0 < \alpha < 1$, then

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_0^x \frac{f(t)dt}{(x-t)^\alpha}.$$

The fractional derivative has the following property ([8]):

$$D^\alpha x^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} x^{\beta-\alpha-1}, \quad \alpha, \beta > 0. \tag{3}$$

It follows from (3) that the fractional derivative for the entire function (1) is defined as

$$|z|^\alpha D^\alpha f(z) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} z^n.$$

Theorem 2. *Let $v \in V$ and $\kappa(t) = 4\sqrt{v(t) \ln v(t)}$. Suppose that f is an entire function, a value r is normal and large enough, $|z_0| = r$,*

$$|f(z_0)| \geq \eta M(r,f), \quad v^{-2}(\nu(r,f)) \leq \eta \leq 1$$

holds, and

$$r\left(1 - \frac{1}{40\kappa(\nu)}\right) < \rho < r\left(1 + \frac{1}{40\kappa(\nu)}\right), \quad \nu = \nu(r,f).$$

Then if $q > 0$ we have for $|z| = \rho$:

$$\frac{\rho^q D^q f(z)}{\nu^q} = f(z) + O\left(\frac{\kappa(\nu)}{\nu}\right) M(\rho, f). \tag{4}$$

In particular, if $\ln \rho - \ln r = o\left(\frac{1}{\kappa(\nu)}\right)$, then

$$\begin{aligned} M(\rho, D^q f(z)) &= \left(\frac{\nu}{\rho}\right)^q \left\{1 + O\left(\frac{\kappa(\nu)}{\nu}\right)\right\} M(\rho, f) \\ &= (1 + o(1)) \left(\frac{\nu}{r}\right)^q M(r,f) \end{aligned} \tag{5}$$

as $r \to +\infty$ outside a set of finite logarithmic measure.

For $\rho \in [0; +\infty)$ we set $\mu(r, \rho, f) = |a_{\nu(r,f)}|\rho^{\nu(r,f)}$.

To prove Theorem 2, we need the following statements.

Lemma 1. [11, Lemma 3.4], cf. [4, Lemma 2] *Let $v \in V$ and $\kappa(t) = 4\sqrt{v(t)\ln v(t)}$. Then we have for any fixed positive q and for all ρ, $|\ln\rho - \ln r| \le \frac{1}{\kappa(\nu)}$,*

$$\sum_{|n-\nu|>\kappa(\nu)} n^q |a_n|\rho^n = o\left(\frac{\nu^q \mu(r,\rho,f)}{v(\nu)^3}\right), \quad \nu = \nu(r,f), \tag{6}$$

as $r \to +\infty$ outside a set of finite logarithmic measure.

Lemma 2. [11, Lemma 3.5], cf. [4, Lemma 7] *Suppose that P is a polynomial of degree m and $|P(z)| \le M$ for $|z| \le r$. Then for $R \ge r$ we have*

$$|P'(z)| \le \frac{eMmR^{m-1}}{r^m}, \quad |z| < R.$$

Theorem 3. [11, Lemma 3.7], cf. [4, Theorem 10] *Let $v \in V$ and $\kappa(t) = 4\sqrt{v(t)\ln v(t)}$. Suppose that f is an entire function, a value r is normal and enough large, $|z_0| = r$,*

$$|f(z_0)| \ge \eta M(r,f), \quad v^{-2}(\nu(r,f)) \le \eta \le 1.$$

Then, if $z = z_0 e^\tau$, $|\tau| \le \frac{\eta}{18\kappa(\nu)}$, $\nu = \nu(r,f)$, we have

$$\ln\frac{f(z)}{f(z_0)} = (\nu(r,f) + \varphi_1)\tau + \varphi_2\tau^2 + \delta(\tau),$$

where

$$|\varphi_j| \le 2,2\left(\frac{18\kappa(\nu)}{\eta}\right)^j, \quad (j = 1,2), \quad |\delta(\tau)| \le 8,8\left(\frac{18\kappa(\nu)\tau}{\eta}\right)^3.$$

Proof. Let

$$\nu_1 = \min\{n : |n-\nu| \le \kappa(\nu)\}, \quad \nu_2 = \max\{n : |n-\nu| \le \kappa(\nu)\}.$$

We write

$$f(z) = P(z)z^{\nu_1} + R(z), \tag{7}$$

where

$$P(z) = \sum_{|n-\nu|\le\kappa(\nu)} |a_n| z^{n-\nu_1}. \tag{8}$$

Since $\mu(r,\rho,f)\le M(\rho,f)$, from Lemma 1 with $q=0$ for all ρ, $|\ln\rho-\ln r|\le \frac{1}{\kappa(\nu)}$, we obtain for $|z|=\rho$

$$f(z)=P(z)z^{\nu_1}+o\left(\frac{\mu(r,\rho,f)}{v(\nu)^3}\right)=P(z)z^{\nu_1}+o\left(\frac{M(\rho,f)}{v(\nu)^3}\right), \tag{9}$$

as $r\to+\infty$ outside a set E of finite logarithmic measure.

In particular, from (9) with $\rho=r$ we have $|P(z)|r^{\nu_1}\le(1+o(1))M(r,f)$, i.e. for all sufficiently large $r\notin E$

$$|P(z)|\le\frac{1{,}01\,M(r,f)}{r^{\nu_1}}=:M^*(r),\quad |z|=r. \tag{10}$$

We need the asymptotic representation for Gamma functions ([5])

$$\frac{\Gamma(t+a)}{\Gamma(t+b)}=t^{a-b}\left(1+O\left(\frac{1}{t}\right)\right),\quad t\to+\infty,\quad b,a\in\mathbf{R}. \tag{11}$$

First we estimate the fractional derivative of order q for $R(z)$. From (3) and Lemma 1 we deduce

$$|\rho^q D^q R(z)|=\left|\rho^q\sum_{|n-\nu|>\kappa(\nu)}\frac{\Gamma(1+n)}{\Gamma(1+n-q)}a_n\rho^{n-q}e^{in\theta}\right|$$

$$\le C\sum_{|n-\nu|>\kappa(\nu)}n^q|a_n|\rho^n=o\left(\frac{\nu^q\mu(r,\rho,f)}{v(\nu)^3}\right), \tag{12}$$

where $\nu=\nu(r,f)$, $r\to+\infty$, $r\notin E$, and $C=\sup\limits_n\{2,\frac{\Gamma(n+1)}{\Gamma(n+1-q)}n^{-q}\}$.

Repeated application of Lemma 2 shows that for any $q\in\mathbf{Z}_+$ and $|z|=\rho$

$$|P^{(q)}(z)|\le\left(\frac{6\kappa(\nu)}{r}\right)^q M^*(r). \tag{13}$$

In fact,

$$|P'(z)|\le\frac{eM^*(r)2\kappa(\nu)\rho^{\nu_2-\nu_1-1}}{r^{\nu_2-\nu_1}}$$

$$\le\frac{2eM^*(r)\kappa(\nu)}{\varrho}\left(1+\frac{1}{40\kappa(\nu)}\right)^{2\kappa(\nu)}\le\frac{6\kappa(\nu)}{r}M^*(r),\quad r\to+\infty$$

and, similarly,

$$|P^{(j)}(z)|\le\frac{e\,\max\{|P^{(j-1)}(z)|:|z|\le\rho\}(2\kappa(\nu)-j+1)\rho^{\nu_2-\nu_1-j}}{r^{\nu_2-\nu_1-j+1}}$$

$$\le \left(\frac{6\kappa(\nu)}{r}\right)^j M^*(r).$$

We need generalized Leibniz's formula for fractional derivatives to estimate the first summand in (7). Let $f(x)$ and $g(x)$ be analytic functions on $[a,b]$, then ([8, p. 278])

$$D^\alpha(f\cdot g)=\sum_{k=0}^{+\infty}\binom{\alpha}{k}(D^{\alpha-k}f)g^{(k)}, \tag{14}$$

where $\binom{\alpha}{k}=\dfrac{(-1)^k\alpha\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)}$.

It follows from (3) and (14) that

$$\rho^q D^q(z^{\nu_1}P(z))=\rho^q\sum_{m=0}^{+\infty}\binom{q}{m}D^{q-m}z^{\nu_1}P^{(m)}(z)$$

$$=\sum_{m=0}^{\nu_2-\nu_1}\binom{q}{m}\frac{\Gamma(\nu_1+1)}{\Gamma(\nu_1+1-q+m)}z^{\nu_1}\rho^m P^{(m)}(z)$$

$$=\frac{\Gamma(\nu_1+1)}{\Gamma(\nu_1+1-q)}z^{\nu_1}\left(P(z)+\sum_{m=1}^{\nu_2-\nu_1}\binom{q}{m}\frac{\Gamma(\nu_1+1-q)}{\Gamma(\nu_1+1-q+m)}\rho^m P^{(m)}(z)\right).$$

We now estimate the second term in parentheses taking into account (13) and (11)

$$\left|\sum_{m=1}^{2\kappa(\nu)}\binom{q}{m}\frac{\Gamma(\nu_1+1-q)}{\Gamma(\nu_1+1-q+m)}\rho^m P^{(m)}(z)\right|$$

$$\le\sum_{m=1}^{2\kappa(\nu)}\frac{q\Gamma(m-q)\Gamma(\nu_1+1-q)}{\Gamma(1-q)\Gamma(m+1)\Gamma(\nu_1+1-q+m)}\left(\frac{6\rho\kappa(\nu)}{r}\right)^m M^*(r)$$

$$\le C(q)\sum_{m=1}^{2\kappa}\frac{1}{m^{1+q}}\nu_1^{-m}C^m\kappa(\nu)^m M^*(r)$$

$$\le C(q)\frac{\kappa(\nu)}{\nu}M^*(r)\sum_{m=1}^{2\kappa(\nu)}\frac{1}{m^{1+q}}=O\left(\frac{\kappa(\nu)}{\nu}\right)M^*(r).$$

Therefore, in view of (9) and the previous estimate we have

$$\rho^q D^q(P(z)z^{\nu_1}) = \frac{\Gamma(\nu_1+1)}{\Gamma(\nu_1+1-q)} z^{\nu_1}\left(P(z) + O\left(\frac{\kappa(\nu)}{\nu}\right)M^*(r)\right)$$
$$= \frac{\Gamma(\nu_1+1)}{\Gamma(\nu_1+1-q)}\left(f(z) + o\left(\frac{\mu(r,\rho,f)}{v(\nu)^3}\right) + O\left(\frac{\kappa(\nu)}{\nu}M^*(r)\rho^{\nu_1}\right)\right). \quad (15)$$

Since $\frac{1}{v(t)^3} = o\left(\frac{\kappa(t)}{t}\right)$, $t \to +\infty$, using (15) and (12) we have for $|z| = \rho$:

$$\rho^q D^q f(z) = \frac{\Gamma(\nu_1+1)}{\Gamma(\nu_1+1-q)}\left(f(z) + O\left(\frac{\mu(r,\rho,f)}{v(\nu)^3}\right)\right.$$
$$\left.+O\left(\frac{\kappa(\nu)}{\nu}M^*(r)\rho^{\nu_1}\right)\right) = \frac{\Gamma(\nu_1+1)}{\Gamma(\nu_1+1-q)}\left(f(z) + o\left(\frac{\kappa(\nu)}{\nu}M(\rho,f)\right)\right.$$
$$\left.+O\left(\frac{\kappa(\nu)}{\nu}M(r,f)\left(\frac{\rho}{r}\right)^{\nu_1}\right)\right), \quad (16)$$

as $r \to +\infty$ outside a set of finite logarithmic measure.

Next we choose z_0 so that $|f(z_0)| = M(r,f)$ and take $\eta = 1$, $\tau = \ln(\rho/r)$. Then Theorem 3 gives

$$\ln\left|f\left(\frac{\rho}{r}z_0\right)\right| = \ln|f(z_0)| + \nu\tau + O(1), \quad |\tau| \le \frac{1}{18\kappa(\nu)},$$

so that

$$\ln M(\rho,f) \ge \ln M(r,f) + \nu\ln(\rho/r) + O(1).$$

Since $(\rho/r)^{\nu_1-\nu} = \exp\{\tau(\nu_1-\nu)\} = O(1)$, we have

$$\left(\frac{\rho}{r}\right)^{\nu_1} M(r,f) = \left(\frac{\rho}{r}\right)^{\nu}\left(\frac{\rho}{r}\right)^{\nu_1-\nu} M(r,f)$$
$$= O\left(\left(\frac{\rho}{r}\right)^{\nu} M(r,f)\right) = O(M(\rho,f)).$$

Thus, (16) yields

$$\rho^q D^q f(z) = \frac{\Gamma(\nu_1+1)}{\Gamma(\nu_1+1-q)}\left(f(z) + O\left(\frac{\kappa(\nu)}{\nu}M(\rho,f)\right)\right). \quad (17)$$

According to (11) we have

$$\frac{\Gamma(\nu_1+1)}{\Gamma(\nu_1+1-q)} = (1+o(1))\nu^q\left(1 + O\left(\frac{1}{\nu}\right)\right), \quad \nu \to +\infty \quad (18)$$

Hence

$$\rho^q D^q f(z) = \nu^q \left(1 + O\left(\frac{1}{\nu}\right)\right)\left(f(z) + O\left(\frac{\kappa(\nu)}{\nu} M(\rho, f)\right)\right)$$
$$= \nu^q \left(f(z) + O\left(\frac{\kappa(\nu)}{\nu} M(\rho, f)\right)\right)$$

when $r \to +\infty$ outside a set of finite logarithmic measure, that is (4).

We choose z in (4) in turn so as to make $|f(z)|$ and $|D^q f(z)|$ maximal and deduce that

$$M(\rho, D^q f) \leq \left(1 + O\left(\frac{\kappa(\nu)}{\nu}\right)\right)\left(\frac{\nu}{\rho}\right)^q M(\rho, f)$$

and

$$M(\rho, D^q f) \geq \left(1 + O\left(\frac{\kappa(\nu)}{\nu}\right)\right)\left(\frac{\nu}{\rho}\right)^q M(\rho, f)$$

so that

$$M(\rho, D^q f) = \left(1 + O\left(\frac{\kappa(\nu)}{\nu}\right)\right)\left(\frac{\nu}{\rho}\right)^q M(\rho, f).$$

To complete the proof of (5) it remains to show that

$$\ln M(\rho, f) = \ln M(r, f) + \nu \ln(\rho/r) + o(1).$$

To see this we note that (7) and (12) yield for our range of ρ

$$\ln M(\rho, f) = \nu_1 \ln \rho + \ln M(\rho, P) + o(1).$$

On the other hand it follows from Lemma 2 that

$$M(\rho, P) = M(r, P)\left(1 + O\left(\frac{(\rho - r)\kappa(\nu)}{r}\right)\right) \sim M(r, P)$$

if $\kappa(\nu) \ln(\rho/r) = o(1)$, and now the second equality of (5) also follows and the proof of Theorem 2 is complete. □

Remark 1. $D^q(\rho^q f(z))$ has the same asymptotic estimate as $\rho^q D^q f(z)$, thus under the conditions of Theorem 2 for $|z| = \rho$ we have

$$D^q(\rho^q f(z)) = \nu^q \left(f(z) + O\left(\frac{\kappa(\nu)}{\nu} M(\rho, f)\right)\right), \tag{19}$$

as $r \to +\infty$ outside a set of finite logarithmic measure. Note that the operator $D^q(\rho^q f(\rho e^{i\varphi}))$ keeps analyticity and have other nice properties (see [3, Ch. IX]).

3. An Application to Fractional Differential Equations

It is known [16, 6, 2] that every nontrivial solution of the equation

$$f^{(q)}(z) + a(z)f(z) = 0, \tag{20}$$

where $a(z)$ is a polynomial of degree m, is an entire function of order $\rho[f] = 1 + \frac{m}{q}$, where

$$\rho[f] = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

On the other hand, for fractional values of $q \in (0,1)$ equation (20) with $a(t) = A(t^q)$, where A is a polynomial of degree m, admits a solution of the form $f(t) = v(t^q)$, $t \geq 0$, where v is entire with $\rho[v] \leq \frac{1+m}{q}$ ([5]).

It is not possible to estimate the growth of solutions of (20) using Theorem 2, because it would require an asymptotics for the Gelfond-Leontiev differential operators (see [5]), which is more general than D^q. Nevertheless we can obtain an asymptotic of solutions for some class of fractional equations.

We consider the fractional differential equation in the form

$$\frac{\widetilde{D}^q(r^q f(z))}{z} + a(z)f(z) = 0, \tag{21}$$

where the coefficient $a(z)$ is an entire function, $q > 0$, and

$$\widetilde{D}^q f(z) = D^q f(z) - \Gamma(q+1)f(0). \tag{22}$$

Remark 2. The analog of the operator (22) can be found in ([3, Chap.9]). This definition provides that $\tilde{D}(r^q f(re^{i\varphi}))\Big|_{r=0} = 0.$

The proofs of the following theorems are standard ([6]).

Theorem 4. *The equation (21) with the initial condition $f(0) = f_0$ has an entire solution.*

Theorem 5. *Let $a(z)$ be a polynomial of degree $m \geq 0$. Then all nottrivial solutions f of the equation (21) have the order of growth $\varrho = \frac{m+1}{q}$.*

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