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# GENERALIZATION OF RIEMANN-HILBERT BOUNDARY VALUE PROBLEM FOR A FIRST ORDER NONLINEAR COMPLEX PARTIAL DIFFERENTIAL EQUATION

N. Taghizadeh[1], H. Hoseini[2 §]
[1,2]Department of Mathematics
Faculty of Mathematical Sciences
University of Guilan
P.O. Box 1914, Rasht, IRAN

**Abstract:** In this paper we discuss on the existence and uniqueness solution of the Riemann-Hilbert boundary value problem in the form:

$$\frac{\partial w}{\partial \bar{z}} = F(z, w, \frac{\partial w}{\partial z}) + G(z, w, \bar{w}), \qquad z \in D, \tag{1}$$

$$Re(a + ib)w = g \qquad \text{on} \quad \partial D \tag{2}$$

in $C_{1,\alpha}(\bar{D})$, where $a$, $b$ and $g$ are given Holder continuously differentiable real-valued functions of a real parameter $t$ on $\partial D$. We shall assume that $a^2 + b^2 = 1$ everywhere on $\partial D$.

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§Correspondence author

## 1. Introduction

Suppose that $D$ is a bounded domain belonging to the class $C_{1,\alpha}$ in the complex plane and $F(z,w,\frac{\partial w}{\partial z}) \in C_{\alpha}(\bar{D}), 0<\alpha<1$, we define the weakly and strongly singular operators $T_D$ and $\prod_D$ in the form:

$$T_D f(z) = -\frac{1}{\pi}\int\int_D \frac{1}{\xi - z} f(\xi) d\zeta d\eta,$$

$$\prod_D f(z) = -\frac{1}{\pi}\int\int_D \frac{1}{(\xi - z)^2} f(\xi) d\zeta d\eta,$$

where $\xi = \zeta + i\eta$, $z = x+iy$, and if $f \in L_p(D)$ then $T_D f$ is bounded and Holder continuous, [1], so that

$$\frac{\partial T_D f(z)}{\partial \bar{z}} = f(z),$$

$$\frac{\partial T_D f(z)}{\partial z} = \prod_D f(z).$$

Furthermore, we assume that $w \in C_{\alpha}(\bar{D}), 0<\alpha<1$, is an arbitrary solution of:

$$\frac{\partial w}{\partial \bar{z}} = F(z,w,\frac{\partial w}{\partial z}) + G(z,w,\bar{w}).$$

We define a function $\Phi$ as follows:

$$\Phi(z) = w(z) - T_D[F(z,w,\frac{\partial w}{\partial z}) + G(z,w,\bar{w})], \tag{3}$$

on differentiating $\Phi$ partially with respect to $\bar{z}$ and $z$ respectively, we obtain the following

$$\begin{cases} \frac{\partial \Phi}{\partial \bar{z}} = \frac{\partial w}{\partial \bar{z}} - [F(z,w,\frac{\partial w}{\partial z}) + G(z,w,\bar{w})] \\ \frac{\partial \Phi}{\partial z} = \frac{\partial w}{\partial z} - \prod_D[F(z,w,\frac{\partial w}{\partial z}) + G(z,w,\bar{w})] \end{cases}. \tag{4}$$

Furthermore, since $F(z,w,\frac{\partial w}{\partial z}) \in C_{\alpha}(\bar{D}), 0<\alpha<1$, the following estimates hold:

$$\|\Phi\|_{\alpha,D} \le \|w\|_{\alpha,D} + \|T_D(F+G)\|_{\alpha,D},$$

$$\|\frac{\partial \Phi}{\partial z}\|_{\alpha,D} \le \|\frac{\partial w}{\partial z}\|_{\alpha,D} + \|\prod_D(F+G)\|_{\alpha,D}.$$

It follows from the first equation (4) and Wely's lemma [6] that $\Phi$ is holomorphic function in $D$, it belong to the class $C_\alpha(\bar{D}), 0 < \alpha < 1$. Moreover, we deduce that, if $w$ is a solution of (1), then $w$ necessarily is of the form

$$w(z) = \Phi(z) + T_D[F(z, w, \frac{\partial w}{\partial z}) + G(z, w, \bar{w})], \tag{5}$$

where $\Phi$ is holomorphic in $D$. We now suppose that $(w, h)$ is a solution of the following system:

$$\begin{cases} w(z) = \Phi(z) + T_D[F(z, w, h) + G(z, w, \bar{w})] \\ h(z) = \Phi'(z) + \prod_D[F(z, w, h) + G(z, w, \bar{w})] \end{cases} . \tag{6}$$

Then $w$ is a solution to the given differential equation(1). On substituting $h = \frac{\partial w}{\partial z}$ in (6) we obtain the following result.

**Theorem 1.1.** *A function $w \in C_{1,\alpha}(\bar{D})$ in the form in (5), is a solution to the partial differential equation (1) if and only if, for a holomorphic function $\Phi \in C_{1,\alpha}(\bar{D})$, $(w, h)$ is the solution of the system (6).*

## 2. Existence of a General Solution in $C_{1,\alpha}(D)$

In order to determine the existence of a solution $w \in C_{1,\alpha}(\bar{D})$, we shall work in the following space $\mathcal{J}_\alpha(\bar{D})$, $0 < \alpha < 1$: we denote by $\mathcal{J}_\alpha(\bar{D})$ the set of all pairs $(w, h)$ for which both $w$ and $h$ belong to the space $C_\alpha(\bar{D})$. The norm shall be defined as follows:

$$\|(w, h)\|_{\alpha,D} := \max(\|w\|_{\alpha,D}, \|h\|_{\alpha,D}),$$

thus the $\mathcal{J}_\alpha(\bar{D})$ becomes a Banach space.

We impose the following assumptions on the complex differential equation (1):

I. The domain $D$ is bounded and belongs to the class $C_{1,\alpha}$, $0 < \alpha < 1$.

II. The right hand side $F(z, w, h)$ is a continuous function of $z \in D$, $w$, $h$.

III. The function $F(z, w, h) \in C_\alpha(\bar{D})$ if $w, h \in C_\alpha(\bar{D})$.

IV. $F(z, w, h)$ satisfies a Lipschitz condition in the metric of Hölder norm:

$$\|F(z, w, h) - F(z, \tilde{w}, \tilde{h})\|_{\alpha,D} \leq L_1\|(w, h) - (\tilde{w}, \tilde{h})\|_{\alpha,D}.$$

$$\|G(z, w, \bar{w}) - G(z, \tilde{w}, \bar{\tilde{w}}\|_{\alpha,D} \leq L_2\|w - \bar{w}\|_{\alpha,D}.$$

The assumption III is satisfied if the following Lipschitz condition is satisfied:

$$|F(z,w,h)-F(\tilde{z},\tilde{w},\tilde{h})|\leq L[|z-\tilde{z}|^{\alpha}+\max(|w-\tilde{w}|,|h-\tilde{h}|)]$$

$$z,\tilde{z}\in\bar{D}\quad\text{and}\quad(w,h),(\tilde{w},\tilde{h})\in\mathcal{J}_{\alpha}(\bar{D}).$$

In this case we have

$$|F(z,w,h)-F(\tilde{z},\tilde{w},\tilde{h})|\leq L_1[1+\max(H(\alpha,w),H(\alpha,h))]|z-\tilde{z}|^{\alpha},$$

where

$$H(\alpha,h)=\sup\frac{|h(z)-h(\tilde{z})|}{|z-\tilde{z}|^{\alpha}},\qquad z,\tilde{z}\in D.$$

With the aid of the right hand side of (5) we now define an operator $Q$ as follows: For $(w,h)\in\mathcal{J}_{\alpha}(\bar{D})$, let $Q(w,h)=(W,H)$:

$$W(z)=\Phi(z)+T_D[F(z,w,h)+G(z,w,\bar{w})],$$

$$H(z)=\Phi'(z)+\prod_D[F(z,w,h)+G(z,w,\bar{w})],$$

where $\Phi$ is a holomorphic function in $D$ and $\Phi\in C_{1,\alpha}(\bar{D})$.

By the properties of the integral operators $T_D$, $\prod_D$ in $C_{\alpha}(\bar{D})$, we conclude that the $Q$ maps $\mathcal{J}_{\alpha}(\bar{D})$ into itself. The following estimates hold:

$$\|W\|_{\alpha,D}\leq\|\Phi\|+\|T_D(F+G)\|_{\alpha,D},$$

$$\|H\|_{\alpha,D}\leq\|\Phi'\|+\|\prod_D(F+G)\|_{\alpha,D}.$$

In order to be able to apply the Banach fixed point theorem, we now compare the distance between two elements $(w,h),(\tilde{w},\tilde{h})\in\mathcal{J}_{\alpha}(\bar{D})$ and that between their corresponding image $(W,H),(\tilde{W},\tilde{H})$ under the mapping $Q$. Thus,

$$\begin{cases}W=\Phi+T_D[F(z,w,h)+G(z,w,\bar{w})]\\ \tilde{W}=\Phi+T_D[F(z,\tilde{w},\tilde{h})+G(z,w,\bar{\tilde{w}})]\end{cases},$$

$$\begin{cases}H=\Phi'+\prod_D[F(z,w,h)+G(z,w,\bar{w})]\\ \tilde{H}=\Phi'+\prod_D[F(z,\tilde{w},\tilde{h})+G(z,w,\bar{\tilde{w}})]\end{cases}.$$

It is an immediate consequence that

$$\|W-\tilde{W}\|_{\alpha}\leq\|T_D\|_{\alpha}\|[F(z,w,h)+G(z,w,\bar{w})]-[F(z,\tilde{w},\tilde{h})+G(z,w,\bar{\tilde{w}})]\|_{\alpha}$$

$$\leq L_1K_1(\alpha,D)\|(w,h)-(\tilde{w},\tilde{h})\|_{\alpha}+L_2K_2(\alpha,D)\|w-\tilde{w}\|_{\alpha,D},$$

$$\|H-\tilde{H}\|_\alpha \le \|\prod_D\|_\alpha \|[F(z,w,h)+G(z,w,\bar{w})]-[F(z,\tilde{w},\tilde{h})+G(z,w,\bar{\tilde{w}})\|_\alpha$$

$$\le L_1K_3(\alpha,D)L\|(w,h)-(\tilde{w},\tilde{h})\|_\alpha + L_2K_4(\alpha,D)\|w-\tilde{w}\|_{\alpha,D},$$

where the constants $K_1, K_2, K_3$ and $K_4$ are depending to $\alpha$ and $D$. Then

$$\|(W,H)-(\tilde{W},\tilde{H})\|_\alpha \le \max(L_1K_1+L_2K_2, L_1K_3+L_2K_4)\|(w,h)-(\tilde{w},\tilde{h})\|_\alpha.$$

If

$$0<\max(L_1K_1+L_2K_2, L_1K_3+L_2K_4)<1,$$

then the $Q$ is contractive in $\mathcal{J}_\alpha(\bar{D})$. Consequently by fixed point theorem $Q$ has exactly one fixed element $(w,h)$, with the condition $Q(w,h)=(w,h)$ so that

$$w=\Phi+T_D[F(z,w,h)+G(z,w,\bar{w})], \qquad h=\Phi'+\prod_D[F(z,w,h)+G(z,w,\bar{w})].$$

By Theorem 1.1, the corresponding $w$ is then a general solution of the partial complex differential equation (1). We now consider the modified Riemann-Hilbert boundary value problem. We determine the solution $w$ satisfying the following conditions:

$$\frac{\partial w}{\partial \bar{z}} = F(z,w,\frac{\partial w}{\partial z}) + G(z,w,\bar{w}), \qquad z\in D, \tag{7}$$

$$Re(a+ib)w = g \qquad \text{on} \quad \partial D, \tag{8}$$

where $a$, $b$, $\varphi$ and $g$ are given Hölder continuously differentiable real-valued function of a real parameter $t$ on $\partial D$, and we shall assume that $a^2+b^2=1$ everywhere on $\partial D$.

It was shown earlier that a general solution $w$ of the partial differential equation (1) has the form

$$w=\Phi+T_D[F(z,w,\frac{\partial w}{\partial z})+G(z,w,\bar{w})], \tag{9}$$

where $\Phi$ is any function in $C_{1,\alpha}(\bar{D})$, $0<\alpha<1$, and holomorphic in $D$. We replace $\Phi$ in (9) by a sum of two holomorphic functions $\phi_g$ and $\phi_{(w,h)}$. It follows then from (7) and (8) that

$$Re(a+ib)[\phi_g+\phi_{(w,h)}+T_D(F+G)]=g,$$

or

$$Re(a+ib)\phi_g + Re(a+ib)\phi_{(w,h)} = g - Re(a+ib)T_D(F+G).$$

Thus the modified Riemann-Hilbert problem for $w$ reduces to a similar problem for the holomorphic functions $\phi_g$ and $\phi_{(w,h)}$. That is, these functions should satisfy the following conditions:
I. $Re(a+ib)\phi_g = g$,
II. $Re(a+ib)\phi_{(w,h)} = -Re(a+ib)T_D[F(z,w,h)+G(z,w,\bar{w})]$.
Since both $g$ and $-Re(a+ib)T_D[F(z,w,h)+G(z,w,\bar{w})]$ are in $C_{1,\alpha}(\partial D)$, both problems have a unique solution in $C_{1,\alpha}(\bar{D})$, [1].

## 3. Conclusion

In this paper we have discussed the modified Riemann-Hilbert boundary value problem for complex partial differential equation (1) and (2) in $C_{1,\alpha}$. We can discuss on the existence and uniqueness of the boundary value problem in the Sobolev space.

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