International Journal of Applied Mathematics

Volume 28 No. 5 2015, 579-591

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v28i5.10

UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS BASED ON ORDER OF CONVOLUTION CONSISTENCE

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Abstract: In this paper we consider the modified Hadamard product or convolution of analytic functions with negative coefficients, combined with an Sălăgean integral operator. We discuss when it is a given class. Following idea of U. Bednarz and J. Sokól we shall determine the order of convolution consistence for certain analytic functions with negative coefficients.

AMS Subject Classification: 30C45, 30C50
Key Words: Hadamard product, analytic functions with negative coefficient, order of convolution, starlike function of order α, convex function of order α, extreme points

1. Introduction and Preliminaries

Let $\mathcal{H}(\mathcal{U})$ be the set of all functions which are regular in the unit disk $\mathcal{U} = \{z : |z| < 1\}$,

$$\mathcal{A} = \{f \in \mathcal{H}(\mathcal{U}) : f(0) = f'(0) - 1 = 0\},$$

and $\mathcal{S} = \{f \in \mathcal{A} : \ f \textit{ is univalent in } \mathcal{U}\}$.

Received: July 20, 2015

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In [12] the subfamily $\mathcal{N}$ of $\mathcal{S}$ consisting the functions of f of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0, \quad k = 2, 3, ..., \quad z \in \mathcal{U} \tag{1}$$

has been considered.

Let D^n be the *Sălăgean* differential operator (see [5], [9]) $D^n : \mathcal{A} \longrightarrow \mathcal{A}$, $n \in \mathbb{N}$, defined as

$$D^0 f(z) = f(z), \qquad D^1 f(z) = Df(z) = zf'(z), \qquad D^n f(z) = D(D^{n-1} f(z))$$

for all $z \in \mathcal{U}$.

Definition 1.1. ([5], [9]) Let $\alpha \in [0, 1)$ and $n \in \mathbb{N}$. The class $\mathcal{S}_n(\alpha)$ of the n-starlike functions of order α is defined by

$$\mathcal{S}_n(\alpha) = \left\{ f \in \mathcal{A} : Re \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha, \ z \in \mathcal{U} \right\}.$$

The class $\mathcal{S}_n(0)$ is denoted by $\mathcal{S}_n$. We note that $\mathcal{S}_0 = \mathcal{ST}$ is the class of starlike functions and $\mathcal{S}_1 = \mathcal{CV}$ is the class of convex functions. Further $\mathcal{S}_0(\alpha) = \mathcal{ST}(\alpha)$ is the class of starlike functions of order α and $\mathcal{S}_1(\alpha) = \mathcal{CV}(\alpha)$ is the class of convex functions of order α.

Let $\mathcal{T}_n(\alpha) = \mathcal{S}_n(\alpha) \bigcap \mathcal{N}$ be the class of n-starlike functions of order α with negative coefficients. In particular, $\mathcal{T}_0(\alpha)$ and $\mathcal{T}_1(\alpha)$ are the classes of the starlike functions of order α with negative coefficients and the class of convex functions of order α with negative coefficient, respectively, introduced by H. Silverman [12]. We denote $\mathcal{T}_n(0)$ by $\mathcal{T}_n$. (see also the works [4], [7], [8] for further developments involving each of the classes $\mathcal{S}_n(\alpha)$).

Definition 1.2. ([6]) Let $\alpha \in [0, 1)$, $\beta \in (0, 1]$ and let $n \in \mathbb{N}$; we define the class $\mathcal{T}_n(\alpha, \beta)$ of n-starlike functions of order α and type β with negative coefficients by

$$\mathcal{T}_n(\alpha, \beta) = \{ f \in \mathcal{A} : |J_n(f, \alpha; z)| < \beta, \ z \in \mathcal{U} \},$$

where

$$J_n(f, \alpha; z) = \frac{\frac{D^{n+1} f(z)}{D^n f(z)} - 1}{\frac{D^{n+1} f(z)}{D^n f(z)} + 1 - 2\alpha}, \ z \in \mathcal{U}.$$

$\mathcal{T}_0(\alpha, \beta)$ is the class of starlike functions of order α and type β, and $\mathcal{T}_1(\alpha, \beta)$ is the class of convex functions of order α and type β, further $\mathcal{T}_n(\alpha, 1) = \mathcal{T}_n(\alpha)$ is the class of n-starlike functions of order α with negative coefficients.

Theorem 1.3. ([6]) *Let $\alpha \in [0, 1)$, $\beta \in (0, 1]$ and $n \in \mathbb{N}$. The function f of the form (1) is in $\mathcal{T}_n(\alpha, \beta)$ if and only if*

$$\sum_{k=2}^{\infty} k^n \big(k - 1 + \beta(k + 1 - 2\alpha)\big) a_k \leq 2\beta(1 - \alpha).$$

This result is sharp.

From Definition 1.2 and Theorem 1.3, we have the following theorem:

Theorem 1.4. *For $f(z)$ of the form (1), we have $f \in \mathcal{T}_n(\alpha, 1) = \mathcal{T}_n(\alpha)$ if and only if*

$$\sum_{k=2}^{\infty} k^n (k - \alpha) a_k \leq 1 - \alpha, \quad \text{where } \alpha \in [0, 1). \tag{2}$$

This result is sharp.

The convolution or the Hadamard product of two functions f and g in $\mathcal{A}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

is the function $(f * g)$ defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Let use consider the Sălăgean integral operator (see [2], [3], [5]) $I^s : \mathcal{A} \longrightarrow \mathcal{A}$, $s \in \mathbb{R}$, such that

$$I^s f(z) = I^s \left(z + \sum_{k=2}^{\infty} a_k z^k \right) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^s} z^k.$$

Definition 1.5. ([2]) Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be subset of $\mathcal{A}$. We say that the triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is $\mathcal{S}$-closed under the convolution if there exists a number $S(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ such that

$$S(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = \min\{s \in \mathbb{R} : I^s(f * g) \in \mathcal{Z}, \ \forall f \in \mathcal{X}, \ \forall g \in \mathcal{Y}\}.$$

The number $S(\mathcal{X},\ \mathcal{Y},\ \mathcal{Z})$ is called the order of convolution consistence of the triple $(\mathcal{X},\ \mathcal{Y},\ \mathcal{Z})$.

In [2] U. Bednarz and J. Sokól obtained the order of convolution consistence concerning certain class of univalent functions (starlike, convex,...). Moreover, in [1] the authors studied the properties of the integral convolution of the neighborhoods of these classes.

The modified Hadamard product or $\circledast$-convolution of two functions f and g in $\mathcal{N}$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad and \quad g(z) = z - \sum_{k=2}^{\infty} b_k z^k \tag{3}$$

is the function $(f \circledast g)$ defined by

$$(f \circledast g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k. \quad (see[11]) \tag{4}$$

Definition 1.6. ([10]) The order of $\circledast$-convolution consistence of the triple $(\mathcal{X},\ \mathcal{Y},\ \mathcal{Z})$, where $\mathcal{X},\ \mathcal{Y}$ and $\mathcal{Z}$ are subsets of $\mathcal{N}$, is denoted by $S_{\circledast}$, where

$$S_{\circledast}(\mathcal{X},\ \mathcal{Y},\ \mathcal{Z}) = \min\{s \in \mathbb{R} : I^s(f \circledast g) \in \mathcal{Z},\ \forall f \in \mathcal{X},\ \forall g \in \mathcal{Y}\}.$$

G. Sălăgean and A. Taut in [10] obtained the order of convolution consistence concerning the classes of starlike functions with negative coefficients and convex functions with negative coefficients. They proved the following theorem:

Theorem 1.7. *We have the following $\circledast$-convolution consistence*

$$\begin{aligned}
&(a) \quad S_{\circledast}(\mathcal{T}_0, \mathcal{T}_0, \mathcal{T}_0) = -1,\\
&(b) \quad S_{\circledast}(\mathcal{T}_0, \mathcal{T}_0, \mathcal{T}_1) = 0,\\
&(c) \quad S_{\circledast}(\mathcal{T}_1, \mathcal{T}_0, \mathcal{T}_0) = -2,\\
&(d) \quad S_{\circledast}(\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_0) = -3,\\
&(e) \quad S_{\circledast}(\mathcal{T}_1, \mathcal{T}_0, \mathcal{T}_1 = -1,\\
&(f) \quad S_{\circledast}(\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1) = -2.
\end{aligned}$$

We note that $\mathcal{T}_0 = \mathcal{ST} \bigcap \mathcal{N}$ *and* $\mathcal{T}_1 = \mathcal{CV} \bigcap \mathcal{N}$.

In this paper we obtain the order of $\circledast$-convolution concerning the class $\mathcal{T}_n(\alpha)$.

2. Main Result

Theorem 2.1. *Let $0 < \alpha < 1$, if $f \in \mathcal{T}_{n+p}(\alpha)$ and $g \in \mathcal{T}_{n+q}(\alpha)$, then $I^s(f \circledast g) \in \mathcal{T}_{n+r}(\alpha)$, where $p,\ q,\ r,\ n\ \in \mathbb{N}$ and*

$$s \geq r - p - q - n - \log_2 \frac{2-\alpha}{1-\alpha}. \tag{5}$$

This result is sharp, and we have

$$S_{\circledast}(\mathcal{T}_{n+p}(\alpha), \mathcal{T}_{n+q}(\alpha), \mathcal{T}_{n+r}(\alpha)) = r - n - p - q - \log_2 \frac{2-\alpha}{1-\alpha}. \tag{6}$$

Proof. Since $f \in \mathcal{T}_{n+p}(\alpha)$ and $g \in \mathcal{T}_{n+q}(\alpha)$. If f and g have the form (3), then from (2) in Theorem 1.4, we have

$$\sum_{k=2}^{\infty} k^{n+p} \frac{k-\alpha}{1-\alpha} a_k \leq 1, \qquad \sum_{k=2}^{\infty} k^{n+q} \frac{k-\alpha}{1-\alpha} b_k \leq 1,$$

and by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=2}^{\infty} k^{n+\frac{p+q}{2}} \frac{k-\alpha}{1-\alpha} \sqrt{a_k b_k} \leq 1. \tag{7}$$

We need to find conditions on s, r, p, q, n such that

$$\sum_{k=2}^{\infty} k^{n+r-s} \frac{k-\alpha}{1-\alpha} a_k b_k \leq 1.$$

Thus, it is sufficient to show that

$$k^{n+r-s} \frac{k-\alpha}{1-\alpha} a_k b_k \leq k^{n+\frac{p+q}{2}} \frac{k-\alpha}{1-\alpha} \sqrt{a_k b_k}, \qquad k \in \{2, 3, ...\}.$$

that is,

$$\sqrt{a_k b_k} \leq k^{s-r+\frac{p+q}{2}}, \qquad k \in \{2, 3, ...\}.$$

From (7), we know that

$$\sqrt{a_k b_k} \leq k^{-n-\frac{p+q}{2}} \frac{1-\alpha}{k-\alpha}, \qquad k \in \{2, 3, ...\}.$$

Consequently, it is sufficiently to have

$$k^{-n-\frac{p+q}{2}} \frac{1-\alpha}{k-\alpha} \leq k^{s-r+\frac{p+q}{2}}, \qquad k \in \{2, 3, ...\}.$$

or, equivalently,

$$(1-\alpha)\frac{k^{r-s-n-p-q}}{k-\alpha} \leq 1, \qquad k \in \{2,3,....\}. \tag{8}$$

Letting $\phi(x) = \frac{x^{r-s-n-p-q}}{x-\alpha}$, $x \geq 2$, we obtain

$$\begin{aligned}\phi'(x) &= \frac{(r-s-n-p-q)x^{r-n-s-p-q-1}(x-\alpha) - x^{r-n-s-p-q}}{(x-\alpha)^2} \\ &= \frac{x^{r-n-s-p-q}}{(x-\alpha)^2}\Big((r-s-n-p-q)(\frac{x-\alpha}{x}) - 1\Big).\end{aligned}$$

Hence, $\phi'(x) \leq 0$ for all $x \leq 2$, or, $\phi(x)$ is a decreasing function on x. Consequently, from (8) it is sufficiently to have

$$\frac{1-\alpha}{2-\alpha}2^{r-s-n-p-q} \leq 1. \tag{9}$$

But the inequality (9) holds for s, r, p, q, n satisfying (5) and this show that

$$S_{\circledast}(\mathcal{T}_{n+p}(\alpha), \mathcal{T}_{n+q}(\alpha), \mathcal{T}_{n+r}(\alpha)) \leq r - p - q - n - \log_2 \frac{2-\alpha}{1-\alpha}. \tag{10}$$

Finally, by using the extremal functions

$$f_2(z) = z - \frac{1-\alpha}{2^{n+p}(2-\alpha)}z^2 \in \mathrm{T}_{n+p}(\alpha) \text{ and } g_2(z) = z - \frac{1-\alpha}{2^{n+q}(2-\alpha)}z^2 \in \mathrm{T}_{n+q}(\alpha),$$

From (4), we can see that

$$I^s(f \circledast g) = z - \frac{(1-\alpha)^2}{2^{2n+q+q+s}(2-\alpha)^2}z^2 \in \mathrm{T}_{n+r}(\alpha).$$

But from (2) in Theorem 1.4 we deduce

$$I^s(f \circledast g) = z - \frac{1-\alpha}{2^{n+r}(2-\alpha)}z^2 \in \mathrm{T}_{n+r}(\alpha), \tag{11}$$

and (11) show that the inequality (5) is sharp and we have,

$$S_{\circledast}(\mathcal{T}_{n+p}(\alpha), \mathcal{T}_{n+q}(\alpha), \mathcal{T}_{n+r}(\alpha)) \geq r - n - p - q - \log_2 \frac{2-\alpha}{1-\alpha}. \tag{12}$$

Therefore from (10) and (12), the relation (6) holds true. The proofs runs as in the previous proof. □

Corollary 2.2. *Let $0 < \alpha < 1$. We have the following $\circledast$-convolution consistence*

$$\begin{aligned}
&(a) \quad S_{\circledast}(\mathcal{T}_0(\alpha),\mathcal{T}_0(\alpha),\mathcal{T}_0(\alpha)) = -log_2\frac{2-\alpha}{1-\alpha},\\
&(b) \quad S_{\circledast}(\mathcal{T}_0(\alpha),\mathcal{T}_0(\alpha),\mathcal{T}_1(\alpha)) = 1-log_2\frac{2-\alpha}{1-\alpha},\\
&(c) \quad S_{\circledast}(\mathcal{T}_1(\alpha),\mathcal{T}_0(\alpha),\mathcal{T}_0(\alpha)) = -1-log_2\frac{2-\alpha}{1-\alpha},\\
&(d) \quad S_{\circledast}(\mathcal{T}_1(\alpha),\mathcal{T}_1(\alpha),\mathcal{T}_0(\alpha)) = -2-log_2\frac{2-\alpha}{1-\alpha},\\
&(e) \quad S_{\circledast}(\mathcal{T}_1(\alpha),\mathcal{T}_0(\alpha),\mathcal{T}_1(\alpha)) = -log_2\frac{2-\alpha}{1-\alpha},\\
&(f) \quad S_{\circledast}(\mathcal{T}_1(\alpha),\mathcal{T}_1(\alpha),\mathcal{T}_1(\alpha)) = -1-log_2\frac{2-\alpha}{1-\alpha}.
\end{aligned}$$

Theorem 2.3. *Let $0 \le \alpha,\beta,\gamma < 1,\ \alpha \ne \beta,\ \gamma \le \alpha,\ \gamma \le \beta$. If $f \in \mathcal{T}_{n+p}(\alpha)$ and $g \in \mathcal{T}_{n+q}(\beta)$, then $I^s(f \circledast g) \in \mathcal{T}_{n+r}(\gamma)$, where $p,\ q,\ r,\ n\ \in \mathbb{N}$ and*

$$s \ge r-p-q-n-\log_2\frac{2-\alpha}{1-\alpha}-\log_2\frac{2-\beta}{1-\beta}+\log_2\frac{2-\gamma}{1-\gamma}. \tag{13}$$

This result is sharp, and we have

$$S_{\circledast}(\mathcal{T}_{n+p}(\alpha),\mathcal{T}_{n+q}(\beta),\mathcal{T}_{n+r}(\gamma)) \\ = r-p-q-n-\log_2\frac{2-\alpha}{1-\alpha}-\log_2\frac{2-\beta}{1-\beta}+\log_2\frac{2-\gamma}{1-\gamma}. \tag{14}$$

Proof. Since $f \in \mathcal{T}_{n+p}(\alpha)$ and $g \in \mathcal{T}_{n+q}(\beta)$, if f and g have the form (3), then from (2) in Theorem1.4, we have

$$\sum_{k=2}^{\infty} k^{n+p}\frac{k-\alpha}{1-\alpha}a_k \le 1, \qquad \sum_{k=2}^{\infty} k^{n+q}\frac{k-\beta}{1-\beta}b_k \le 1,$$

and by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=2}^{\infty} k^{n+\frac{p+q}{2}}\sqrt{\frac{(k-\alpha)(k-\beta)}{(1-\alpha)(1-\beta)}}\sqrt{a_k b_k} \le 1. \tag{15}$$

We need to find conditions on s,r,p,q,n such that

$$\sum_{k=2}^{\infty} k^{n+r-s}\frac{k-\gamma}{1-\gamma}a_k b_k \le 1.$$

Thus, it is sufficient to show that

$$k^{n+r-s}\frac{k-\gamma}{1-\gamma}a_kb_k \le k^{n+\frac{p+q}{2}}\sqrt{\frac{(k-\alpha)(k-\beta)}{(1-\alpha)(1-\beta)}}\sqrt{a_kb_k},$$

that is,

$$\sqrt{a_kb_k} \le k^{s-r+\frac{p+q}{2}}\sqrt{\frac{(k-\alpha)(k-\beta)}{(1-\alpha)(1-\beta)}}\frac{1-\gamma}{k-\gamma}, \qquad k\in\{2,3,....\}.$$

From (15), we know that

$$\sqrt{a_kb_k} \le k^{-n-\frac{p+q}{2}}\sqrt{\frac{(1-\alpha)(1-\beta)}{(k-\alpha)(k-\beta)}}, \qquad k\in\{2,3,....\}.$$

Consequently, it is sufficiently to have,

$$k^{-n-\frac{p+q}{2}}\sqrt{\frac{(1-\alpha)(1-\beta)}{(k-\alpha)(k-\beta)}} \le k^{s-r+\frac{p+q}{2}}\sqrt{\frac{(k-\alpha)(k-\beta)}{(1-\alpha)(1-\beta)}}\frac{1-\gamma}{k-\gamma}, \quad k\in\{2,3,....\},$$

or, equivalently,

$$\frac{(1-\alpha)(1-\beta)(k-\gamma)}{(k-\alpha)(k-\beta)(1-\gamma)}k^{r-s-n-p-q} \le 1, \quad k\in\{2,3,....\}. \tag{16}$$

Letting $\phi(x)=\frac{(x-\gamma)}{(x-\alpha)(x-\beta)}x^{r-s-n-p-q}$, $x\ge 2$, we obtain that

$$\begin{aligned}
\phi'(x) &= \frac{x^{r-n-s-p-q}}{(x-\alpha)^2(x-\beta)^2}\Big((x-\alpha)(x-\beta) \\
&+ \frac{(x-\gamma)(x-\alpha)(x-\beta)(r-n-s-p-q)}{x} \\
&- (x-\alpha)(x-\gamma)-(x-\beta)(x-\gamma)\Big) \\
&\le \frac{x^{r-n-s-p-q}}{(x-\alpha)^2(x-\beta)^2}\Big(2(x-\alpha)(x-\beta)-(x-\alpha)(x-\gamma) \\
&- (x-\beta)(x-\gamma)\Big) \le 0.
\end{aligned}$$

Hence, $\phi'(x)\le 0$ for all $x\le 2$, or, $\phi(x)$ is a decreasing function on x. Consequently, from (16) it is sufficient to have

$$2^{r-s-n-p-q} \le \frac{2-\alpha}{1-\alpha}\frac{2-\beta}{1-\beta}\frac{1-\gamma}{2-\gamma}. \tag{17}$$

But the inequality (17) holds for s, r, p, q, n satisfying (13) and this show that

$$S_{\circledast}(\mathcal{T}_{n+p}(\alpha), \mathcal{T}_{n+q}(\alpha), \mathcal{T}_{n+r}(\alpha)) \leq r - n - s - p - q - \log_2 \frac{2-\alpha}{1-\alpha} - \log_2 \frac{2-\beta}{1-\beta} + \log_2 \frac{2-\gamma}{1-\gamma}. \quad (18)$$

Finally, by using the extremal functions

$$f_2(z) = z - \frac{1-\alpha}{2^{n+p}(2-\alpha)} z^2 \in \mathrm{T}_{n+p}(\alpha) \text{ and } g_2(z) = z - \frac{1-\beta}{2^{n+q}(2-\beta)} z^2 \in \mathrm{T}_{n+q}(\beta).$$

From (4) we can see that

$$I^s(f \circledast g) = z - \frac{(1-\alpha)(1-\beta)}{2^{2n+q+q+s}(2-\alpha)(2-\beta)} z^2 \in \mathrm{T}_{n+r}(\alpha). \quad (19)$$

But from (2) in Theorem 1.4 we deduce

$$I^s(f \circledast g) = z - \frac{1-\gamma}{2^{n+r}(2-\gamma)} z^2 \in \mathrm{T}_{n+r}(\alpha), \quad (20)$$

and (20) show that the inequality (12) is sharp and we have,

$$S_{\circledast}(\mathcal{T}_{n+p}(\alpha), \mathcal{T}_{n+q}(\alpha), \mathcal{T}_{n+r}(\alpha)) \geq r - n - s - p - q - \log_2 \frac{2-\alpha}{1-\alpha} - \log_2 \frac{2-\beta}{1-\beta} + \log_2 \frac{2-\gamma}{1-\gamma}. \quad (21)$$

Therefore from (18) and (21) the relation (14) holds. The proof goes as the previous one. □

Corollary 2.4. *Let $0 \leq \alpha, \beta, \gamma < 1$, $\alpha \neq \beta$, $\gamma \leq \alpha$, $\gamma \leq \beta$. We have the following $\circledast$-convolution consistence:*

$$(a) \quad S_{\circledast}(\mathcal{T}_0(\alpha), \mathcal{T}_0(\beta), \mathcal{T}_0(\gamma)) = -log_2 \frac{2-\alpha}{1-\alpha} - log_2 \frac{2-\beta}{1-\beta} + log_2 \frac{2-\gamma}{1-\gamma},$$

$$(b) \quad S_{\circledast}(\mathcal{T}_0(\alpha), \mathcal{T}_0(\beta), \mathcal{T}_1(\gamma)) = 1 - log_2 \frac{2-\alpha}{1-\alpha} - log_2 \frac{2-\beta}{1-\beta} + log_2 \frac{2-\gamma}{1-\gamma},$$

$$(c) \quad S_{\circledast}(\mathcal{T}_1(\alpha), \mathcal{T}_0(\beta), \mathcal{T}_0(\gamma)) = -1 - log_2 \frac{2-\alpha}{1-\alpha} - log_2 \frac{2-\beta}{1-\beta} + log_2 \frac{2-\gamma}{1-\gamma},$$

$$(d) \quad S_{\circledast}(\mathcal{T}_1(\alpha), \mathcal{T}_1(\beta), \mathcal{T}_0(\gamma)) = -2 - log_2 \frac{2-\alpha}{1-\alpha} - log_2 \frac{2-\beta}{1-\beta} + log_2 \frac{2-\gamma}{1-\gamma},$$

$$(e) \quad S_{\circledast}(\mathcal{T}_1(\alpha), \mathcal{T}_0(\beta), \mathcal{T}_1(\gamma)) = -log_2 \frac{2-\alpha}{1-\alpha} - log_2 \frac{2-\beta}{1-\beta} + log_2 \frac{2-\gamma}{1-\gamma},$$

$$(f) \quad S_{\circledast}(\mathcal{T}_1(\alpha), \mathcal{T}_1(\beta), \mathcal{T}_1(\gamma)) = -1 - log_2 \frac{2-\alpha}{1-\alpha} - log_2 \frac{2-\beta}{1-\beta} + log_2 \frac{2-\gamma}{1-\gamma}.$$

Theorem 2.5. *Let $0 \leq \alpha \leq 1$, if $f \in \mathcal{T}_{n+p}(\alpha)$ and $g \in \mathcal{T}_{n+q}$ Then $I^s(f \circledast g) \in \mathcal{T}_{n+r}(\alpha)$, where $p,\ q,\ r,\ n\ \in \mathbb{N}$ and*

$$s \geq r - p - q - n - 1.$$

This result is sharp, and we have

$$S_{\circledast}(\mathcal{T}_{n+p}(\alpha), \mathcal{T}_{n+q}, \mathcal{T}_{n+r}(\alpha)) = r - p - q - n - 1.$$

Proof. In Theorem (2.3), we set $\alpha = \gamma$ and $\beta = 0$. □

Corollary 2.6. *Let $0 < \alpha < 1$, we have the following $\circledast$-convolution consistence*

$$\begin{aligned}
&(a) \quad S_{\circledast}(\mathcal{T}_0(\alpha), \mathcal{T}_0, \mathcal{T}_0(\alpha)) = -1,\\
&(b) \quad S_{\circledast}(\mathcal{T}_0(\alpha), \mathcal{T}_0, \mathcal{T}_1(\alpha)) = 0,\\
&(c) \quad S_{\circledast}(\mathcal{T}_1(\alpha), \mathcal{T}_0, \mathcal{T}_0(\alpha)) = -2,\\
&(d) \quad S_{\circledast}(\mathcal{T}_1(\alpha), \mathcal{T}_1, \mathcal{T}_0(\alpha)) = -3,\\
&(e) \quad S_{\circledast}(\mathcal{T}_1(\alpha), \mathcal{T}_0, \mathcal{T}_1(\alpha)) = -1,\\
&(f) \quad S_{\circledast}(\mathcal{T}_1(\alpha), \mathcal{T}_1, \mathcal{T}_1(\alpha)) = -2.
\end{aligned}$$

Theorem 2.7. *Let $0 < \alpha < 1$, if $f \in \mathcal{T}_{n+p}(\alpha)$ and $g \in \mathcal{T}_{n+q}$ Then $I^s(f \circledast g) \in \mathcal{T}_{n+r}$, where $p,\ q,\ r,\ n\ \in \mathbb{N}$ and*

$$s \geq r - p - q - n - \log_2 \frac{2-\alpha}{1-\alpha}.$$

This result is sharp, and we have

$$S_{\circledast}(\mathcal{T}_{n+p}(\alpha), \mathcal{T}_{n+q}, \mathcal{T}_{n+r}) = r - p - q - n - \log_2 \frac{2-\alpha}{1-\alpha}.$$

Proof. In Theorem (2.3), we set $\beta = \gamma = 0$. □

Corollary 2.8. *Let $0 < \alpha < 1$,. We have the following $\circledast$-convolution* *consistence*

$$\begin{aligned}
&(a) \quad S_{\circledast}(\mathcal{T}_0(\alpha), \mathcal{T}_0, \mathcal{T}_0) = -log_2 \frac{2-\alpha}{1-\alpha},\\
&(b) \quad S_{\circledast}(\mathcal{T}_0(\alpha), \mathcal{T}_0, \mathcal{T}_1) = 1 - log_2 \frac{2-\alpha}{1-\alpha},\\
&(c) \quad S_{\circledast}(\mathcal{T}_1(\alpha), \mathcal{T}_0, \mathcal{T}_0) = -1 - log_2 \frac{2-\alpha}{1-\alpha},\\
&(d) \quad S_{\circledast}(\mathcal{T}_1(\alpha), \mathcal{T}_1, \mathcal{T}_0) = -2 - log_2 \frac{2-\alpha}{1-\alpha},\\
&(e) \quad S_{\circledast}(\mathcal{T}_1(\alpha), \mathcal{T}_0, \mathcal{T}_1) = -log_2 \frac{2-\alpha}{1-\alpha},\\
&(f) \quad S_{\circledast}(\mathcal{T}_1(\alpha), \mathcal{T}_1, \mathcal{T}_1) = -1 - log_2 \frac{2-\alpha}{1-\alpha}.
\end{aligned}$$

Theorem 2.9. *Let $0 \le \alpha < 1$, $0 < \beta \le 1$, if $f \in \mathcal{T}_{n+p}(\alpha, \beta)$ and $g \in \mathcal{T}_{n+q}(\alpha, \beta)$. Then $I^s(f \circledast g) \in \mathcal{T}_{n+r}(\alpha, \beta)$, where $p,\ q,\ r,\ n\ \in \mathbb{N}$ and*

$$s \ge r - p - q - n - \log_2 \frac{1+\beta(3-2\alpha)}{2\beta(1-\alpha)}. \tag{22}$$

This result is sharp, and we have

$$S_{\circledast}(\mathcal{T}_{n+p}(\alpha), \mathcal{T}_{n+q}(\beta), \mathcal{T}_{n+r}(\gamma)) = r - p - q - n - \log_2 \frac{1+\beta(3-2\alpha)}{2\beta(1-\alpha)}. \tag{23}$$

Proof. By applying the Cauchy-Schwarz inequality and similar technique as in the proof of Theorem 2.1, we obtain that

$$\frac{k^{r-s-n-p-q} 2\beta(1-\alpha)}{(k-1)+\beta(k+1-2\alpha)} \le 1, \qquad k \in \{2, 3, ...\}. \tag{24}$$

Letting $\phi(x) = \frac{x^{r-s-n-p-q}}{(x-1)+\beta(x+1-2\alpha)}$, $x \ge 2$ and we obtain that,

$$\phi'(x) = \frac{\big(r-s-n-p-q\big)\big(x(\beta+1)+\beta(1-2\alpha)-1\big)x^{r-n-s-p-q-1} - (\beta+1)x^{r-n-s-p-q}}{\big(x(\beta+1)+\beta(1-2\alpha)-1\big)^2}$$

$$= \frac{x^{r-n-s-p-q}}{(x(\beta+1)+\beta(1-2\alpha)-1)^2}\Big((r-s-n-p-q)(\frac{x(\beta+1)+\beta(1-2\alpha)-1}{x}) - (\beta+1)\Big).$$

Hence, $\phi'(x) \le 0$ for all $x \le 2$, or, $\phi(x)$ is a decreasing function on x. Consequently, from (24) it is sufficient to have

$$2^{r-s-n-p-q} \le \frac{1+\beta(3-2\alpha)}{2\beta(1-\alpha)}. \tag{25}$$

But the inequality (25) holds for s, r, p, q, n satisfying (22) and this show that

$$S_{\circledast}(\mathcal{T}_{n+p}(\alpha), \mathcal{T}_{n+q}(\alpha), \mathcal{T}_{n+r}(\alpha)) \leq r - p - q - n - \log_2 \frac{1+\beta(2-\alpha)}{2\beta(1-\alpha)}. \tag{26}$$

Finally, by using the extremal functions

$$f_2(z) = z - \frac{2\beta(1-\alpha)}{2^{n+p}(1+\beta(3-2\alpha))} z^2 \in \mathrm{T}_{n+p}(\alpha, \beta),$$
$$g_2(z) = z - \frac{2\beta(1-\alpha)}{2^{n+q}(1+\beta(3-2\alpha))} z^2 \in \mathrm{T}_{n+q}(\alpha, \beta).$$

From (4) we can see that

$$I^s(f \circledast g) = z - \frac{(2\beta(1-\alpha))^2}{2^{2n+q+q+s}(1+\beta(3-2\alpha)))^2} z^2 \in \mathrm{T}_{n+r}(\alpha)$$

But from (2) in Theorem 1.4 we deduce

$$I^s(f \circledast g) = z - \frac{2\beta(1-\alpha)}{2^{n+r}(1+\beta(3-2\alpha))} z^2 \in \mathrm{T}_{n+r}(\alpha), \tag{27}$$

and (27) show that the inequality (22) is sharp and we have,

$$S_{\circledast}(\mathcal{T}_{n+p}(\alpha), \mathcal{T}_{n+q}(\alpha), \mathcal{T}_{n+r}(\alpha)) \geq r - n - p - q - \log_2 \frac{1+\beta(3-2\alpha)}{2\beta(1-\alpha)}. \tag{28}$$

Therefore from (26) and (28) the relation (23) holds. Those proofs run as the previous ones. □

Corollary 2.10. *Let $0 \leq \alpha < 1$, $0 < \beta \leq 1$. We have the following $\circledast$-convolution consistence*

$$(a) \quad S_{\circledast}(\mathcal{T}_0(\alpha,\beta), \mathcal{T}_0(\alpha,\beta), \mathcal{T}_0(\alpha,\beta)) = -\log_2 \frac{1+\beta(3-2\alpha)}{2\beta(1-\alpha)},$$
$$(b) \quad S_{\circledast}(\mathcal{T}_0(\alpha,\beta), \mathcal{T}_0(\alpha,\beta), \mathcal{T}_1(\alpha,\beta)) = 1 - \log_2 \frac{1+\beta(3-2\alpha)}{2\beta(1-\alpha)},$$
$$(c) \quad S_{\circledast}(\mathcal{T}_1(\alpha,\beta), \mathcal{T}_0(\alpha,\beta), \mathcal{T}_0(\alpha,\beta)) = -1 - \log_2 \frac{1+\beta(3-2\alpha)}{2\beta(1-\alpha)},$$
$$(d) \quad S_{\circledast}(\mathcal{T}_1(\alpha,\beta), \mathcal{T}_1(\alpha,\beta), \mathcal{T}_0(\alpha,\beta)) = -2 - \log_2 \frac{1+\beta(3-2\alpha)}{2\beta(1-\alpha)},$$
$$(e) \quad S_{\circledast}(\mathcal{T}_1(\alpha,\beta), \mathcal{T}_0(\alpha,\beta), \mathcal{T}_1(\alpha,\beta)) = -\log_2 \frac{1+\beta(3-2\alpha)}{2\beta(1-\alpha)},$$
$$(f) \quad S_{\circledast}(\mathcal{T}_1(\alpha,\beta), \mathcal{T}_1(\alpha,\beta), \mathcal{T}_1(\alpha,\beta)) = -1 - \log_2 \frac{1+\beta(3-2\alpha)}{2\beta(1-\alpha)}.$$

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