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SOLUTION OF HIGHER ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS BY LEGENDRE WAVELETS

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Abstract: In this paper the solution of higher order Volterra integro-differential equations using Legendre wavelets is presented. The problem is reduced into the solution of non-linear algebraic equations using Legendre wavelets. A reliable approach is also discussed for the convergence of the Legendre wavelets. To demonstrate the validity and applicability of the proposed method, some examples have been discussed. The results in these examples demonstrate the reliability and efficiency of the proposed method. The results obtained by the Legendre wavelet method are very near to exact solution.

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1. Introduction

Most of the real life problems using mathematical modeling are converted into functional equations like partial differential equations, integro-differential equation, stochastic equations and others. The mathematical formulations of many physical phenomena result into integro-differential equations.

An integro-differential equation is an equation which contains derivatives and integrations of unknown function. These equations arise in fluid dynamics, biological models and chemical kinetics. In the past several decades for the numerical solution of linear and non linear differential equations many effective methods have been presented like variational iteration method [7], Adomian decomposition method [1], homotopy analysis method [8], homotopy perturbation method [5], He's homotopy perturbation method [2] and wavelet method [4].

But there are very few methods for the numerical solution of the boundary value problems for higher order integro differential equations. The boundary value problems for higher order integro differential equations had been investigated by Agrawal [3], Morchalo [11], [12], Karimpour [9], Wazwaz [14], Yusufoglu [15] and Zhao, Corless [16].

In the recent years, the wavelets have started to play an important role into many different fields of science and engineering. Many researchers have started to use different types of wavelets to solve differential equations and found wavelets a powerful and effective tool for analyzing problems of greater computational complexity.

Lepik [10] used Haar wavelet method to solve non-linear integro differential equations. Ghasemi et al. [5] presented the comparison between Wavelet - Galerkin method and homotopy perturbation method for the non - linear integro differential equations. Ghasemi et al. also provided numerical solution of linear integro differential equations by using sine-cosine wavelet method [6].

In the present paper, the Legendre wavelet method (LWM) is applied to find numerical solution of m-th order integro-differential equation of the form

$$y^{(m)}(x) = f(x) + \int_0^x k(x,t)F\big(y(t)\big)dt, \quad 0 < x < b \tag{1}$$

with the boundary conditions

$$y^{(j)}(0) = \alpha_j, \ j = 0, 1, 2, ..., r-1,$$
$$y^{(j)}(b) = \beta_j, \ j = r, r+1, ..., m-1,$$

where $y^{(m)}(x)$ indicates the m-th order derivative of $y(x)$ and $F\big(y(x)\big)$ is a nonlinear function, $K(x,t)$ is the kernel and $f(x)$ is a function of x. $y(x)$ and $f(x)$ are real and can be differentiated any number of times for $x \in [0,b]$ and α_j, and β_j are real finite constants.

The Legendre wavelet method converts the integro differential equation into integral equation and using Legendre wavelets expand the solution having unknown coefficients. The approximate solution of equation (1) is obtained by evaluating unknown coefficients using the properties of Legendre wavelets along with the Gaussian integration formula.

2. Preliminary Concepts

2.1. Wavelets and Legendre Wavelets

The wavelets are families of functions constructed from dilation parameter a and translation parameter b of a single function called the 'mother wavelet' $\Psi(t)$. They are defined by

$$\Psi_{a,b}(t) = \frac{1}{\sqrt{\mid a \mid}} \Psi\left(\frac{t-b}{a}\right), \quad a,b \in R,\ a \neq 0.$$

Now for the discrete values of a and b, $a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 0$, where n and k are positive integers. We have the following family of discrete wavelets:

$$\Psi_{k,n}(t) = \mid a \mid^{-1/2} \Psi\Big(a_0^k t - nb_0\Big),$$

where $\Psi_{k,n}(t)$ forms a basis of $L^2(R)$.

Legendre Wavelets. The Legendre wavelet $\Psi_{nm}(t) = \Psi(k,\hat{n},m,t)$ have four arguments $\hat{n} = 2n-1, n = 1,2,3,\cdots,2^{k-1}$, k can be any positive integer, m is the order of the Legendre polynomials and t is the normalized time. They are defined on the interval $[0,1)$ by

$$\Psi_{nm}(t) = \begin{cases} 2^{k/2}\sqrt{m+\frac{1}{2}}P_m\big(2^k t - \hat{n}\big) & \text{for } \frac{\hat{n}-1}{2^k} \leq t \leq \frac{\hat{n}+1}{2^k}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is 2^{-k} and the translation parameter is $\hat{n}2^{-k}$. Here $P_m(t)$ are the well known Legendre

polynomials of order m which are orthogonal with respect to the weight function $w(t) = 1$ on the interval $[-1, 1]$ and satisfy the following formulae:

$$P_0(t) = 1, \quad P_1(t) = t$$

and

$$P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t\, P_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t), \quad m = 1, 2, 3, \cdots .$$

2.2. Function Approximation

A function defined over [0, 1] may be expressed as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{nm} \Psi_{nm}(t). \tag{3}$$

Here

$$C_{nm} = \langle f(t), \Psi_{nm}(t) \rangle$$

$\langle ., . \rangle$ denotes the inner product. On truncating the infinite series in equation (3), we have

$$f(t) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{nm} \Psi_{nm}(t) = C^T \Psi(t),$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$\begin{aligned} C = \big[&C_{10}, C_{11}, ..., C_{1\,M-1}, C_{20}, ..., \\ &C_{2\,M-1}, ..., C_{2^{k-1}0}, ..., C_{2^{k-1}M-1}\big]^T \end{aligned} \tag{4}$$

and

$$\begin{aligned} \psi(t) = \big[&\psi_{10}(t), \psi_{11}(t), ..., \psi_{1\,M-1}(t), \psi_{20}(t), ..., \\ &\psi_{2\,M-1}(t), ..., \psi_{2^{k-1}0}(t), ..., \psi_{2^{k-1}M-1}\big]^T. \end{aligned} \tag{5}$$

3. Legendre Wavelet Method (LWM) for Higher Order Integro-Differential Equations with Given Boundary Conditions

Consider the integro-differential equation given in equation (1). Now define $L = \frac{d^m}{dx^m}$. So

$$L^{-1}y(x) = \int_0^x dx \int_0^x dx ... \int_0^x y(x)dx \quad (m\text{-times}).$$

By equation (1),

$$L^{-1}\big\{y^m(x)\big\} = L^{-1}\big\{f(x)\big\} + L^{-1}\left\{\int_0^x K(x,t)F\big(y(t)\big)dt\right\}$$

$$y(x) = g(x) + h(x) + L^{-1}\left\{\int_0^x K(x,t)F\big(y(t)\big)dt\right\},$$

where $h(x) = L^{-1}\big\{f(x)\big\}$ and $g(x)$ is a function of x along with constants.

Now suppose

$$\varnothing(x) = g(x) + h(x),$$

then

$$y(x) = \varnothing(x) + \int_0^x (x-t)^m K(x,t)F\big(y(t)\big)dt. \tag{6}$$

Let

$$y(x) = C^T\Psi(x). \tag{7}$$

So we have

$$C^T\Psi(x) = \varnothing(x) + \int_0^x (x-t)^m K(x,t)F\big(C^T\Psi(t)\big)dt. \tag{8}$$

Now collocate equation (8) at $2^{k-1}M$ points x_i as

$$C^T\Psi(x_i) = \varnothing(x_i) + \int_0^{x_i} (x-t)^m K(x,t)F\big(C^T\Psi(t)\big)dt. \tag{9}$$

On taking the zeros of the Chebyshev polynomials as collocation points, we have

$$x_i = \cos\big((2i+1)\pi/2^kM\big), \quad i = 1, 2, ..., 2^{k-1}M.$$

For using the Gaussian integration formula in equation (9), the interval $[0, x_i]$ is transferred into the interval [-1,1] by means of the transformation $\tau = \frac{2}{x_i}t - 1$. Then, equation (9) may be written as

$$C^T\Psi(x_i) = \varnothing(x_i) + \frac{x_i}{2}\int_{-1}^{1}\Big(x - \frac{x_i}{2}(\tau+1)\Big)^m \times K\Big(x, \frac{x_i}{2}(\tau+1)\Big)F\Big(C^T\Psi\big(\frac{x_i}{2}(\tau+1)\big)\Big)d\tau.$$

Now on using the Gaussian integration formula, we have

$$C^T\Psi(x_i) = \varnothing(x_i) + \frac{x_i}{2}\sum_{j=1}^{s} W_j\Big(x - \frac{x_i}{2}(\tau_j+1)\Big)^m \times K\Big(x, \frac{x_i}{2}(\tau_j+1)\Big)F\Big(C^T\Psi\big(\frac{x_i}{2}(\tau_j+1)\big)\Big)d\tau, \tag{10}$$

where τ is a zero of the Legendre polynomials P_{s+1}, and W_j are the corresponding weights. The Gaussian integration formula is used for the polynomials of degree not exceeding $2s+1$. The weight W_j can be evaluated by the formula

$$W_j = \int_{-1}^{1}\prod_{j=0, j\neq i}^{s}\Big(\frac{\tau-\tau_j}{\tau_i-\tau_j}\Big)d\tau.$$

From equation (10), $2^{k-1}M$ non-linear equations are obtained which can be solved for the elements of C in equation (7) using Newton's iterative method.

4. Theorems on Convergence and Error Estimation

In this section, some theorems on convergence analysis and error estimation of proposed method are given.

Theorem 4.1. *The solution of problem (1), given by series solution equation (3), using the Legendre wavelet method converges towards $u(x)$.*

Proof. Suppose $\Psi_{k,n}(t) = \mid a \mid^{-1/2} \Psi(a_0^k t - nb_0)$, where $\Psi_{k,n}(t)$ form a basis of $L^2(R)$ and let $L^2(R)$ be a Hilbert space. In particular, for $a_0 = 2$ and $b_0 = 1, \Psi_{k,n}(t)$ forms an orthonormal basis. □

Let

$$u(x) = \sum_{i=1}^{M-1} C_{1i}\Psi_{1i}(x),$$

where

$C_{1i} = \langle u(x), \Psi_{1i}(x)\rangle$ for $k = 1$ and $\langle ., .\rangle$ represents an inner product.

Now

$$u(x) = \sum_{i=1}^{n} \langle u(x), \Psi_{1i}(x)\rangle \Psi_{1i}(x).$$

Let us denote

$$\Psi_{1i}(x) \text{ as } \Psi(x) \text{ and } \alpha_j = \langle u(x), \Psi(x)\rangle.$$

Suppose $\{S_n\}$ is the sequence of partial sums of $\big(\alpha_j \Psi(x_j)\big)$ and S_n, S_m are arbitrary partial sums with $n \geq m$.

Now we prove $\{S_n\}$ is a Cauchy sequence in Hilbert space.

Let

$$S_n = \sum_{j=1}^{n} \alpha_j \Psi(x_j).$$

So

$$\langle u(x), S_n\rangle = \left\langle u(x), \sum_{j=1}^{n} \alpha_j \Psi(x_j) \right\rangle$$
$$= \sum_{j=1}^{n} \overline{\alpha_j} \langle u(x), \Psi(x_j)\rangle = \sum_{j=1}^{n} \overline{\alpha_j}\alpha_j = \sum_{j=1}^{n} |\ \alpha_j\ |^2 .$$

Now we claim that

$$\|\ S_n - S_m\ \|^2 = \sum_{j=m+1}^{n} |\ \alpha_j\ |^2 \quad \text{for } n > m.$$

Then,

$$\|\ \sum_{j=m+1}^{n} \alpha_j \Psi(x_j)\ \|^2 = \left\langle \sum_{i=m+1}^{n} \alpha_i \Psi(x_i), \sum_{j=m+1}^{n} \alpha_j \Psi(x_j) \right\rangle$$
$$= \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \alpha_i \overline{\alpha_j} \langle \Psi(x_i), \Psi(x_j)\rangle = \sum_{j=m+1}^{n} \alpha_j \overline{\alpha_j} = \sum_{j=m+1}^{n} |\ \alpha_j\ |^2 .$$

Hence $\| S_n - S_m \|^2 = \sum_{j=m+1}^{n} | \alpha_j |^2$ for $n > m$.

By the Bessel inequality, we have $\sum_{j=1}^{\infty} | \alpha_j |^2$ is convergent and hence $\| S_n - S_m \|^2 \to 0$ as $m, n \to \infty$. Now $\| S_n - S_m \|$ converges to 0 and $\{S_n\}$ is a Cauchy sequence so suppose it converges to $'s'$. We prove that $u(x) = s$,

$$\langle s - u(x), \Psi(x_j) \rangle = \langle s, \Psi(x_j) \rangle - \langle u(x), \Psi(x_j) \rangle$$

$$= \langle \lim_{n \to \infty} S_n, \Psi(x_j) \rangle - \alpha_j = \lim_{n \to \infty} \langle S_n, \Psi(x_j) \rangle - \alpha_j = \alpha_j - \alpha_j$$

$$\Rightarrow \langle s - u(x), \Psi(x_j) \rangle = 0.$$

Hence $u(x) = s$ and $\sum_{j=1}^{n} \alpha_j \Psi(x_j)$ converges to $u(x)$.

4.1. Error Estimation

Error estimation for the approximate solution of equation (6) is discussed in this part using the method presented in [13].

Suppose $\overline{u}(x)$ is the approximate solution for $u(x)$ and $E_n(x) = u(x) - \overline{u}(x)$ is the error function,

$$\overline{u}(x) = \varnothing(x) + \int_0^x (x-t)^m K(x,t) F\big(y(t)\big) dt + H_n(x),$$

where $H_n(x)$ is the perturbation term,

$$H_n(x) = \overline{u}(x) - \varnothing(x) - \int_0^x (x-t)^m K(x,t) F\big(y(t)\big) dt. \tag{11}$$

Now find an approximation $\overline{E_n}(x)$ to the error function $E_n(x)$ in the same way as we did before the solution of the problem. Subtracting equation (11) from equation (6), the error function $E_n(t)$ satisfies the problem

$$E_n(x) + \int_0^x (x-t)^m K(x,t) F\big(y(t)\big) dt = -H_n(x). \tag{12}$$

Equation (12) is recalculated in the same way as we did before the solution of equation (7) for the construction of $\overline{E_n}(x)$ to $E_n(x)$. Hence the stability of the Legendre wavelet method is established through this convergence theorem and error estimation.

5. Numerical Examples

In the following examples, the proposed Legendre wavelet method is discussed to find the numerical solution of two boundary value problems of fourth order integro-differential equations.

Example (i). In the following example, we consider linear boundary value problem for the integro-differential equation

$$y''''(x) = 5e^x - 1 + \int_0^x y(x)dx, \;\; 0 < x < 1, \tag{13}$$

subject to the boundary conditions

$$y(0) = 0, y'(0) = 1, y(1) = e, y'(1) = 2e. \tag{14}$$

We apply the Legendre wavelet method presented in this paper and solve the equation (13) for $k = 1$ and $M = 4$.

On applying L^{-1} both sides of equation (13), we have

$$L^{-1}[y''''(x)] = L^{-1}[5e^x] - L^{-1}[-1] + L^{-1}\left[\int_0^x y(x)dx\right]$$

$$y(x) = x + \frac{Ax^2}{2} + \frac{Bx^3}{6} + \frac{x^4}{24} + L^{-1}[5e^x] + L^{-1}\left[\int_0^x y(x)dx\right],$$

where $y''(0) = A,\ y'''(0) = B$,

$$y(x) = -5 - 4x + \frac{(A-5)x^2}{2} + \frac{(B-5)x^3}{6} + \frac{x^4}{24} + 5e^x + \int_0^x (x-t)^4 y(t)dt.$$

On replacing $y(x)$ by $C^T\Psi(x)$,

$$C^T\Psi(x) = -5 - 4x + \frac{(A-5)x^2}{2} + \frac{(B-5)x^3}{6} + \frac{x^4}{24} + 5e^x + \int_0^x (x-t)^4 C^T\Psi(t)dt. \tag{15}$$

On solving equation (15), we have

$$C_{10} = \frac{-B - 7(2A+B)}{24}, \;\; C_{11} = \frac{-4 - (2A+B)}{4\sqrt{3}},$$
$$C_{12} = \frac{2A+B}{24\sqrt{5}}, \;\; C_{13} = \frac{B}{120\sqrt{7}}.$$

On using the boundary conditions, we have $A = 2, B = 3$. Now by equation (7) we have

$$y(x) = C_{10}\Psi_{10} + C_{11}\Psi_{11} + C_{12}\Psi_{12} + C_{13}\Psi_{13}.$$

Hence,

$$y(x) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!}.$$

Therefore, we have $y(x) = xe^x$ which is the exact solution.

Table 1 shows the error for different values of M compared with the exact solution for Example (i).

Example (ii). Now we consider the non-linear boundary value problem for integro-differential equation

$$y''''(x) = 3e^x - \frac{x^3}{3} - \frac{e^{2x}}{2} - 2xe^x - \frac{3}{2} + \int_0^x y^2 dx, \tag{16}$$

subject to the boundary conditions $y(0) = 1, y(1) = 1 + e, y'(0) = 2, y'(1) = 1 + e$. For $k = 1, M = 4$, we have by equation (16)

$$L^{-1}[y''''(x)] = L^{-1}\Big[3e^x - \frac{x^3}{3} - \frac{e^{2x}}{2} - 2xe^x - \frac{3}{2}\Big] + L^{-1}\left[\int_0^x y^2 dx\right]$$
$$y(x) = 1 + 2x + \frac{Ax^2}{2} + \frac{Bx^3}{6}$$
$$+L^{-1}\Big[3e^x - \frac{x^3}{3} - \frac{e^{2x}}{2} - 2xe^x - \frac{3}{2}\Big] + L^{-1}\left[\int_0^x y^2 dx\right],$$

where $y''(0) = A, y'''(0) = B$,

$$y(x) = \frac{193}{32} - \frac{111}{16}x + \frac{8A-55}{16}x^2 + \frac{4B-19}{24}x^3 - \frac{1}{16}x^4$$
$$-\frac{1}{2520}x^7 + 11e^x - \frac{1}{32}e^{2x} - 2xe^x + \int_0^x (x-t)^4 \big(y(t)\big)^2 dt.$$

Table 1 The error for example (i) at M = 4, 10, 15

M	X	Exact Solution	LWM Solution	Error between Exact and LWM solution
4	0.0	0.0	0.0	0.0
	0.2	0.24428054	0.2442666	0.00001394
	0.4	0.59672984	0.59626667	0.00046317
	0.6	1.09327128	1.0896	0.00367128
	0.8	1.78043272	1.76426667	0.01616605
	1.0	2.71828182	2.66666667	0.05161515
10	0.0	0.0	0.0	0.0
	0.2	0.24428054	0.24428058	0.00000004
	0.4	0.59672984	0.59672972	0.00000012
	0.6	1.09327128	1.09327356	−0.00000228
	0.8	1.78043272	1.78043666	−0.00000394
	1.0	2.71828182	2.71828889	−0.00000707
15	0.0	0.0	0.0	0.0
	0.2	0.24428054	0.24428054	0.0
	0.4	0.59672984	0.59672978	0.00000006
	0.6	1.09327128	1.09327112	0.00000016
	0.8	1.78043272	1.78043298	−0.00000026
	1.0	2.71828182	2.71828334	−0.00000152

On replacing $y(x)$ by $C^T\Psi(x)$ and solving it, we have

$$C_{10} = \frac{-B - 7(2A+B)}{24}, \quad C_{11} = \frac{-4 - (2A+B)}{4\sqrt{3}},$$
$$C_{12} = \frac{2A+B}{24\sqrt{5}}, \quad C_{13} = \frac{B}{120\sqrt{7}}.$$

By using the boundary conditions, we have $A = 1, B = 1$. Now by equation (7) we have

$$y(x) = C_{10}\Psi_{10} + C_{11}\Psi_{11} + C_{12}\Psi_{12} + C_{13}\Psi_{13}.$$

Hence,

$$y(x) = 1 + 2x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

Therefore, we have $y(x) = x + e^x$ which is the exact solution.

Table 2 shows the error for different values of M with the exact solution for Example (ii).

Table 2 The error for example (ii) at M = 4, 10, 15

M	X	Exact Solution	LWM Solution	Error between Exact and LWM solution
4	0.0	1.0000000	1.0000000	0.0
	0.2	1.4214027	1.4213333	0.0000694
	0.4	1.8918246	1.8906667	0.0011579
	0.6	2.4221188	2.4160000	0.0061188
	0.8	3.0255409	3.0053333	0.0202076
	1.0	3.7182818	3.6666667	0.0516151
10	0.0	1.0000000	1.0000000	0.0
	0.2	1.4214027	1.4214011	0.0000016
	0.4	1.8918246	1.8918111	0.0000135
	0.6	2.4221188	2.4221011	0.0000177
	0.8	3.0255409	3.0255222	0.0000187
	1.0	3.7182818	3.7183333	−0.0000515
15	0.0	1.0000000	1.0000000	0.0
	0.2	1.4214027	1.4214022	0.0000005
	0.4	1.8918246	1.8918233	0.0000013
	0.6	2.4221188	2.4221111	0.0000077
	0.8	3.0255409	3.0255455	−0.0000046
	1.0	3.7182818	3.7182889	−0.0000071

6. Conclusion

In this paper the Legendre wavelet method (LWM) has been proposed for the solutions of boundary value problems of higher order linear and nonlinear Volterra integro-differential equations. The Gaussian integration method and the properties of the Legendre wavelets are used to reduce the problem into nonlinear algebraic equations. For the given function approximation, the convergence of the Legendre wavelet method is also proved. Illustrative examples are given to demonstrate the validity, accuracy and correctness of the proposed method. The error between the approximate solution and exact solution decreases when the degree of approximation increases.

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