OPTIMAL INVESTMENT PROBLEM
WITH OPTION

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Abstract: In this paper, the aim of the investor is to maximize expected return for a given level of risk. The model is based on a particular risk measure conditional value-at-risk (CVaR), the expected loss exceeding Value-at-Risk. The portfolio is optimized for investment in equity, debt and option on equity. In order to enhance the return potential, the expected return of intermittent re-investment payment obtained from option is also optimized. We develop a method to deal with the maximization of return of a portfolio in a two period context extending the work of Korn and Zeytun [4].

1. Introduction

Investment in financial asset classes, subject to uncertain prices, leads to an

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expected return distribution. The deviation from expectation is characterized as the risk of investment (Artzner et al. [1]) contends, risk is related to the variability of the future value of a position, due to market changes or more generally to uncertain events, it is better to instead consider future values only. The risk measure used can be relative or absolute. The former category includes measures like variance, standard deviation or absolute deviation, whereas latter category includes measures like value at risk and its variants like conditional value at risk. The former measures are also termed as volatility measures.

Portfolio is allocation of investment budget into competing asset classes with a stated objective, which could be either expected return maximization or risk minimization. These are competing objectives requiring a trade-off on the part of risk-averse utility maximizer investor, thereby leading to concept of portfolio optimization and making of economic choice under uncertainty and risk. Portfolios are essentially built to reduce the risk for a given level of return. They essentially try to lever the co-variance characteristics of the constituent securities and it is contended that the portfolio risk is an amalgam of individual as well as interactive risk. The earliest theory of (Markowitz [7]) addressed the issue by referring to a relative measure of risk, variance essentially for normal distribution. It contended that when portfolios are created, expected return is weighted average where as risk defined as variance (or its square root i.e., standard deviation) is not. Efficient diversification can reduce the portfolio risk significantly by playing on the covariance characteristics and the relative weight of the securities simultaneously. We have other mean-variance models as well like Kan and Smith [3], Liu [6] addresses the problem by optimizing mean-expected absolute deviation.

A challenge arises when we address non-normal distributions with prominent tail characteristics. There is nothing efficient about an optimized portfolio obtained by ignoring the tails. In fact, by incorporating the tails into risk-return frontiers hitherto ignored, efficient portfolios become inefficient (Rockafellar et al. [10]). In the second category, we take absolute risk measure models. In this category, we have value-at-risk (VaR) models and Conditional Value-at-Risk (CVaR) models (Artzner et al. [1]). The essential advantage of using CVaR as a risk measure is that it can be expressed in a linear format (Rockafellar et al. [9]) and then the portfolio performance can be optimized using standard linear programming technique.

It is very important to decide which risk measure should be taken into account. A risk measure shall ideally be coherent (Artzner et al. [1]). According to (Artzner et al. [1]), Coherence: a risk measure satisfying the four axioms of translation invariance, subadditivity, positive homogeneity, and monotonicity,

is called coherent. Although there are several risk measures proposed so far in the literature, none of them have superiority in all aspects. VaR and CVaR are becoming more and more popular in portfolio management. Value-at-Risk (VaR) means the amount of money that expresses the maximum expected loss from an investment over a specific investment horizon for a given confidence level. But VaR does not give any information beyond this amount of money (Baweja et al. [2]). Further it has undesirable mathematical characteristics such as lack of subaddititivty (Artzner et al. [1]) and Convexity (Rockafellar et al. [9]). By contrast CVaR is considered a more consistent measure of risk than VaR. Conditional Value-at-Risk is defined as the weighted average of VaR and losses strictly exceeding VaR (Rockafellar et al. [9]).

Hence in the present paper we intend to take a coherent risk measure CVaR. In this paper we assume a financial market with three asset classes bond, stock and an option on stock. We essentially try to modify (Korn et al. [4]) the model by introducing one more optimization loop at an intermittent time when the option matures. We impose a constraint on CVaR as is done in (Martinelli et al. [8]). It has been observed that the use of option with higher strike price leads to higher expected return while keeping the risk constant (Korn et al. [4]). We consider the aspect of using an option that matures before investor's horizon time. Then, the one-period (Martinelli et al. [8]) problem gets a dynamic aspect, the problem of optimally reinvesting the intermediate payments resulting from the option. It is essentially at this point that our model works differently from (Korn et al. [4]). It essentially converts the original one period problem into a two period problem.

The rest of the paper is organized as follows. For better understanding of the paper, Section 2 introduces basic information on option, stochastic process, VaR, CVaR and also linearization method of Rockafellar and Uryasev [9]. Based on CVaR risk measure, the crisp form of the model is presented in Section 3. In Section 4, we make experiment to examine its parameters and do sensitivity analysis for the model. Concluding remarks are given in Section 5.

2. Basic Concepts

2.1. Option

Options are one of the more popular financial derivatives currently available. They are far more popular than the underlying, on which they are written. Options are essentially dichotomized into Call, an option to buy the underlying

and Put, an option to sell the underlying ; at a predetermined exercise price up to or at a predetermined time. Option exercisable only at the expiry of a predetermined time are called European and option exercisable any time up to the predetermined time are called American option. In this paper, we only focus on European call option. A European call option is a contract that gives the holder the right, but not the obligation, to buy one unit of a stock for a predetermined strike price K on the maturity time T. The payoff of a call option is

$$p(S_T) = \begin{cases} 0 & \text{if } S_T \leq K \\ S_T - K & \text{if } S_T > k, \end{cases}$$

where S_T is the price of the underlying asset at maturity time T with strike price K.

2.2. Stochastic Process [5]

Any variable whose value changes over time in an uncertain way is said to follow a stochastic process. Stochastic processes can be classified as discrete time or continuous time. In this paper, we are using discrete-time stochastic process (Brownian motion). A discrete-time stochastic process is one where value of the variable can change only at certain fixed points in time such as every day. A variable follows Brownian motion if the change ΔZ during a small period of time Δt is given by

$$\Delta Z = \varepsilon \sqrt{\Delta t},$$

where ε has a standard normal distribution $\phi(0, 1)$.

2.3. Linearization of CVaR Method of Rockafellar and Uryasev [9]

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be the loss function which depends upon the control vector $w \in \mathbb{R}^n$ and the random vector $y \in \mathbb{R}^m$. The random vector y has the probability distribution function $p : \mathbb{R}^m \to \mathbb{R}$. However, the existence of the density is not critical for the considered approach, this assumption can be relaxed. Let $\psi(w, \alpha)$ denote the probability function, given by

$$\psi(w, \alpha) = \int_{f(w,y) \leq \alpha} p(y) dy$$

which is the probability that the loss function $f(w, y)$ does not exceed threshold value α.

The VaR function $\alpha(w, \beta) = \min\{\alpha \in \mathbb{R} : \psi(w, \alpha) \geq \beta\}$ where β stands for confidence level.

The CVaR function $\phi_\beta(w) = (1 - \beta)^{-1} \int_{f(w,y) \geq \alpha(w,\beta)} f(w, y) p(y) dy$.

Here $\phi_\beta(w)$ is the conditional expected value of loss function $F(w, y)$ under the condition that it exceeds VaR function $\alpha(w, \beta)$. It was shown in (Rockafellar et al. [9]) that the minimization of CVaR function $\phi_\beta(w)$ on the feasible set $X \subseteq \mathbb{R}^n$ can be reduced to the minimization of the function $F_\beta(w, \alpha)$ given by

$$F_\beta(w, \alpha) = \alpha + (1 - \beta)^{-1} \int_{y \in \mathbb{R}^m} X^+ p(y) dy,$$

on the set $X \times \mathbb{R}$, where $X^+ = (f(w, y) - \alpha)^+ = \max\{0, (f(w, y) - \alpha)\}$.

Consequently, using equality

$$\min_{\alpha \in \mathbb{R}} F_\beta(w, \alpha) = \phi_\beta(w),$$

$$\min_{\substack{w \in X \\ \alpha \in \mathbb{R}}} F_\beta(w, \alpha) = \min_{w \in X} F_\beta(w, \alpha(w, \beta)) = \min_{w \in X} \phi_\beta(w).$$

Note that the optimal solution α of this problem is VaR and under general conditions, the function $F_\beta(w, \alpha)$ is smooth. Thus, we can simultaneously find VaR and CVaR by minimizing function $F_\beta(w, \alpha)$.

We can use various approaches to calculate integral function $F_\beta(w, \alpha)$. In this paper, we use method proposed by Rockafellar and Uryasev [10] to approximate the integral appearing in $F_\beta(w, \alpha)$ by using a sample from the distribution of uncertainty vector y. The integral can be replaced by a summation, and in this case

$$\min_{\substack{w \in X \\ \alpha \in \mathbb{R}}} F_\beta(w, \alpha) = \min_{w \in X} \left[\alpha + \frac{1}{N(1 - \beta)} \sum_{i=1}^{N} x_i^+ \right],$$

$$\text{subject to constraints} \quad x_i \geq f(w, y_i) - \alpha,$$

$$x_i \geq 0, \quad i = 1, 2, \ldots, N,$$

where $x_i^+ = \max\{0, f(w, y_i) - \alpha\}$ and N is the size of the sample and x_i's are dummy variables.

Now $R_i(w)$ is the return associated with portfolio w, and the relation between return function and loss function $f(w, y_i)$ is given by

$$f(w, y_i) = -R_i(w), \quad i = 1, 2, \ldots, N,$$

where $R(w) = \dfrac{\text{final wealth} - \text{initial wealth}}{\text{initial wealth}}$.

3. Formation of Model

We assume a simple investment problem where besides stocks and bonds, the investor can also include options into the investment portfolio with investment horizon time T. R_T^S, R_T^B and $R_T^{O,V}$ denote return from stock, bond and European call option respectively. The option matures at time $T_1 = T/4$ and the vector $w = (w^S, w^B, w^O)$ represents weights of the stock, the bond and the option in the portfolio. $R_T^{W,V}$ is return of the portfolio at time T and it is defined by

$$R_T^{W,V} = w^S R_T^S + w^B R_T^B + w^O R_T^{O,V}.$$

The aim of the investor is to maximize the expected return with risk (CVaR) kept under the control. In our problem CVaR does not exceed an upper bound U. Since one financial instrument (option) matures before horizon time T, the investor faces the problem of re-investing these intermittent payments in the remaining two investment opportunities (stock and bond) at time T_1. As a consequence, the one-period problem has turned into a two-period problem. The investor again optimizes the re-investment portfolio (subportfolio) using weight vector $V = (V^S, V^B)$ to enhance expected return of re-investment $r^{O,V}$. Here V^S and V^B represent the weights of the stock and the bond respectively for time $T - T_1$. π^0 denotes the European call option's return at time $T_1 = T/4$. So π^0 is given by

$$\pi^0 = \frac{(\max\{0, S_{T/4} - K\}) - C(S_0, K, T/4)}{C(S_0, K, T/4)},$$

$C(S_0, K, T/4)$ is the premium of the option with initial stock price. S_0, strike price K and maturity time $T/4$. The return of the option $r^{O,V}$ is obtained by re-investing payoff of the option at time T_1 and $r^{O,V}$ satisfies

$$r^{O,V} = (1 + \pi^0)[V^S(1 + r^S) + V^B(1 + r^B)] - 1,$$

where r^S and r^B denote the return of the stock and the bond for time $T - T/4$.

The inner optimization loop optimizes expected return of subportfolio for the optimal choice of re-investment strategy V and the optimal value of objective function gives us return of option $R_T^{O,V}$. The outer optimization loop optimizes expected return of main portfolio for the optimal choice of initial

portfolio weight vector w. So our portfolio optimization model reads as follows:

Main Portfolio

Subportfolio

$$\max_{W \in \mathbb{R}^3} \frac{1}{N} \sum_{i=1}^{N} R_{T,i}^{W,V}, \tag{1}$$

subject to the constraints

$$R_{T,i}^{W,V} = w^S R_{T,i}^S + w^B R_{T,i}^B + w^O R_{T,i}^{O,V}, \ i = 1, 2, \ldots, N, \tag{2}$$

$$R_{T,i}^{O,V} = \max_{V \in \mathbb{R}^2} \frac{1}{N} \sum_{j=1}^{N} r_{i,j}^{O,V} \tag{3}$$

subject to the constraints

$$r_{i,j}^{O,V} = (1 + \pi_{i,j}^0)[V^S(1 + r_{i,j}^S) + V^B(1 + r_{i,j}^B)] - 1,$$
$$j = 1, 2, \ldots, N, \tag{4}$$

$$V^S + V^B = 1 \tag{5}$$

$$V^S, V^B \geq 0 \tag{6}$$

$$\alpha + \frac{1}{N(1-\beta)} \sum_{i=1}^{N} x_i^+ \leq U \tag{7}$$

$$x_i = -R_{T,i}^{w,V} - \alpha, \ i = 1, 2, \ldots, N, \tag{8}$$

$$x_i^+ = \max\{0, x_i\} \tag{9}$$

$$w^S + w^B + w^O = 1 \tag{10}$$

$$w^S, w^B, w^O \geq 0 \tag{11}$$

$$\alpha \text{ is free.} \tag{12}$$

Here, the subscripts i and j indicate the values of the indexed variables corresponding to simulation run number. N is number of simulated paths. Eq. (1) is the objective function with goal to maximize expected return of main portfolio. Eq. (2) represents portfolio return at time T with initial weight vector w and re-investment weight vector v, for each scenario. Eq. (3) is objective function of nested subportfolio to maximize expected return of option for ith scenario. Eq. (4) represents option's return at time T for ith scenario. Eq. (5) and Eq. (10) make sure that the portfolio weights add up to 1. Eq. (6) and Eq. (11) guarantee that short selling is not allowed. Eq. (8), Eq. (9) and Eq. (12) are needed to control CVaR of the portfolio. Eq. (7) gives CVaR constraint with upper bound "U".

Eq. (8), Eq. (9) and Eq. (12), these three constraints guarantee that Eq. (7)

gives CVaR and the corresponding value of α will be equal to VaR. If there are many optimal values of α than required VaR is the left end-point of the optimal interval.

4. Numerical Application of the Model

We use stochastic simulation optimization method to solve the model by Analytic Solver Platform of Microsoft Excel. Real market data of S&P 500 from 1/1/2013 to 31/12/2013 is used to predict future stock price with the help of standard financial market model for geometric Brownian motion

$$dS_t = (\mu - \sigma^2/2)S_t dt + \sigma S_t dw_t,$$
$$S_{t+1} - S_t = (\mu - \sigma^2/2)S_t dt + \sigma S_t dw_t,$$
$$S_{t+1} = S_t + (\mu - \sigma^2/2)S_t dt + \sigma S_t dw_t.$$

Here S_t is stock price at time t with constant volatility σ. dw_t is a standard Wiener process (Brownian motion) with zero mean and unit rate of variance.

In this paper, the present value (PV) of bond with yearly coupons is given by

$$PV = \sum_{t=1}^{N} \frac{C}{(1+r_t)^t} + \frac{D}{(1+r_N)^N},$$

where D is bond's face value, C is bond's coupon payment, and N is bond's maturity and r_t is market interest rate at time t. Interest rates and their dynamics provide probably the most computationally difficult part of the modern financial theory. The stochastic process for short-term market interest rates are assumed to follow Vasicek model (discrete version)

$$\Delta r = \alpha(b - r)\Delta t + \sigma \varepsilon \sqrt{\Delta t},$$

r is current market rate of interest with volatility σ. b is long-run mean of interest rate and α is speed of mean-reversion.

A fundamental principle of bond-investing is that market interest rate and bond prices move in opposite directions. Fig. 1 and Fig. 2 can help to visualize the relationship between market interest rates and present bond prices. These figures show that market interest rate and present value of bond (bond price) are inversely proportional.
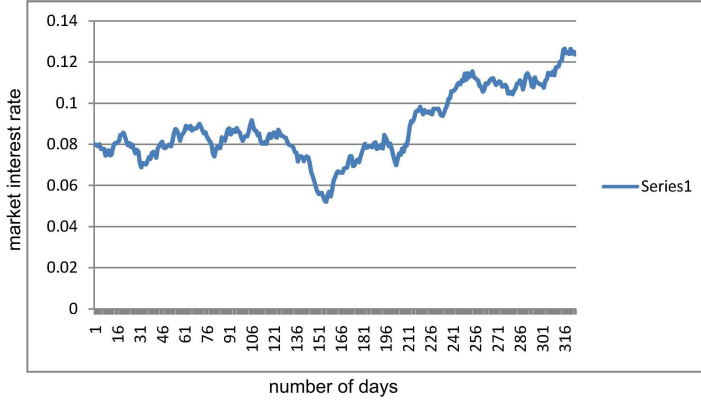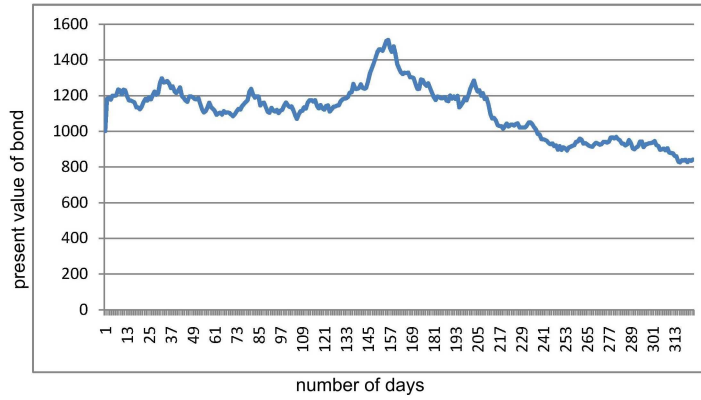
Figure 1: Graph of market interest rate

Figure 2: Graph of present value of bond

For calculating the premium of non-dividend paying call option, Black-Scholes model is used which is described as

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

$$d_1 = \frac{\log(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t},$$

where S is stock price at time t, T is the maturity date, K is strike price, $N(d_1)$ is cumulative normal distribution, σ is volatility.

Investment horizon (T)	=	4 years
Size of time steps in discretization (Δt)	=	0.004
Number of scenarios (N)	=	100
Initial stock price (S_0)	=	1782.59
variance of stock return (σ_s^2)	=	4.62469E-05
volatility of stock return (σ_s)	=	0.00680051
Mean of daily Stock return (μ_s)	=	0.000957
Initial interest rate (r_0)	=	0.08
Long-run mean of interest rate (b)	=	0.06
volatility of interest rate (σ)	=	0.03
Speed of mean reversion of interest rate (α)	=	0.07
coupon rate of bond	=	0.1
Time to maturity (bond)	=	16 years
Face value of bond	=	1000
initial weight vector (w)	=	$(1/3, 1/3, 1/3)$
Initial weight vector for reinvestment (v)	=	$(1/2, 1/2)$
Time to maturity (option) t	=	1 year
Strike price of option (K)	=	1800
Premium of option $C(S_0, K, T/4)$	=	34
Confidence level (β)	=	0.95
Upper bound of CVaR	=	10%

Table 1: Parameter values (assuming each year has 250 trading days)

We consider an investment problem with a four year investment horizon. The call option expires in one year. If option ends up in-the-money, then we re-invest this payoff to the stock and bond with weight vector $V = (V^S, V^B)$. We simulate 100 paths for the stock price and interest rate.

The parameter values used in the optimization problem are given in above Table 1.

By using Standard Evolutionary Engine to the simulated scenarios, we optimize our subportfolio for re-investment weights for the payoff of the option. The optimal solution of subportfolio is given in Table 2.

The optimal value of objective function of subportfolio is option's return, which is used to optimize main portfolio for initial weights. The optimal solution

$R^{o,v}$	7.06628106
$V^s =$	0.987657396
$V^B =$	0.012342604

Table 2: Optimal solution of the subportfolio

$R^{W,V}$	1.363102227
W^S	0.67200713
W^B	0.166504847
W^O	0.161488023

Table 3: Optimal main portfolio with option

$R^{W,V}$	0.762589582
W^s	1
W^B	0

Table 4: Optimal main portfolio without option

of main portfolio is given in Table 3.

The result for the same optimization problem when the option does not exist are given in Table 4.

From Tables 3 and 4, it can be seen that the inclusion of an option in the portfolio increases the quality of the portfolio based on the risk-return trade off.

The results for different strike prices of the call option are given in Table 5.

The case study show that the optimization algorithm, which is based on linear programming techniques, is very stable and efficient.CVaR risk management constraint (reduce to linear constraint) can be used in various applications to bound percentiles of loss distributions.

So, above results show that by using a call option with high strike price, we can increase expected return of our portfolio. But after a certain level of strike price, the call option leads to total loss as the option ends up out-of-money. In

K	$C(S_0, K)$	$E(R)$	CVaR	w^s	w^B	w^0
1650	165	0.9438	0.057	0.6000	0	0.3999
1700	116	1.0044	0.076	0.3868	0.243	0.369
1750	68	1.0970	0.075	0.3788	0.3655	0.2556
1785	37	1.3566	0.093	0.6110	0.212	0.176
1800	27	1.3702	0.098	0.6430	0.187	0.168

Table 5

this case the call investment is the most risky investment. Another remarkable result is that risk of losing from stock investment is highly correlated with the risk of losing from call investment. If the option ends up in-the-money, the risk of losing from stock investment has decreased substantially as the stock has already done well until maturity of option. Also, the first two columns of Table 5 show that as the strike price of option increases; the call option premium decreases. Consequently, for the same amount of money, more options can be bought. That is one of the reasons for the higher strike price leads to higher expected return.

5. Conclusion

The paper implements a solution to investment situations in a portfolio with investment in stock, bonds and options on the underlying stock. The investor's portfolio is analyzed in terms of expected return and CVaR, and the optimal weights are determined in a two-stage procedure. The final stage determines the optimal value of initial weights. The fact that the final stage is dependent on other stages, makes this approach very interesting. It has been observed that the introduction of option on equity has enhanced the risk-return tradeoff of the portfolio tremendously. It has also been observed that the use of option with higher strike price (which are normally priced cheaper than lower strike price options) improve the return performance of the portfolio much more than they impact the risk performance of the porfolio. The optimization model can handle trading constraints, such as short-selling restriction, while still retaining

in the class of linear optimization programs. This model has the advantage that the problem is solvable in reasonable time. The model shows that rebalancing at intermediate time points is necessary in order to meet the investor's risk requirement and to maximize the reward potential of the portfolio.By keeping a tab on the CVaR at 10% and by successively investing in options with enhanced strike price (From Rs. 1650 to Rs. 1800) it has been observed that the expected return of the portfolio increased from 0.9438 to 1.3702. The model indicates that option can be a potent investment vehicle for investors searching for low risk alternatives with a limited downside. It is strongly recommended to use this modified Korn and Zeytun framework to allocate the portfolio with structured product (option), since it generates well-balanced portfolio providing superior risk and return trade-off.

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