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TRAVELLING WAVES OF A DELAYED PREDATOR-PREY MODEL WITH STAGE STRUCTURE

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Abstract: This paper investigates the existence of travelling wave solutions of a stage-structured predator-prey model with spatial diffusion and time delay. By using the cross iteration method and Schauder's fixed point theorem, we reduce the existence of travelling wave solutions to the existence of a pair of upper-lower solutions.

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1. Introduction

Predator-prey systems describing the population dynamics between different species are very important in ecology and mathematical ecology. In the nature world, many species experience two or more life stages as they proceed from birth to death where they have different reactions to the environment. Recently, population models with stage structure have received much attention (see, for example, [3]-[4]). In [7], Yu et al. investigated a predator-prey model with stage

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structure for the predator in which feeding on prey can only make contribution to the increasing of the physique of the predator and does not make contribution to the reproductive ability. Letting $x(t)$ represent the density of the prey populations at time t, $y_1(t)$ and $y_2(t)$ represent the densities of the immature and the mature predator populations at time t, respectively, the strengthen type predator-prey model with stage structure is as follows:

$$\begin{aligned} \dot{x}(t) =& x(t)(r - ax(t) - a_1 y_2(t)), \\ \dot{y}_1(t) =& e y_2(t) - (D + r_1) y_1(t), \\ \dot{y}_2(t) =& D y_1(t) - r_2 y_2(t) + a_2 x(t) y_2(t). \end{aligned} \tag{1.1}$$

In (1.1), the parameters $a, e, r, a_1, a_2, r_1, r_2$ and D are positive constants in which a is the intra-specific competition rate of the prey, a_1 is the capturing rate of mature predators, a_2/a_1 is the conversion rate of the mature predator by consuming prey, e is the birth rate of predators, r is the intrinsic growth rate of the prey, D represents the transformation of the immature predator into the mature predator, r_1 and r_2 denote the death rates of the immature predator and the mature predator, respectively. In [7], the global asymptotic stability of the coexistence equilibrium was established by constructing suitable Lyapunov functions.

In reality, the species may disperse spatially as well as evolving in time. This spatial dispersal or diffusion arises from the tendency of certain species to migrate towards regions of lower population density, mainly due to resource limitation. In this situation the governing equations for the population densities become a system of reaction-diffusion equations (see, for example, [2],[5],[6]). In addition, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate(see, for example, [1],[5]). Time delay due to gestation is a common example, because generally the consumption of prey by the predator throughout its past history governs the present birth rate of the predator.

In this paper, motivated by the works mentioned above, we are concerned with a stage-structured predator-prey model with spatial diffusion and time delay due to gestation of the predators:

$$\begin{aligned} \frac{\partial u}{\partial t} =& d_1 \frac{\partial^2 u}{\partial x^2} + u(x,t)(r - au(x,t) - a_1 v_2(x,t)), \\ \frac{\partial v_1}{\partial t} =& d_2 \frac{\partial^2 v_1}{\partial x^2} + e v_2(x,t) - (D + r_1) v_1(x,t), \\ \frac{\partial v_2}{\partial t} =& d_2 \frac{\partial^2 v_2}{\partial x^2} + D v_1(x,t) + v_2(x,t)(-r_2 + a_2 u(x,t-\tau) - b v_2(x,t)) \end{aligned} \tag{1.2}$$

for $t > 0, x \in (-\infty, \infty)$, with initial conditions

$$\begin{aligned} &u(x,t) = \rho_1(x,t), v_1(x,t) = \rho_2(x,t), \\ &v_2(x,t) = \rho_3(x,t), t \in [-\tau, 0], x \in \overline{\Omega}. \end{aligned} \tag{1.3}$$

In problem (1.2)-(1.3), Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. The functions $\rho_i(x,t)(i = 1,2,3)$ are nonnegative and Hölder continuous and satisfy $\partial\rho_i/\partial x = 0$ in $[-\infty, 0] \times \overline{\Omega}$. The parameters d_1 and d_2 represent the diffusion rates of the prey and predator populations, respectively. $u(x,t), v_1(x,t)$ and $v_2(x,t)$ represent the densities of prey, immature predator and mature predator populations at location x and time t, respectively. The positive constant b is the intra-specific competition rate of the mature predators. $\tau > 0$ is a constant delay due to the gestation of the mature predators. The other parameters match the same meaning to those in system (1.1).

The organization of this paper is as follows. In the next section, we employ a new cross iteration method and Schauder's fixed point theorem in a profile set to obtain the existence of travelling wave solutions for system (1.2). In Section 3, by constructing a pair of upper-lower solutions, we use the result derived in Section 2 to prove the existence of travelling wave solutions of system (1.2).

2. Preliminaries

In this section, we apply Schauder's fixed point theorem to study the existence of travelling waves of system (1.2) connecting the trivial steady state and the positive steady state.

The system (1.2) always has a trivial steady state $E_0(0,0,0)$. If the following conditions hold

(A1) $(a_1r_2 + br)(D + r_1) > a_1De, \quad a_2r(D + r_1) + aDe > ar_2(D + r_1),$

system (1.2) has a positive steady state $E^*(k_1, k_2, k_3)$, where

$$\begin{aligned} k_1 &= \frac{(a_1r_2 + br)(D + r_1) - a_1De}{(a_1a_2 + ab)(D + r_1)}, \\ k_2 &= \frac{(a_2r - ar_2)(D + r_1) + aDe}{(a_1a_2 + ab)(D + r_1)}, \\ k_3 &= \frac{e[(a_2r - ar_2)(D + r_1) + aDe]}{(a_1a_2 + ab)(D + r_1)^2}. \end{aligned}$$

A travelling wave solution of (1.2) is a special solution of the form $u(x,t) = \phi(x+ct), v_1(x,t) = \varphi(x+ct), v_2(x,t) = \psi(x+ct), c > 0$ where $(\phi, \varphi, \psi) \in C^2(\mathbb{R}, \mathbb{R}^3)$ is the profile of the wave that propagates through the one-dimensional spatial domain at a constant speed c. Substituting $u(x,t) = \phi(x+ct), v_1(x,t) = \varphi(x+ct), v_2(x,t) = \psi(x+ct)$ into (1.2) and denoting $x+ct$ by t, we derive the following system

$$\begin{aligned} &d_1\phi''(t) - c\phi'(t) + \phi(t)(r - a\phi(t) - a_1\psi(t)) = 0, \\ &d_2\varphi''(t) - c\varphi'(t) + e\psi(t) - (D + r_1)\varphi(t) = 0, \\ &d_2\psi''(t) - c\psi'(t) + D\varphi(t) + \psi(t)(-r_2 + a_2\phi(t - c\tau) - b\psi(t)) = 0, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} f_{c1}(\phi_t, \varphi_t, \psi_t) =& \phi(0)(r - a\phi(0) - a_1\psi(0)), \\ f_{c2}(\phi_t, \varphi_t, \psi_t) =& e\psi(0) - (D + r_1)\varphi(0), \\ f_{c3}(\phi_t, \varphi_t, \psi_t) =& D\varphi(0) + \psi(0)(-r_2 + a_2\phi(-\tau) - b\psi(0)). \end{aligned}$$

Then (1.2) has a travelling wave solution if and only if there exists a solution of the equation (2.1) satisfying asymptotic boundary conditions

$$\lim_{t\to-\infty}(\phi(t), \varphi(t), \psi(t)) = (0,0,0), \lim_{t\to+\infty}(\phi(t), \varphi(t), \psi(t)) = (k_1, k_2, k_3). \tag{2.2}$$

Now, we give the definition of upper and lower solutions of system (2.1).

Definition 1. The continuous functions $\overline{\Phi} = (\overline{\phi}, \overline{\varphi}, \overline{\psi})$ and $\underline{\Phi} = (\underline{\phi}, \underline{\varphi}, \underline{\psi}) \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^3)$ are called a pair of upper-lower solutions of system (2.1), if there exist constants $T_i (i = 1, 2, \cdots, m)$, such that $\overline{\Phi}$ and $\underline{\Phi}$ are twice differential in $\mathbb{R}\backslash\{T_i : i = 1, 2, \cdots, m\}$ and they are essentially bounded on $\mathbb{R}^3$, and there hold

$$\begin{aligned} &d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + f_{c1}(\overline{\phi}_t, \overline{\varphi}_t, \underline{\psi_t}) \leqslant 0, \\ &d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + f_{c2}(\overline{\phi}_t, \overline{\varphi}_t, \overline{\psi}_t) \leqslant 0, \quad t \in \mathbb{R}\backslash\{T_i : i = 1, 2, \cdots, m\}, \\ &d_3\overline{\psi}''(t) - c\overline{\psi}'(t) + f_{c3}(\overline{\phi}_t, \overline{\varphi}_t, \overline{\psi}_t) \leqslant 0, \end{aligned}$$

and

$$\begin{aligned} &d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + f_{c1}(\underline{\phi}_t, \underline{\varphi}_t, \overline{\psi_t}) \geqslant 0, \\ &d_2\underline{\varphi}''(t) - c\underline{\varphi}'(t) + f_{c2}(\underline{\phi}_t, \underline{\varphi}_t, \underline{\psi}_t) \geqslant 0, \quad t \in \mathbb{R}\backslash\{T_i : i = 1, 2, \cdots, m\}. \\ &d_3\underline{\psi}''(t) - c\underline{\psi}'(t) + f_{c3}(\underline{\phi}_t, \underline{\varphi}_t, \underline{\psi}_t) \geqslant 0. \end{aligned}$$

Let

$$\begin{aligned}C_{[0,M]}(\mathbb{R},\mathbb{R}^3) =&\{(\phi,\varphi,\psi)\in C(\mathbb{R},\mathbb{R}^3):\\ &\mathbf{0}\leqslant(\phi(s),\varphi(s),\psi(s))\leqslant(M_1,M_2,M_3), s\in\mathbb{R}\}.\end{aligned}$$

In order to formulate the main result, we begin with some lemmas.

Lemma 1. *f_{c1}, f_{c2} and f_{c3} satisfy the following conditions:*

(A2) There exist three positive constants $\beta_1, \beta_2, \beta_3 > 0$ such that

$$\begin{aligned}&f_{c1}(\phi_1,\varphi_1,\psi_2)-f_{c1}(\phi_2,\varphi_2,\psi_1)+\beta_1[\phi_1(0)-\phi_2(0)]\geqslant 0,\\ &f_{c2}(\phi_1,\varphi_1,\psi_1)-f_{c2}(\phi_2,\varphi_2,\psi_2)+\beta_2[\varphi_1(0)-\varphi_2(0)]\geqslant 0,\\ &f_{c3}(\phi_1,\varphi_1,\psi_1)-f_{c3}(\phi_2,\varphi_2,\psi_2)+\beta_3[\psi_1(0)-\psi_2(0)]\geqslant 0,\end{aligned} \tag{2.3}$$

where $(\phi_i,\varphi_i,\psi_i)\in C_{[0,M]}(\mathbb{R},\mathbb{R}^3)(i=1,2)$ with $\phi_2\leqslant\phi_1, \varphi_2\leqslant\varphi_1$ and $\psi_2\leqslant\psi_1$.

Proof. It is not difficult to verify that

$$\begin{aligned}&f_{c1}(\phi_1,\varphi_1,\psi_2)-f_{c1}(\phi_2,\varphi_2,\psi_1)\\ =&r(\phi_1-\phi_2)+a(\phi_2^2-\phi_1^2)+a_1(\phi_2\psi_1-\phi_1\psi_2)\\ \geqslant&[r-a(\phi_1+\phi_2)-a_1\psi_1](\phi_1(0)-\phi_2(0)),\end{aligned}$$

$$\begin{aligned}f_{c2}(\phi_1,\varphi_1,\psi_1)-f_{c2}(\phi_2,\varphi_2,\psi_2)&=-(D+r_1)(\varphi_1-\varphi_2)+e(\psi_1-\psi_2)\\ &\geqslant-(D+r_1)(\varphi_1(0)-\varphi_2(0)),\end{aligned}$$

$$\begin{aligned}&f_{c3}(\phi_1,\varphi_1,\psi_1)-f_{c3}(\phi_2,\varphi_2,\psi_2)\\ =&-r_2(\psi_1-\psi_2)+b(\psi_2^2-\psi_1^2)+D(\varphi_1-\varphi_2)\\ &+a_2(\phi_1(-\tau)\psi_1-\phi_2(-\tau)\psi_2)\\ \geqslant&[-r_2-b(\psi_1+\psi_2)](\psi_1(0)-\psi_2(0)).\end{aligned}$$

Letting $\beta_1=-r+2aM_1+a_1M_3>0, \beta_2=D+r_1>0$ and $\beta_3=r_2+2bM_3>0$, the proof is complete. □

For the constants $\beta_i>0(i=1,2,3)$ above, define

$$\begin{aligned}H_1(\phi,\varphi,\psi)(t)&=f_{c1}(\phi_t,\varphi_t,\psi_t)+\beta_1\phi(t),\\ H_2(\phi,\varphi,\psi)(t)&=f_{c2}(\phi_t,\varphi_t,\psi_t)+\beta_2\varphi(t),\\ H_3(\phi,\varphi,\psi)(t)&=f_{c3}(\phi_t,\varphi_t,\psi_t)+\beta_3\psi(t),\end{aligned}$$

and

$$\begin{aligned}F_i(\phi,\varphi,\psi)(t)=&\frac{1}{d_i(\lambda_{i2}-\lambda_{i1})}\int_{-\infty}^{t}e^{\lambda_{i1}(t-s)}H_i(\phi,\varphi,\psi)(s)ds\\ &+\frac{1}{d_i(\lambda_{i2}-\lambda_{i1})}\int_{t}^{+\infty}e^{\lambda_{i2}(t-s)}H_i(\phi,\varphi,\psi)(s)ds,\end{aligned} \tag{2.4}$$

where

$$\lambda_{i1} = \frac{c - \sqrt{c^2 + 4\beta_i d_i}}{2d_i}, \quad \lambda_{i2} = \frac{c + \sqrt{c^2 + 4\beta_i d_i}}{2d_i} \quad (i = 1, 2, 3).$$

Then F is well defined such that

$$d_i F_i''(\phi, \varphi, \psi)(t) - cF_i'(\phi, \varphi, \psi)(t) - \beta_i F_i(\phi, \varphi, \psi)(t) + H_i(\phi, \varphi, \psi)(t) = 0.$$

Therefore, a fixed point of the operator F is a solution of system (2.1), which is a travelling wave solution of (1.2) connecting $\mathbf{0} = (0, 0, 0)$ and $\mathbf{K} = (k_1, k_2, k_3)$ if it satisfies (2.2).

From Lemmas 3.1-3.6 of [2], we have the following result.

Lemma 2. *Assume that (A1)-(A2) hold. If* (2.1) *has a pair of upper-lower solutions* $\overline{\Phi}(t) = (\overline{\phi}, \overline{\varphi}, \overline{\psi})$ *and* $\underline{\Phi}(t) = (\underline{\phi}, \underline{\varphi}, \underline{\psi}) \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^3)$ *such that (P1)-(P3) hold, then* (1.2) *has a travelling wave solution satisfying* (2.2), *where*

(P1) $\mathbf{0} \leq \underline{\Phi} \leq \overline{\Phi} \leq \mathbf{M} = (M_1, M_2, M_3)$;

(P2) $\lim_{t \to -\infty} \overline{\Phi}(t) = \mathbf{0}, \lim_{t \to \infty} \underline{\Phi}(t) = \lim_{t \to \infty} \overline{\Phi}(t) = \mathbf{K}$;

(P3) $\overline{\Phi}'(t+) \leq \overline{\Phi}'(t-), \underline{\Phi}'(t+) \geq \underline{\Phi}'(t-), t \in \mathbb{R}$.

3. Existence of Travelling Waves of System (1.2)

In this section, we investigate the existence of travelling wave solutions to problem (1.2) using the result in Section 2.

In the following, we will construct a pair of upper-lower solutions for (2.1) satisfying (P1)-(P3). Denote

$$\kappa = \frac{e}{D + r_1}, \quad A_1 = 4d_1 r, \quad A_2 = 4d_2(D\kappa - r_2 + a_2 M_1).$$

Since $D\kappa - r_2 + a_2 M_1 \geqslant bk_3$, it is readily seen that $A_2 > 0$. Letting

$$c > c^* = \max\{\sqrt{A_1}, \sqrt{A_2}\},$$

then we introduce the following positive numbers $\lambda_i (1 \leqslant i \leqslant 4)$ satisfying

$$\lambda_{1,2} = \frac{c \mp \sqrt{c^2 - A_1}}{2d_1}, \quad \lambda_{3,4} = \frac{c \mp \sqrt{c^2 - A_2}}{2d_2}.$$

We can choose $\varepsilon_i > 0 (i = 1, 2, 3, 4, 5, 6)$ such that

$$\begin{aligned} &a\varepsilon_1 - a_1\varepsilon_6 > 0, \quad \varepsilon_2 - \kappa\varepsilon_3 > 0, \\ &a\varepsilon_4 - a_1\varepsilon_3 > 0, \quad \varepsilon_5 - \kappa\varepsilon_6 > 0, \\ &D(\kappa\varepsilon_3 - \varepsilon_2) + (k_3 + \varepsilon_3)(b\varepsilon_3 - a_2\varepsilon_1) > 0, \\ &D(\kappa\varepsilon_6 - k_2) + (k_3 - \varepsilon_6)(b\varepsilon_6 - a_2\varepsilon_4) > 0. \end{aligned} \tag{3.1}$$

For the above constants and suitable constants $t_i (i = 1, 2, \cdots, 6)$ satisfying $t_2 > t_3, \min\{t_1, t_5\} > t_6$ and $t_6 - c\tau > t_4$, we define the continuous functions $\Phi(t) = (\phi_1(t), \varphi_1(t), \psi_1(t))$ and $\Psi(t) = (\phi_2(t), \varphi_2(t), \psi_2(t))$ as follows:

$$\phi_1(t) = \begin{cases} e^{\lambda_1 t}, & t \leqslant t_1; \\ k_1 + \varepsilon_1 e^{-\lambda t}, & t > t_1, \end{cases} \quad \phi_2(t) = \begin{cases} 0, & t \leqslant t_4; \\ k_1 - \varepsilon_4 e^{-\lambda t}, & t > t_4, \end{cases}$$
$$\varphi_1(t) = \begin{cases} \kappa e^{\lambda_3 t}, & t \leqslant t_2; \\ k_2 + \varepsilon_2 e^{-\lambda t}, & t > t_2, \end{cases} \quad \varphi_2(t) = \begin{cases} 0, & t \leqslant t_5; \\ k_2 - \varepsilon_5 e^{-\lambda t}, & t > t_5, \end{cases}$$
$$\psi_1(t) = \begin{cases} e^{\lambda_3 t}, & t \leqslant t_3; \\ k_3 + \varepsilon_3 e^{-\lambda t}, & t > t_3, \end{cases} \quad \psi_2(t) = \begin{cases} 0, & t \leqslant t_6; \\ k_3 - \varepsilon_6 e^{-\lambda t}, & t > t_6. \end{cases}$$

We now prove that continuous functions $\Phi(t)$ and $\Psi(t)$ are an upper solution and a lower solution of (2.1), respectively.

Lemma 6. *Assume that (A1) and* (3.1) *hold.* $\Phi(t) = (\phi_1(t), \varphi_1(t), \psi_1(t))$ *is an upper solution of system* (2.1).

Proof. Denote

$$\begin{aligned} p_1(t) :=& d_1\phi_1''(t) - c\phi_1'(t) + \phi_1(t)(r - a\phi_1(t) - a_1\psi_2(t)), \\ p_2(t) :=& d_2\varphi_1''(t) - c\varphi_1'(t) + e\psi_1(t) - (D + r_1)\varphi_1(t), \\ p_3(t) :=& d_2\psi_1''(t) - c\psi_1'(t) + D\varphi_1(t) \\ &+ \psi_1(t)(-r_2 + a_2\phi_1(t - c\tau) - b\psi_1(t)). \end{aligned}$$

For $\phi_1(t)$, we need to prove that $p_1(t) \leqslant 0$.

If $t \leqslant t_1, \phi_1(t) = e^{\lambda_1 t}$. Taking account of $\psi_2(t) \geqslant 0$ and $d_1\lambda_1^2 - c\lambda_1 + r = 0$, we have that

$$\begin{aligned} p_1(t) &\leqslant d_1\phi_1''(t) - c\phi_1'(t) + r\phi_1(t) \\ &= e^{\lambda_1 t}(d_1\lambda_1^2 - c\lambda_1 + r) \\ &= 0. \end{aligned}$$

If $t > t_1, \phi_1(t) = k_1 + \varepsilon_1 e^{-\lambda t}, \psi_2(t) = k_3 - \varepsilon_6 e^{-\lambda t}$. It follows that

$$\begin{aligned} p_1(t) =& e^{-\lambda t}(d_1\varepsilon_1\lambda^2 + c\varepsilon_1\lambda) \\ &+ (k_1 + \varepsilon_1 e^{-\lambda t})[r - a(k_1 + \varepsilon_1 e^{-\lambda t}) - a_1(k_3 - \varepsilon_6 e^{-\lambda t})] \\ =& e^{-\lambda t}[d_1\varepsilon_1\lambda^2 + c\varepsilon_1\lambda + (k_1 + \varepsilon_1 e^{-\lambda t})(a_1\varepsilon_6 - a\varepsilon_1)] \\ \triangleq& I_1(\lambda). \end{aligned}$$

Since $a\varepsilon_1 - a_1\varepsilon_6 > 0$, which implies $I_1(0) < 0$, then there exists a $\lambda_1^* > 0$ such that $p_1(t) \leqslant 0$ for all $\lambda \in (0, \lambda_1^*)$.

For $\varphi_1(t)$, we need to prove that $p_2(t) \leqslant 0$.

If $t \leqslant t_3, \varphi_1(t) = \kappa e^{\lambda_3 t}, \psi_1(t) = e^{\lambda_3 t}$. Considering the definition of λ_3, it follows that

$$\begin{aligned} p_2(t) &= d_2\kappa\lambda_3^2 e^{\lambda_3 t} - c\kappa\lambda_3 e^{\lambda_3 t} + e \cdot e^{\lambda_3 t} - e \cdot e^{\lambda_3 t} \\ &= \kappa e^{\lambda_3 t}(d_2\lambda_3^2 - c\lambda_3) \\ &< 0. \end{aligned}$$

If $t_3 < t \leqslant t_2, \varphi_1(t) = \kappa e^{\lambda_3 t}, \psi_1(t) = k_3 + \varepsilon_3 e^{-\lambda t}$. By the case above, it is obvious that $p_2(t) \leqslant 0$ since $k_3 + \varepsilon_3 e^{-\lambda t} \leqslant e^{\lambda_3 t}$ for $t_3 < t \leqslant t_2$.

If $t > t_2, \varphi_1(t) = k_2 + \varepsilon_2 e^{-\lambda t}, \psi_1(t) = k_3 + \varepsilon_3 e^{-\lambda t}$. We obtain that

$$\begin{aligned} p_2(t) &= e^{-\lambda t}(d_2\varepsilon_2\lambda^2 + c\varepsilon_2\lambda) + e(k_3 + \varepsilon_3 e^{-\lambda t}) - (D + r_1)(k_2 + \varepsilon_2 e^{-\lambda t}) \\ &= e^{-\lambda t}[d_2\varepsilon_2\lambda^2 + c\varepsilon_2\lambda + e\varepsilon_3 - (D + r_1)\varepsilon_2] \\ &\triangleq I_2(\lambda). \end{aligned}$$

Since $(D + r_1)\varepsilon_2 - e\varepsilon_3 > 0$, it is easy to know that $I_2(0) < 0$. Thus, there exists a $\lambda_2^* > 0$ such that $p_2(t) \leqslant 0$ for all $\lambda \in (0, \lambda_2^*)$.

For $\psi_1(t)$, we need to prove that $p_3(t) \leqslant 0$.

If $t \leqslant t_3, \psi_1(t) = e^{\lambda_3 t}, \phi_1(t - c\tau) \leqslant M_1, \varphi_1(t) = \kappa e^{\lambda_3 t}$. We deserve that,

$$\begin{aligned} p_3(t) &\leqslant e^{\lambda_3 t}(d_2\lambda_3^2 - c\lambda_3) + D\kappa e^{\lambda_3 t} + e^{\lambda_3 t}(-r_2 + a_2 M_1) \\ &= e^{\lambda_3 t}(d_2\lambda_3^2 - c\lambda_3 + D\kappa - r_2 + a_2 M_1). \end{aligned}$$

Since $d_2\lambda_3^2 - c\lambda_3 + D\kappa - r_2 + a_2 M_1 = 0$, it is clear that $p_3(t) \leqslant 0$ when $t \leqslant t_3$.

If $t > t_3, \psi_1(t) = k_3 + \varepsilon_3 e^{-\lambda t}$, $\varphi_1(t) \leqslant k_2 + \varepsilon_2 e^{-\lambda t}, \phi_1(t - c\tau) \leqslant k_1 + \varepsilon_1 e^{-\lambda(t - c\tau)}$. Noting that $Dk_2 + k_3(-r_2 - bk_3 + a_2 k_1) = 0$ and $-r_2 - bk_3 + a_2 k_1 =$

$-D\kappa$, by calculation we can derive that

$$\begin{aligned}p_3(t) &\leqslant e^{-\lambda t}(d_2\varepsilon_3\lambda^2 + c\varepsilon_3\lambda) + D(k_2 + \varepsilon_2 e^{-\lambda t}) - r_2(k_3 + \varepsilon_3 e^{-\lambda t})\\ &\quad - b(k_3 + \varepsilon_3 e^{-\lambda t})^2 + a_2(k_1 + \varepsilon_1 e^{-\lambda(t-c\tau)})(k_3 + \varepsilon_3 e^{-\lambda t})\\ &= e^{-\lambda t}(d_2\varepsilon_3\lambda^2 + c\varepsilon_3\lambda) + D\varepsilon_2 e^{-\lambda t} + \varepsilon_3 e^{-\lambda t}(-r_2 - bk_3 + a_2k_1)\\ &\quad + e^{-\lambda t}(k_3 + \varepsilon_3 e^{-\lambda t})(a_2\varepsilon_1 e^{\lambda c\tau} - b\varepsilon_3)\\ &= e^{-\lambda t}[d_2\varepsilon_3\lambda^2 + c\varepsilon_3\lambda + D\varepsilon_2 - D\kappa\varepsilon_3]\\ &\quad + e^{-\lambda t}(k_3 + \varepsilon_3 e^{-\lambda t})(a_2\varepsilon_1 e^{\lambda c\tau} - b\varepsilon_3)\\ &\triangleq I_3(\lambda).\end{aligned}$$

From (3.1), we see that $I_3(0) = D(\varepsilon_2 - \kappa\varepsilon_3) + (k_3 + \varepsilon_3)(a_2\varepsilon_1 - b\varepsilon_3) < 0$. Therefore, there exists a $\lambda_3^* > 0$ such that $p_3(t) \leqslant 0$ for all $\lambda \in (0, \lambda_3^*)$.

For the above argument, we see that $\Phi(t) = (\phi_1(t), \varphi_1(t), \psi_1(t))$ is an upper solution of system (2.1) for $\lambda \in (0, \min\{\lambda_1^*, \lambda_2^*, \lambda_3^*\})$.

This completes the proof. □

Lemma 7. *Assume that (A1) and* (3.1) *hold.* $\Psi(t) = (\phi_2(t), \varphi_2(t), \psi_2(t))$ *is a lower solution of system* (2.1).

Proof. Denote

$$\begin{aligned}q_1(t) :=& d_1\phi_2''(t) - c\phi_2'(t) + \phi_2(t)(r - a\phi_2(t) - a_1\psi_1(t)),\\ q_2(t) :=& d_2\varphi_2''(t) - c\varphi_2'(t) + e\psi_2(t) - (D + r_1)\varphi_2(t),\\ q_3(t) :=& d_2\psi_2''(t) - c\psi_2'(t) + D\varphi_2(t)\\ &+ \psi_2(t)(-r_2 + a_2\phi_2(t - c\tau) - b\psi_2(t)).\end{aligned}$$

For $\phi_2(t)$, we need to prove that $q_1(t) \geqslant 0$.

If $t \leqslant t_4, \phi_2(t) = 0, q_1(t) = 0$.

If $t > t_4, \phi_2(t) = k_1 - \varepsilon_4 e^{-\lambda t}$. Whether $t > t_3$ or not, the inequality $\psi_1(t) \leqslant k_3 + \varepsilon_3 e^{-\lambda t}$ always holds. We obtain that

$$\begin{aligned}q_1(t) \geqslant& e^{-\lambda t}(-d_1\varepsilon_4\lambda^2 - c\varepsilon_4\lambda) + (k_1 - \varepsilon_4 e^{-\lambda t})[r - a(k_1 - \varepsilon_4 e^{-\lambda t})]\\ &- a_1(k_1 - \varepsilon_4 e^{-\lambda t})(k_3 + \varepsilon_3 e^{-\lambda t})\\ =& e^{-\lambda t}[-d_1\varepsilon_4\lambda^2 - c\varepsilon_4\lambda + (k_1 - \varepsilon_4 e^{-\lambda t})(a\varepsilon_4 - a_1\varepsilon_3)]\\ \triangleq& I_4(\lambda).\end{aligned}$$

Thus $I_4(0) = (k_1 - \varepsilon_4)(a\varepsilon_4 - a_1\varepsilon_3) > 0$, which implies that there exists a $\lambda_4^* > 0$ such that $q_1(t) \geqslant 0$ for all $\lambda \in (0, \lambda_4^*)$.

For $\varphi_2(t)$, we need to prove that $q_2(t) \geqslant 0$.

If $t \leqslant t_5, \varphi_2(t) = 0, q_2(t) = e\psi_2(t) \geqslant 0$.

If $t > t_5, \varphi_2(t) = k_2 - \varepsilon_5 e^{-\lambda t}, \psi_2(t) = k_3 - \varepsilon_6 e^{-\lambda t}$. By calculation, we deserve that

$$\begin{aligned} q_2(t) =& e^{-\lambda t}(-d_2\varepsilon_5\lambda^2 - c\varepsilon_5\lambda) + e(k_3 - \varepsilon_6 e^{-\lambda t}) \\ & - (D + r_1)(k_2 - \varepsilon_5 e^{-\lambda t}) \\ =& e^{-\lambda t}[-d_2\varepsilon_5\lambda^2 - c\varepsilon_5\lambda - e\varepsilon_6 + (D + r_1)\varepsilon_5] \\ \triangleq& I_5(\lambda). \end{aligned}$$

Clearly, $I_5(0) = (D + r_1)\varepsilon_5 - e\varepsilon_6 > 0$. Therefore, there exists a $\lambda_5^* > 0$ such that $q_2(t) \geqslant 0$ for all $\lambda \in (0, \lambda_5^*)$.

For $\psi_2(t)$, we need to prove that $q_3(t) \geqslant 0$.

If $t \leqslant t_6, \psi_2(t) = 0, q_3(t) = D\varphi_2(t) \geqslant 0$.

If $t_6 < t \leqslant t_5, \psi_2(t) = k_3 - \varepsilon_6 e^{-\lambda t}, \varphi_2(t) = 0$. Since $t_6 - c\tau > t_4$, then $\phi_2(t - c\tau) = k_1 - \varepsilon_4 e^{-\lambda(t-c\tau)}$. Thus, by calculation, we get that

$$\begin{aligned} q_3(t) =& e^{-\lambda t}(-d_2\varepsilon_6\lambda^2 - c\varepsilon_6\lambda) + (k_3 - \varepsilon_6 e^{-\lambda t})[-r_2 - b(k_3 - \varepsilon_6 e^{-\lambda t})] \\ & + a_2(k_1 - \varepsilon_4 e^{-\lambda(t-c\tau)})(k_3 - \varepsilon_6 e^{-\lambda t}) \\ =& e^{-\lambda t}(-d_2\varepsilon_6\lambda^2 - c\varepsilon_6\lambda) - Dk_2 - \varepsilon_6 e^{-\lambda t}[-r_2 - bk_3 + a_2k_1] \\ & - b\varepsilon_6 e^{-\lambda t}(\varepsilon_6 e^{-\lambda t} - k_3) + a_2\varepsilon_4 e^{-\lambda(t-c\tau)}(\varepsilon_6 e^{-\lambda t} - k_3) \\ =& e^{-\lambda t}[-d_2\varepsilon_6\lambda^2 - c\varepsilon_6\lambda - D(k_2 e^{\lambda t} - \kappa\varepsilon_6)] \\ & - e^{-\lambda t}(k_3 - \varepsilon_6 e^{-\lambda t})(a_2\varepsilon_4 e^{\lambda c\tau} - b\varepsilon_6) \\ \triangleq& I_6(\lambda). \end{aligned}$$

Since $I_6(0) = D(\kappa\varepsilon_6 - k_2) + (k_3 - \varepsilon_6)(b\varepsilon_6 - a_2\varepsilon_4) > 0$, there exists a $\lambda_6^* > 0$ such that $q_3(t) \geqslant 0$ for all $\lambda \in (0, \lambda_6^*)$.

If $t > t_5, \psi_2(t) = k_3 - \varepsilon_6 e^{-\lambda t}, \varphi_2(t) = k_2 - \varepsilon_5 e^{-\lambda t}, \phi_2(t - c\tau) = k_1 - \varepsilon_4 e^{-\lambda(t-c\tau)}$. Thus, by the case above, we get that $q_3(t) \geqslant 0$.

Obviously, for all $\lambda \in (0, \min\{\lambda_4^*, \lambda_5^*, \lambda_6^*\}), q_i(t) \geqslant 0 (i = 1, 2, 3)$. The proof is complete. □

Combining Lemma 1 and Lemma 2, we have the following conclusion.

Theorem 1. *Suppose that (A1) holds. For every $c > c^*$, system* (1.2) *always has a travelling wave solution with speed c connecting the trivial steady state $E_0(0, 0, 0)$ and the positive steady state $E^*(k_1, k_2, k_3)$.*

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