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FIXED POINT THEOREMS IN THE STUDY OF POSITIVE SOLUTIONS FOR SYSTEMS OF EQUATIONS IN ORDERED BANACH SPACES

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Abstract: In this paper we are interested in giving some sufficient conditions for the existence of coexistence states to systems of the form

$$x = F_1(x, y)$$

$$y = F_2(x, y),$$

where F_1 and F_2 satisfy some conditions. The proofs will be based on the fixed point index.

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1. Introduction

In this paper we give some abstract fixed point theorems for systems of equations in ordered Banach spaces. This can be done by using topological methods, in particular, the fixed point index.

Let us consider the following system

$$x = F_1(x, y)$$

$$y = F_2(x, y)$$

in $E \times E$, where E is an appropriate ordered Banach space with cone P. We are interested in producing sufficient conditions for the existence of a particular solutions called "coexistence states", i.e. solution (x, y) with both components nonnegative and nontrivial ($(x, y) \in P \setminus \{0\} \times P \setminus \{0\}$). These solutions are of special importance. Semitrivial solutions, i.e. solutions (x, y) with exactly one component nonnegative and nontrivial, are also of interest.

Note that a direct application of Amann's results in [1] in the Banach space $(E \times E, P \times P)$ for the map $F = (F_1, F_2)$ implies the existence of a solution $(x, y) \in P \times P \setminus \{(0, 0)\}$, this means that $(x, y) \neq (0, 0)$ but some component of the fixed point (x, y) may be trivial. To solve this problem, we give some new conditions concerning the partial derivative of F_1 and F_2 to assure that each component of (x, y) belongs to $P \setminus \{0\}$. Furthermore, if we suppose that F verifies the hypothesis

$$F_1(0, y) = F_2(x, 0) = 0 \quad \forall (x, y) \in E \times E,$$

we assure the existence of "semitrivial solutions". Hence, we deduce the existence of four fixed points in $P \times P : (0, 0), (x_0, 0), (0, y_0), (x_1, y_1)$ such that

$$x_j, y_j \in P \setminus \{0\} \tag{1}$$

for $j = 0, 1$.

Finally, we close this paper by an application of these abstract results to some problem arising in the theory of epidemics, where the existence of the "coexistence states" and "semitrivial solutions" is obtained.

Note that these abstract results may also be applied to some other situations such as nonlinear boundary value problems for elliptic systems, and other kinds of systems of nonlinear integral equations. Obviously, with our approach, systems of n equations may also be included. This will be done elsewhere.

2. Abstract Fixed Point Theorems

Let $(E, \|.\|_E)$ a real Banach space and P be a nonempty closed convex set in E.

P is called a cone if it satisfies the following two conditions:

(i) : $x \in P, \lambda \geq 0 \Longrightarrow \lambda x \in P$

(ii) : $x \in P, -x \in P \Longrightarrow x = \theta$, where θ denotes the zero element in E.

The cone P defines a linear ordering in E by

$$x \leq y \quad \text{iff} \quad y - x \in P.$$

For every open subset U of P (from now on, the topological notions of subsets of P refer to the relative topology of P as a topological subspace of E) and every compact map $F : \bar{U} \to P$ (F is continuous and $F(\bar{U})$ is relatively compact), which has no fixed points on ∂U, there exists an integer, $i_p(F, U)$, called the fixed point index of F on U with respect to P, satisfying the usual properties of the Leray-Schauder degree.

It is trivial that $P \times P$ is a cone in the Banach space $(E \times E, \|.\|_{E\times E})$ where, for each $(x, y) \in E \times E$

$$\|(x, y)\|_{E\times E} = \max\{\|x\|_E, \|y\|_E\}.$$

If $r > 0$, we denote

$$P_r = \{x \in P : \|x\|_E < r\}, \quad S_r = \{x \in P : \|x\|_E = r\},$$

and for any two real numbers $0 < \alpha < \beta$, we denote by $R_{\alpha,\beta}$ the set

$$R_{\alpha,\beta} = \{(x, y) \in P \times P : \|x\|_E < \alpha, \|y\|_E < \beta\}.$$

The cone $P \times P$ defines a linear ordering in $E \times E$ by

$$(x_1, y_1) \leq (x_2, y_2) \quad \text{iff} \quad x_2 - x_1 \in P \quad \text{and} \quad y_2 - y_1 \in P.$$

Define the operator $F = (G, H) : P \times P \to P \times P$, where $G : P \times P \to P$ and $H : P \times P \to P$ verifying the following hypotheses

(F1) : For every $y \in P$, G has a right partial derivative $G_x(\infty, y)$, such that $G_x(\infty, y) = G_x(\infty, \infty)$ and $G(x, y) = G_x(\infty, y)x + r(x, y)$, where r is $o(\|x\|_E)$ for $x \in P$ near $+\infty$ uniformly in $y \in P$.

(F2) : For every $x \in P$, H has a right partial derivative $H_y(x, \infty)$, such that $H_y(x, \infty) = H_y(\infty, \infty)$ and $H(x, y) = H_y(x, \infty)y + r'(x, y)$, where r' is $o(\|y\|_E)$ for $y \in P$ near $+\infty$ uniformly in $x \in P$.

Now we present and prove our main results.

Theorem 1. *Let $F : P \times P \to P \times P$ be a completely continuous map verifying the previous hypotheses (F1)-(F2) and*

(H1) There exists $(p_1, p_2) \in P \setminus \{0\} \times P \setminus \{0\}$ and $\sigma > 0$ such that

$$x - G(x, y) \neq \lambda p_1, \quad \forall (x, y) \in S_\sigma \times P, \quad \forall \lambda \geq 0$$

and

$$y - H(x, y) \neq \lambda p_2, \quad \forall (x, y) \in P \times S_\sigma, \quad \forall \lambda \geq 0,$$

(H2) 1 is neither an eigenvalue of $G_x(\infty, \infty)$ nor of $H_y(\infty, \infty)$, and both $G_x(\infty, \infty)$ and $H_y(\infty, \infty)$ possess no positive eigenvector to an eigenvalue greater than one.

Then F has at least one fixed points in $P \times P : (x_1, y_1)$ verifying (1).

Proof. We shall use the following notation

$$U = R_{\sigma,\sigma}.$$

First, we prove the existence of a fixed point (x_1, y_1) of F ($F(x_1, y_1) = (x_1, y_1)$) satisfying (1). In fact, the proof is based on the following steps:

a)

$$i_{P \times P}(F, U) = 0.$$

In fact, choose a real number μ such that

$$\mu > \sup_{(x,y) \in U} \frac{\|(x, y) - F(x, y)\|}{\|(p_1, p_2)\|}$$

and define $h : [0, 1] \times \bar{U} \to P \times P$ by

$$h(\lambda, x, y) = F(x, y) + \lambda\mu(p_1, p_2).$$

It is clear that h is completely continuous and from (H1) we have

$$h(\lambda, x, y) \neq (x, y), \quad \forall(\lambda, x, y) \in [0,1] \times \partial U.$$

Hence, by homotopy invariance property

$$i_{P\times P}(F, U) = i_{P\times P}(h(0, .), U) = i_{P\times P}(h(1, .), U).$$

However

$$i_{P\times P}(h(1, .), U) = 0,$$

since if $i_{P\times P}(h(1, .), U) \neq 0$, the existence property implies that there exists some $(x, y) \in U$ such that

$$(x, y) = F(x, y) + \mu(p_1, p_2),$$

whence

$$\mu = \frac{\|(x, y) - F(x, y)\|}{\|(p_1, p_2)\|},$$

which is a contradiction.

b) For every $y \in P$ define the map $G_y : P \to P$ by $G_y(x) = G(x, y)$. Clearly, G_y is a completely continuous map, $G'_y(\infty) = G_x(\infty, y)$. Then, by Theorem 7.3 in [1], $G_x(\infty, \infty) \setminus P = G_x(\infty, y) \setminus P$ is a completely continuous map, So $id_E - G_x(\infty, \infty)$ is closed on closed subset of P, therefore $(id_E - G_x(\infty, \infty))(S_1)$ is a closed set, and $0 \notin (id_E - G_x(\infty, \infty))(S_1)$ by the hypothesis of the theorem. Hence there exists a positive constant α_1 such that

$$\|x - G_x(\infty, \infty)x\| \geq \alpha_1\|x\| \quad \forall x \in P. \tag{2}$$

Choose $\rho_\infty > \sigma$ such that for all $x \in P$ with $\|x\| \geq \rho_\infty$ and $y \in P$

$$\|G(x, y) - G_x(\infty, y)x\| \leq \alpha_1 \frac{\|x\|}{2}.$$

Since $G_x(\infty, y)x = G_x(\infty, \infty)x$ we have for all $x \in P$ with $\|x\| \geq \rho_\infty$ and $y \in P$

$$\|G(x, y) - G_x(\infty, \infty)x\| \leq \alpha_1 \frac{\|x\|}{2}. \tag{3}$$

Define the map $p : P \times P \to P$ by $p(x, y) = x$, therefore for every $\rho \geq \rho_\infty$ and every $\lambda \in [0, 1]$ the map $(1 - \lambda)(G_x(\infty, \infty)p, H + \mu p_2) + \lambda F = H_\lambda$ possesses no fixed point on ∂U_1 where $R_{\rho,\sigma} = U_1$, where μ is a real number verifying

$$\mu > \sup_{(x,y)\in U_1} \frac{\|y - H(x, y)\|}{\|p_2\|}.$$

Indeed, by taking into account that

$$\partial U_1 =\{(x,y) \in P \times P : \|y\|_E = \sigma, \|x\|_E \leq \rho\} \\ \cup \{(x,y) \in P \times P : \|y\|_E \leq \sigma, \|x\|_E = \rho\},$$

we distinguish two cases:

1) $\|y\|_E = \sigma, \quad \|x\|_E \leq \rho.$

If $H_\lambda(x,y) = (x,y)$, then

$$(1-\lambda)(H(x,y) + \mu p_2) + \lambda H(x,y) = H(x,y) + (1-\lambda)\mu p_2 = y,$$

which contradicts (H1);

2) $\|y\|_E \leq \sigma, \quad \|x\|_E = \rho.$

We get $G_x(\infty,\infty)p(x,y) = G_x(\infty,\infty)x.$

On the other hand

$$\begin{aligned}\|x-(1-\lambda)(G_x(\infty,\infty)x) - \lambda G(x,y)\| &\geq \|x - G_x(\infty,\infty)x\| \\ &- \|G(x,y) - G_x(\infty,\infty)x\| \\ &\geq \rho(\alpha_1 - \frac{\alpha_1}{2}) \\ &> 0,\end{aligned}$$

whence $H_\lambda(x,y) \neq (x,y)$.

Then by the homotopy invariance property

$$i_{P\times P}(F, U_1) = i_{P\times P}((G_x(\infty,\infty)p, H + \mu p_2), U_1).$$

Next, we prove that

$$i_{P\times P}((G_x(\infty,\infty)p, H + \mu p_2), U_1) = 0.$$

If it is not so, then there exists some $(x,y) \in U_1$ verifying

$$y = H(x,y) + \mu p_2.$$

So, that

$$\mu = \frac{\|y - H(x,y)\|}{\|p_2\|},$$

which contradicts the definition of μ.

c) Similarly, we find a positive constants α_2 and ρ'_∞ satisfying

$$\|y - H_y(\infty,\infty)y\| \geq \alpha_2\|y\| \quad \forall y \in P,$$

and for all $y \in P$ with $\|y\| \geq \rho'_\infty$ and $x \in P$

$$\|H(x,y) - H_y(\infty,\infty)y\| \leq \alpha_2 \frac{\|y\|}{2}.$$

Define the map $q : P \times P \to P$ by $q(x,y) = y$, therefore for every $\rho \geq \rho'_\infty$ and every $\lambda \in [0,1]$ the map $(1-\lambda)(G + \mu p_1, H_y(\infty,\infty)q) + \lambda F = H'_\lambda$ possesses no fixed point on ∂U_2 where $R_{\sigma,\rho} = U_2$ where μ is a real number verifying

$$\mu > \sup_{(x,y)\in U_2} \frac{\|x - G(x,y)\|}{\|p_1\|}.$$

Then, from what has already been proved

$$i_{P\times P}(F, U_2) = i_{P\times P}((G + \mu p_1, H_y(\infty,\infty)q), U_2) = 0.$$

d) For a fixed $\rho \geq \max\{\rho_\infty, \rho'_\infty\}$ we shall use the following notation

$$U_3 = R_{\rho,\rho}.$$

Next, we prove

$$i_{P\times P}(F, U_3) = 1.$$

To see this, define the map $(1-\lambda)(G_x(\infty,\infty)p, H_y(\infty,\infty)q) + \lambda F = H''_\lambda$ which has no fixed point on ∂U_3. Indeed, by taking into account that

$$\begin{aligned}\partial U_3 =& \{(x,y) \in P \times P : \|y\|_E = \rho, \|x\|_E \leq \rho\} \\ &\cup \{(x,y) \in P \times P : \|y\|_E \leq \rho, \|x\|_E = \rho\},\end{aligned}$$

we distinguish two cases:

1) $\|y\|_E \leq \rho, \quad \|x\|_E = \rho.$

We have

$$\|x - (1-\lambda)(G_x(\infty,\infty)x) - \lambda G(x,y)\| > 0.$$

Then, in this case $H''_\lambda(x,y) \neq (x,y)$.

2) $\|y\|_E = \rho, \quad \|x\|_E \leq \rho.$

This case is completely analogous to case 1).

Then by the homotopy invariance property

$$i_{P\times P}(F, U_3) = i_{P\times P}((G_x(\infty,\infty)p, H_y(\infty,\infty)q), U_3).$$

Next, we prove that

$$i_{P\times P}((G_x(\infty,\infty)p, H_y(\infty,\infty)q), U_3) = 1.$$

Indeed, observe that by hypothesis (H2), the equation

$$(x,y)-\lambda(G_x(\infty,\infty)x, H_y(\infty,\infty)y)=(0,0)$$

has no solution in ∂U_3 for $\lambda\in[0,1]$. Hence by the homotopy invariance and by the solution property

$$i_{P\times P}((G_x(\infty,\infty)p, H_y(\infty,\infty)q), U_3)=i_{P\times P}((0,0),U_3)=1.$$

e) We shall use the following notation

$$U_4=U_3\setminus\bar{U_1}\cup\bar{U_2}\quad U_5=U_1\setminus\bar{U}\quad U_6=U_2\setminus\bar{U}.$$

Therefore

$$U_4=\{(x,y)\in P\times P:\sigma<\|x\|_E<\rho,\quad \sigma<\|y\|_E<\rho\}.$$

Now, observe that if $\lambda=1, F=H_1=H_1'=H_1''$ has no fixed point on $\partial U_1\cup\partial U_2\cup\partial U_3$.

Since U and U_5 are disjoint open subsets of U_1 such that F has no fixed points on $\bar{U_1}\setminus(U\cup U_5)$, in fact $\bar{U_1}\setminus(U\cup U_5)\subset\partial U_1\cup\partial U_2$. Therefore by the additivity property

$$i_{P\times P}(F,U_5)=i_{P\times P}(F,U_1)-i_{P\times P}(F,U)=0-0=0.$$

Similarly, U and U_6 are disjoint open subsets of U_2 such that F has no fixed points on $\bar{U_2}\setminus(U\cup U_6)$, in fact $\bar{U_2}\setminus(U\cup U_6)\subset\partial U_1\cup\partial U_2$. Therefore by the additivity property

$$i_{P\times P}(F,U_6)=i_{P\times P}(F,U_2)-i_{P\times P}(F,U)=0-0=0.$$

Finally, since $(U\cup U_5\cup U_6)$ and U_4 are disjoint open subsets of U_3 such that F has no fixed points on $\bar{U_3}\setminus(U\cup U_5\cup U_6\cup U_4)$, in fact $\bar{U_3}\setminus(U\cup U_5\cup U_6\cup U_4)\subset(\partial U_3\cup\partial U_1\cup\partial U_2)$. Therefore by the additivity property

$$\begin{aligned}i_{P\times P}(F,U_4)&=i_{P\times P}(F,U_3)-i_{P\times P}(F,U)-i_{P\times P}(F,U_5)\\&-i_{P\times P}(F,U_6)\\&=1-0-0-0=1.\end{aligned}$$

which implies the existence of a fixed point (x_1,y_1) of F satisfying (2.1).

Remark 1. Suppose, in addition, that F verifies the following hypothesis

$$G(0,y) = H(x,0) = 0 \quad \forall (x,y) \in E \times E,$$

then we can prove the existence of two fixed point $(x_0, 0), (0, y_0)$, of F satisfying (1).

In fact, from $G_0(x) = G(x,0)$, we know, $G_0'(\infty) = G_x(\infty, 0)$, and by (H2) and Lemma 13.4 in [1] there exists $\rho_\infty > \sigma$ such that for every $\rho \geq \rho_\infty, i_P(G_0, P_\rho) = 1$. On the other hand from hypothesis (H1), we have $x - G_0(x) \neq \lambda p_1, \quad \forall \lambda \geq 0, \quad \forall x \in S_\sigma$, then $i_P(G_0, P_\sigma) = 0$ (see Lemma 12.1 in [1]). Therefore, by the additivity property $i_P(G_0, P_\rho \setminus \bar{P_\sigma}) = 1$. Consequently, the solution property of the fixed point index implies that G_0 has at least one fixed point x_0 with $\sigma < \|x_0\|_E < \rho$. Now $(x_0, 0)$ is a fixed point of F.

In a similar manner we can prove the existence of $(0, y_0)$.

Remark 2. If P has nonempty interior and $G_x(\infty,\infty)$ and $H_y(\infty,\infty)$ are strongly positive then it is well known that the spectral radius of $G_x(\infty,\infty)$ (or $H_y(\infty,\infty)$) is an eigenvalue to a positive eigenvector, and in fact the only eigenvalue with this property. Then we have the following corollary.

Corollary 2. *Suppose that P has nonempty interior and let $F : P \times P \to P \times P$ a completely continuous map verifying the previous hypotheses (F1)-(F2). Moreover suppose that the right partial derivatives $G_x(\infty,\infty)$ and $H_y(\infty,\infty)$ are strongly positive. Then if*

(H'1)

$$G(x,y) \not\leq x \quad \forall (x,y) \in S_\sigma \times P$$

and

$$H(x,y) \not\leq y \quad \forall (x,y) \in P \times S_\sigma,$$

(H'2) $r(G_x(\infty,\infty)) < 1$ *and* $r(H_y(\infty,\infty)) < 1$,

F has at least one fixed points in $P \times P : (x_1, y_1)$ verifying (1).

Remark 3. If F supposed to be asymptotically linear along P, and if the above hypotheses (H'1) and (H'2) are substituted by

(H"1) $F(x,y) \not\leq (x,y) \quad \forall (x,y) \in \partial R_{\sigma,\sigma}$,

(H"2) $r(F'(\infty,\infty)) < 1$,

then (see Amann [1]) (H"1) and (H"2) imply that F has a fixed point $(x, y) \in P \times P$ verifying

$$\sigma < \|(x, y)\|_E < \rho,$$

but some component of the fixed point (x, y) (x or y) may be trivial.

Remark 4. In [5] we have shown some results similar to Theorem 1 and Corollary 2 but under different assumptions.

3. Application to System of Nonlinear Integral Equations

In this section we shall study the existence of positive solutions of system of nonlinear integral equations of the form

$$\begin{aligned} x(t) &= \int_0^{\tau_1(t)} f(t, s, x(t-s-l), y(t-s-l))\, ds \\ y(t) &= \int_0^{\tau_2(t)} g(t, s, x(t-s-l), y(t-s-l))\, ds \end{aligned} \tag{4}$$

under the following assumptions on functions f and g :
$f, g : \mathbb{R} \times \mathbb{R} \times [0, +\infty[\longrightarrow \mathbb{R}$ are continuous functions with :

(F1) : $f(t, s, 0, y) = g(t, s, x, 0) = 0$ for all $(t, s, x, y) \in \mathbb{R}\times\mathbb{R}\times[0, +\infty[\times[0, +\infty[$,

(F2) : $f(t, s, x, y) \geq 0, g(t, s, x, y) \geq 0, \quad \forall (t, s, x, y) \in \mathbb{R}\times\mathbb{R}\times[0, +\infty[\times[0, +\infty[$ and there exists a positive number w, $(w > 0)$ such that $f(t+w, s, x, y) = f(t, s, x, y)$ and $g(t + w, s, x, y) = g(t, s, x, y)$, $\forall (t, s, x, y) \in \mathbb{R} \times \mathbb{R} \times [0, +\infty[\times[0, +\infty[$,

(F3) : l is a nonnegative constant and $\tau_1, \tau_2 : \mathbb{R} \longrightarrow \mathbb{R}^+$ are a continuous and λ-periodic function $(\lambda > 0)$ such that $\frac{\omega}{\lambda} = \frac{p}{q}, p, q \in \mathbb{N}$.

System (4) includes the system proposed by Cooke and Kaplan [4] as a model to explain the evolution in time of two interacting species when seasonal factors are taken into account. For more details, see [5, 4, 3].

We are interested in the existence of nontrivial, nonnegative, continuous and $q\omega$- periodic solutions. Especially, we are interested in the existence of coexistence states. Also the existence of semitrivial solutions of (4) may be of interest, i.e. solutions with exactly one nontrivial component: this means that one species may survive in the absence of the other one.

Denote by P the cone of nonnegative functions in the real Banach space E, of all real and continuous $q\omega-$ periodic functions defined on $\mathbb{R}$, where if $x \in E$

$$\|x\| = \max_{0 \le t \le q\omega} |x(t)|.$$

Define the operator $F = (G, H) : P \times P \to P \times P$, by

$$F(x, y)(t) = (G(x, y)(t), H(x, y)(t)),$$

where

$$G(x, y)(t) = \int_0^{\tau_1(t)} f(t, s, x(t-s-l), y(t-s-l))\ ds$$

and

$$H(x, y)(t) = \int_0^{\tau_2(t)} g(t, s, x(t-s-l), y(t-s-l))\ ds.$$

It is easily to see that F is completely continuous (see [2]).

Take

$$\min_{t\in\mathbb{R}} \tau_1(t) = \tau_1 \quad \min_{t\in\mathbb{R}} \tau_2(t) = \tau_2$$

and

$$\max_{t\in\mathbb{R}} \tau_1(t) = \tau_1' \quad \max_{t\in\mathbb{R}} \tau_2(t) = \tau_2'.$$

Theorem 3. *Suppose that:*

(H'1) *f is bounded in bounded $x-$intervals uniformly in $(t, s, y) \in [0, q\omega] \times [0, \tau_1'] \times \mathbb{R}$;*

(H'2) *g is bounded in bounded $y-$intervals uniformly in $(t, s, x) \in [0, q\omega] \times [0, \tau_2'] \times \mathbb{R}$;*

(H'3) *there exists a continuous function $a : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$\lim_{x\to 0^+} \frac{f(t, s, x, y)}{x} = a(t, s, y),$$

uniformly in $(t, s, y) \in \mathbb{R} \times \mathbb{R} \times [0, \infty[$;

(H'4) *there exists a continuous function $b : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$\lim_{y\to 0^+} \frac{g(t, s, x, y)}{y} = b(t, s, x),$$

uniformly in $(t, s, x) \in \mathbb{R} \times \mathbb{R} \times [0, \infty[$;

(H'5) *there exists a continuous function* $c : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$\lim_{x \to +\infty} \frac{f(t,s,x,y)}{x} = c(t,s),$$

uniformly in $(t,s,y) \in \mathbb{R} \times \mathbb{R} \times [0,\infty[$;

(H'6) *there exists a continuous function* $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$\lim_{y \to +\infty} \frac{g(t,s,x,y)}{y} = d(t,s),$$

uniformly in $(t,s,x) \in \mathbb{R} \times \mathbb{R} \times [0,\infty[$;

(H'7) $a(t,s,y) \geq a > 0, \quad b(t,s,x) \geq b > 0,$

$$\forall (t,s,x,y) \in \mathbb{R} \times \mathbb{R} \times [0,\infty[\times[0,\infty[.$$

Then if

$$a\tau_1 > 1, \quad b\tau_2 > 1 \tag{5}$$

$$r(L(\tau_1,c)) < 1, \quad \text{and} \quad r(L(\tau_2,d)) < 1,$$

F *has at least four fixed points in* $P \times P : (0,0), (x_0,0), (0,y_0), (x_1,y_1)$ *verifying* (1), *where* $r(L(\tau_1,c))$ *means the spectral radius of the linear operator* $L(\tau_1,c) : E \longrightarrow E$ *defined by*

$$L(\tau_1,c)x(t) = \int_0^{\tau_1(t)} c(t,s)x(t-s-l)\, ds, \quad \forall x \in E,$$

(analogously for $r(L(\tau_2,d))$ *and* $L(\tau_2,d)$*).*

Proof. We are going to prove that all conditions of Theorem 1, Remark 1 and Remark 2 are satisfied. For it, we must observe that (E,P) is an ordered Banach space with $\overset{\circ}{P} \neq \emptyset$.

Select $\varepsilon > 0$ verifying

$$(a-\varepsilon)\tau_1 > 1, \quad (b-\varepsilon)\tau_2 > 1. \tag{6}$$

From hypotheses (H'3) and (H'4), we obtain $\sigma(\varepsilon) > 0$ such that

$$f(t,s,x,y) \geq (a(t,s,y)-\varepsilon)x, \quad \forall x \in [0,\sigma(\varepsilon)]$$

for all $(t,s,y) \in \mathbb{R} \times \mathbb{R} \times [0,\infty[$,

$$g(t,s,x,y) \geq (b(t,s,x) - \varepsilon)y, \quad \forall y \in [0, \sigma(\varepsilon)]$$

for all $(t,s,x) \in \mathbb{R}\times\mathbb{R}\times[0,\infty[$. Now, taking a fixed $\sigma \in (0, \sigma(\epsilon)]$ and $p_1 = p_2 = 1$, we claim that (H1) of theorem (1) is satisfied. In fact if $(x,y) \in S_\sigma \times P$ and $\lambda \geq 0$ such that $x - G(x,y) = \lambda$. From $\|x\| = \sigma > 0$, we affirm that $\min_{t\in[0,q\omega]} x(t) > 0$. To see this, suppose that $x(t_1) = \min_{t\in[0,q\omega]} x(t)$. Then

$$\begin{aligned} x(t_1) &= \int_0^{\tau_1(t_1)} f(t_1, s, x(t_1 - s - l), y(t_1 - s - l))\, ds + \lambda \\ &\geq \int_0^{\tau_1(t_1)} f(t_1, s, x(t_1 - s - l), y(t_1 - s - l))\, ds \\ &\geq \int_0^{\tau_1(t_1)} (a(t_1, s, y(t_1 - s - l)) - \varepsilon) x(t_1 - s - l)\, ds \\ &\geq (a - \varepsilon) \int_0^{\tau_1(t_1)} x(t_1 - s - l)\, ds. \end{aligned}$$

So, if $x(t_1) = 0$, then $x(t_1 - s - l) = 0, \forall s \in [0, \tau_1(t_1)]$
or equivalently

$$x(s - l) = 0, \forall s \in [t_1 - \tau_1(t_1), t_1],$$

which implies that

$$x(s) = 0, \quad \forall s \in [t_1 - \tau_1(t_1) - l, t_1 - l].$$

Iterating the procedure n times we obtain $x(s) = 0, \forall s \in I$ where I is an interval of length at least $n\tau_1(t_1) \geq n\tau_1 \geq q\omega$ and since x is $q\omega$- periodic, x must be zero, which is a contradiction.
Once we have proved that $x(t_1) > 0$, we have

$$x(t_1) \geq (a - \varepsilon)\tau_1 x(t_1)$$

and consequently $1 \geq (a - \varepsilon)\tau_1$ which contradicts (6). One may proceed in an analogous way if $\|y\| = \sigma$ and $x \in P$. Therefore (H1) of Theorem (1) is verified.

Now it is not hard to prove the existence of $G_x(\infty, y)$ and $H_y(x, \infty)$(see [5]).

In fact, for all $x \in P$ and $y \in P$ we have verified that

$$G_x(\infty, y)x(t) = G_x(\infty, \infty)x(t) = L(\tau_1, c)x(t)$$

and

$$\lim_{\substack{\|x\|\longrightarrow\infty \\ x\in P}} \frac{G(x,y)-L(\tau_1,c)x}{\|x\|}=0, \quad \text{uniformly in} \quad y\in P.$$

Also we have

$$H_y(x,\infty)y(t)=H_y(\infty,\infty)y(t)=L(\tau_2,d)y(t)$$

and

$$\lim_{\substack{y\longrightarrow+\infty \\ y\in P}} \frac{H(x,y)-L(\tau_2,d)y}{\|y\|}=0, \quad \text{uniformly in} \quad x\in P.$$

Now, it is easily seen that (see the proof of Theorem (1) in [2]) $L(\tau_1,c)$ and $L(\tau_2,d)$ are strongly positive. Consequently hypothesis (H2) of the Theorem is satisfied.

Now we present an example of Theorem 3.

Example 4. Let $f_1 : [0,+\infty)\times[0,\infty)\to\mathbb{R}^+$ be a continuous function defined by

$$f_1(x,y)=\begin{cases} x(1+\cos^2 y), & 0\le x\le 1, \le y\ge 0\\ \sqrt{x}\cos^2 y+\frac{1}{2}x+\frac{1}{2}, & x\ge 1, y\ge 0\end{cases}$$

and $g_1 : [0,+\infty)\times[0,\infty)\to\mathbb{R}^+$ defined by $g_1(x,y)=f_1(y,x)$.

And take $d,d' : \mathbb{R}\to\mathbb{R}$ a continuous, positive and ω-periodic functions ($\omega>0$) and $l=0$.

Let the system of nonlinear integral equations

$$x(t)=\int_0^{\tau_1(t)} d(t-s)f_1(x(s),y(s))\;ds$$
$$y(t)=\int_0^{\tau_2(t)} d'(t-s)g_1(x(s),y(s))\;ds.$$

If

$$f(t,s,x,y)=d(t-s)f_1(x(s),y(s))$$

for all $(t,s,x,y)\in\mathbb{R}\times\mathbb{R}\times[0,+\infty[\times[0,+\infty[$ and

$$g(t,s,x,y)=d'(t-s)g_1(x(s),y(s))$$

for all $(t,s,x,y)\in\mathbb{R}\times\mathbb{R}\times[0,+\infty[\times[0,+\infty[$, hypotheses (H'1) -(H'7) of Theorem 3 are satisfied with

$a(t,s,y) = d(t-s)(1+\cos^2 y)$, $b(t,s,x) = d'(t-s)(1+\cos^2 x)$, $c(t,s) = \frac{1}{2}d(t-s)$, $d(t,s) = \frac{1}{2}d'(t-s)$.

Consequently, if

$$d(t-s)\min_{t\in\mathbb{R}}\tau_1(t) > 1 \quad \text{and} \quad d'(t-s)\min_{t\in\mathbb{R}}\tau_2(t) > 1 \quad \forall (t,s)\in\mathbb{R}\times\mathbb{R} \tag{7}$$

$$r(L(\tau_1,c)) < 2, \quad r(L(\tau_2,d)) < 2, \tag{8}$$

the above system has at least four fixed points in $P\times P$: $(0,0), (x_0,0), (0,y_0), (x_1,y_1)$ verifying (1). Note that in the particular case where $d(t)\equiv d\in\mathbb{R}^+$ and $d'(t)\equiv d'\in\mathbb{R}^+$ conditions (7) and (8) are satisfied, if we take

$$\frac{1}{d} < \min_{t\in\mathbb{R}}\tau_1(t) \leq \max_{t\in\mathbb{R}}\tau_1(t) < \frac{2}{d}$$

and

$$\frac{1}{d'} < \min_{t\in\mathbb{R}}\tau_2(t) \leq \max_{t\in\mathbb{R}}\tau_1(t) < \frac{2}{d'}.$$

Here we use that fact that (see [10, 2])

$$\min_{t\in\mathbb{R}}\int_0^{\tau_1(t)}\alpha(t,s)ds \leq r(L(\tau_1,\alpha)) \leq \max_{t\in\mathbb{R}}\int_0^{\tau_1(t)}\alpha(t,s)ds$$

for every continuous function $\alpha : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ which is ω-periodic in t.

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