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GLOBAL DYNAMICS OF AN SEIR EPIDEMIC MODEL WITH WANING PREVENTIVE VACCINE

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Abstract: In this paper, an SEIR epidemic model with waning preventive vaccine is investigated. The results of our mathematical analysis indicate that the dynamics of the system is almost determined by the basic reproduction number. If the basic reproduction number is less than unity, it is proven that the disease-free equilibrium is globally asymptotically stable by comparison arguments. If it is greater than unity, there exists a unique endemic equilibrium and sufficient conditions are obtained for the global stability of the endemic equilibrium by the theory of the compound matrices. Numerical simulations are carried out to illustrate the main results.

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1. Introduction

Over the past few decades, many compartmental mathematical models, such as SIS, SIR or SIRS (where S, I, and R denote the populations of susceptible, infectious and recovered), have been used to investigate the spread of the disease (see, for instance, [2, 11, 14] and the the references therein). In these studies, it was assumed that the disease incubation is negligible. However, for some diseases, such as scarlet fever, poliomyelitis and AIDS, on adequate contact

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with an infective, a susceptible individual becomes exposed, that is, infected but not infective. This individual remains in the exposed class for a certain latent period before becoming infective. Hence, a new class E called the exposed class is introduced, in which the susceptible remains for a given length of time before moving into the infective class. This kind of model is called SEIR (see, for instance, [7, 9] and the the references therein).

Emerging infectious diseases have devastating impacts on public health. So it is important to evaluate potential methods for controlling these diseases. Vaccination is a commonly used method for controlling disease, such as pertussis, diphtheria, or influenza. Mathematical models including vaccination aid in deciding on a vaccination strategy and in determining changes in qualitative behavior that could result from such a control measure(see, for instance, [5, 6, 15] and the reference therein). In [1], Gumel et al. considered the following infectious disease model:

$$\dot{S}(t) = \Pi - \beta SI - \xi S - \mu S, \tag{1.1a}$$
$$\dot{V}(t) = \xi S - (1-\tau)\beta VI - \mu V, \tag{1.1b}$$
$$\dot{E}(t) = \beta SI + (1-\tau)\beta VI - \alpha E - \mu E, \tag{1.1c}$$
$$\dot{I}(t) = \alpha E - \delta I - dI - \mu I, \tag{1.1d}$$
$$\dot{R}(t) = \delta I - \mu R, \tag{1.1e}$$

where the total population N is divided into five compartments: susceptible, $S(t)$, vaccinated, $V(t)$, latently infected, $E(t)$, infectious, $I(t)$ and recovered, $R(t)$, individuals. In $(1.1a) - (1.1e)$, the parameters $\Pi, \beta, \xi, \mu, \alpha, \delta$ and d are positive constants. Π denotes the recruitment rate of susceptible humans, μ denotes the natural mortality rate, β denotes the effective contact rate, ξ denotes the vaccination coverage rate, α denotes the rate of development of clinical symptoms, δ denotes the recovery rate, d denotes the disease-induced mortality rate. $0 \leq \tau \leq 1$ denotes the vaccine efficacy($\tau = 1$ represents a vaccine that offers 100% protection against infection, $\tau = 0$ models a vaccine that offers no protection at all). However, some recent clinical studies have shown vaccine-induced immunity can wane in preventive vaccines against infectious diseases such as Hepatitis B, Polio and Mumps. Once a vaccine wanes from the body of the vaccinated person, the person becomes susceptible to the disease again.

Motivated by the work of Gumel et al.[1], we are concerned with the following infectious disease model with a waning preventive vaccine:

$$\dot{S}(t) = \Pi - \beta SI - \xi S - \mu S + \eta V, \tag{1.2a}$$
$$\dot{V}(t) = \xi S - (1-\tau)\beta VI - \mu V - \eta V, \tag{1.2b}$$
$$\dot{E}(t) = \beta SI + (1-\tau)\beta VI - \alpha E - \mu E, \tag{1.2c}$$
$$\dot{I}(t) = \alpha E - \delta I - dI - \mu I, \tag{1.2d}$$
$$\dot{R}(t) = \delta I - \mu R, \tag{1.2e}$$

where η is the rate at which vaccine wanes (that is $1/\eta$ is the duration of the loss of immunity acquired by preventive vaccine or by infection), and the meanings of the other parameters are the same as those of the model $(1.1a) - (1.1e)$.

The initial conditions for the model $(1.2a) - (1.2e)$ take the form

$$S(0) > 0, \;\; V(0) > 0, \;\; E(0) > 0, \;\; I(0) > 0, \;\; R(0) > 0. \tag{1.3}$$

Notice that the recovered population $R(t)$ does not feature in the first four equations of the model, we will only discuss Equations $(1.2a) - (1.2d)$ in the following. The dynamic behaviors of $R(t)$ can be obtained from equation $(1.2e)$.

The paper is organized as follows. In the next section, the ultimate boundedness of the solutions for the model $(1.2a) - (1.2d)$ is presented. In Section 3, by analyzing the corresponding characteristic equations, the local stability of a disease free equilibrium and an endemic equilibrium of the model $(1.2a) - (1.2d)$ is discussed. In Section 4, by using comparison arguments and the theory of compound matrices, sufficient conditions are received for the global asymptotic stability of the disease free equilibrium and the endemic equilibrium, respectively. Numerical simulations are carried out in Section 5 to illustrate the main theoretical results. A brief discussion is given in Section 6 to conclude this work.

2. Preliminary Results

In this section, we study the basic properties of the model $(1.2a) - (1.2d)$.

By the fundamental theory of functional differential equations [4], it is well known that the model $(1.2a) - (1.2d)$ has a unique solution $(S(t), V(t), E(t), I(t))$ satisfying initial conditions in R_4^+. Further, it is easy to show that all solutions of the model $(1.2a) - (1.2d)$ are defined on $[0, +\infty)$.

Theorem 2.1. *For any solution* $(S(t), V(t), E(t), I(t))$ *of the model* (1.2a)–(1.2d), *we have*

$$\limsup_{t\to+\infty} S(t) \le \frac{\Pi(\mu+\eta)}{\mu(\xi+\mu+\eta)}, \quad \limsup_{t\to+\infty} V(t) \le \frac{\xi\Pi}{\mu(\mu+\xi+\eta)},$$
$$\limsup_{t\to+\infty} E(t) \le \frac{\Pi}{\mu}, \qquad \limsup_{t\to+\infty} I(t) \le \frac{\Pi}{\mu}.$$

Proof. Define

$$L(t) = S(t) + V(t) + E(t) + I(t).$$

Calculating the derivative of $L(t)$ along the solutions of the model (1.2a) – (1.2d), it follows that

$$\dot{L}(t) = \Pi - \mu L - dI - \delta I \le \Pi - \mu L,$$

a standard comparison argument shows that $\limsup_{t\to+\infty} L(t) \le \Pi/\mu$. Hence,

$$\limsup_{t\to+\infty} E(t) \le \frac{\Pi}{\mu}, \quad \limsup_{t\to+\infty} I(t) \le \frac{\Pi}{\mu}.$$

By equations (1.2a) – (1.2b), we get

$$\dot{S}(t) \le \Pi - (\xi+\mu)S + \eta V,$$
$$\dot{V}(t) \le \xi S - (\mu+\eta)V.$$

Consider the following auxiliary system

$$\begin{aligned} \dot{z_1}(t) &= \Pi - (\xi+\mu)z_1 + \eta z_2, \\ \dot{z_2}(t) &= \xi z_1 - (\mu+\eta)z_2. \end{aligned} \tag{2.1}$$

It is easy to prove that the positive equilibrium

$$z^*(\Pi(\mu+\eta)/(\mu(\xi+\mu+\eta)), \xi\Pi/(\mu(\mu+\xi+\eta))$$

of system (2.1) is globally asymptotically stable. By comparison, it follows that

$$\limsup_{t\to+\infty} S(t) \le \frac{\Pi(\mu+\eta)}{\mu(\xi+\mu+\eta)}, \quad \limsup_{t\to+\infty} V(t) \le \frac{\xi\Pi}{\mu(\mu+\xi+\eta)}. \tag{2.2}$$

This completes the proof. □

Denote

$$D=\left\{(S,V,E,I)\in R^4_{+0}: S+V+E+I\leq\frac{\Pi}{\mu}, S\leq\frac{\Pi(\mu+\eta)}{\mu(\xi+\mu+\eta)}, V\leq\frac{\xi\Pi}{\mu(\xi+\mu+\eta)}\right\}.$$

Theorem 2.1 implies that the set D is a positively invariant and the attracting region for the disease transmission model given by the model $(1.2a)-(1.2d)$ with initial conditions in R^4_+.

3. Equilibria and Local Stability

In this section, we study the local stability of a disease-free equilibrium and an endemic equilibrium of the model $(1.2a)-(1.2d)$ by analyzing the corresponding characteristic equations, respectively.

The model $(1.2a)-(1.2d)$ always has a disease-free equilibrium

$$P_0=(S_0,V_0,E_0,I_0)=(\Pi(\mu+\eta)/(\mu(\xi+\mu+\eta)),\xi\Pi/(\mu(\xi+\mu+\eta)),0,0).$$

Following the method of next generation matrix by van den Driessche and Watmough [12], one obtains the basic reproduction number for the model $(1.2a)-(1.2d)$ as

$$R_0=\frac{\alpha\beta\Pi(\mu+(1-\tau)\xi+\eta)}{\mu(\alpha+\mu)(\xi+\mu+\eta)(\delta+\mu+d)}.$$

The basic reproduction number R_0 is defined as the expected number of secondary cases produced in an entirely susceptible population by a typical infected individual during its entire infectious period [12].

To calculate the endemic equilibrium $P^*(S^*,V^*,E^*,I^*)$, we solve

$$\Pi-\beta S^*I^*-\xi S^*-\mu S^*+\eta V^*=0, \tag{3.1a}$$
$$\xi S^*-(1-\tau)\beta V^*I^*-\mu V^*-\eta V^*=0, \tag{3.1b}$$
$$\beta S^*I^*+(1-\tau)\beta V^*I^*-\alpha E^*-\mu E^*=0, \tag{3.1c}$$
$$\alpha E^*-\delta I^*-dI^*-\mu I^*=0. \tag{3.1d}$$

From equations $(3.1a)$, $(3.1b)$ and $(3.1d)$, we get

$$S^*=\frac{\Pi+\eta V^*}{\beta I^*+\xi+\mu}, V^*=\frac{\xi S^*}{(1-\tau)\beta I^*+\mu+\eta}, E^*=\frac{\mu+\delta+d}{\alpha}I^*. \tag{3.2}$$

Equation $(3.1c)$ yields

$$E^*=\frac{\beta S^*I^*+(1-\tau)\beta V^*I^*}{\alpha+\mu}. \tag{3.3}$$

Substituting the expressions of S^*, V^*, E^* in equation (3.2) into equation (3.3), which gives

$$Q(I^*) = AI^{*2} + BI^* + C = 0, \tag{3.4}$$

where

$$\begin{aligned} A &= (1-\tau)\beta^2, \\ B &= \beta(\mu{+}\eta)\left[1{+}\frac{\mu(1{-}\tau)(\xi{+}\mu{+}\eta)}{(\mu{+}\eta)(\mu{+}\eta{+}(1-\tau)\xi)}\left(1{-}R_0{+}\frac{\xi((1{-}\tau)(\mu{+}\xi){+}\eta)}{\mu(\xi{+}\mu{+}\eta)}\right)\right], \\ C &= \mu(\xi+\mu+\eta)(1-R_0). \end{aligned} \tag{3.5}$$

The endemic equilibrium of the model $(1.2a)-(1.2d)$ are given by equation (3.2) with the positive root I^* of equation (3.4). Let I_1^*, I_2^* be the roots of equation (3.4), and the conditions for equation (3.4) to have positive roots are determined below.

Suppose $0 \le \tau < 1$, then $A > 0$. If $R_0 > 1$, $C < 0$. Then Equation (3.4) has a unique positive root for $I_1^* I_2^* = C/A < 0$. If $R_0 = 1$, $B > 0, C = 0$. Here, $Q(I) = I(AI + B)$, with $I_1^* = 0, I_2^* = -B/A < 0$. Hence, equation (3.4) has no positive root. If $R_0 < 1$, $A > 0, B > 0, C > 0$. Thus, equation (3.4) has no positive root.

Suppose $\tau = 1$, then $A = 0, B = \beta(\mu + \eta)$. Hence, $Q(I) = BI + C$, with the root $I = -C/B$. If $R_0 > 1$, $C < 0$. Then equation (3.4) has a unique positive root. If $R_0 \le 1$, $C \ge 0$. Then equation (3.4) has no positive root.

In conclusion, we have the following results.

Theorem 3.1. *The model* $(1.2a)-(1.2d)$ *has a unique endemic equilibrium* $P^*(S^*, V^*, E^*, I^*)$ *when* $R_0 > 1$ *and no endemic equilibrium when* $R_0 \le 1$.

Now we study the local stability of the disease-free equilibrium P_0 and the endemic equilibrium P^*.

The characteristic equation of the model $(1.2a) - (1.2d)$ at the equilibrium P_0 is of the form

$$(\lambda{+}\mu)(\lambda{+}\xi{+}\mu{+}\eta)[\lambda^2{+}(\alpha{+}2\mu{+}\delta{+}d)\lambda{+}(\alpha{+}\mu)(\delta{+}d{+}\mu)(1{-}R_0)]{=}0. \tag{3.6}$$

Clearly, equation (3.6) always has two negative real roots $\lambda_1 = -\mu$, $\lambda_2 = -\xi - \mu - \eta$. The other roots λ_3, λ_4 of Equation (3.6) are determined by the following equation

$$\lambda^2 + (\alpha + 2\mu + \delta + d)\lambda + (\alpha + \mu)(\delta + d + \mu)(1 - R_0) = 0. \tag{3.7}$$

If $R_0 > 1$, $\lambda_3\lambda_4 < 0$, then equation (3.7) has one positive real part root. Hence, P_0 is unstable. If $R_0 < 1$, $\lambda_3 + \lambda_4 < 0$, $\lambda_3\lambda_4 > 0$, then the characteristic roots of equation (3.7) have negative real parts. Hence, P_0 is locally asymptotically stable.

By Theorem 3.1, the model $(1.2a)-(1.2d)$ has a unique endemic equilibrium P^* when $R_0 > 1$. The characteristic equation of the model $(1.2a) - (1.2d)$ at the equilibrium P^* takes the form

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{3.8}$$

where

$$\begin{aligned}
a_1 =&\delta + d + \alpha + \xi + 4\mu + \eta + (2-\tau)\beta I^*,\\
a_2 =&(\beta I^* + \mu)(\mu + \eta + (1-\tau)\beta I^*) + \xi(\mu + (1-\tau)\beta I^*)\\
&+ (\alpha + 2\mu + \delta + d)(\xi + 2\mu + \eta + (2-\tau)\beta I^*),\\
a_3 =&(\alpha+2\mu+\delta+d)((\beta I^*+\mu)((1-\tau)\beta I^*+\mu+\eta)+\xi((1-\tau)\beta I^*+\mu))\\
&+ (\alpha+\mu)(\delta+d+\mu)(1-\tau)\beta I^* + \alpha\tau\beta^2 S^* I^*,\\
a_4 =&(\alpha+\mu)(\delta+d+\mu)((1-\tau)\beta I^*(\beta I^*+\xi)+\beta I^*((1-\tau)\mu+\eta))+\alpha\mu\tau\beta^2 SI.
\end{aligned}$$

By calculation,

$$\begin{aligned}
\Delta_1 =&a_1 = \delta + d + \alpha + \xi + 4\mu + \eta + (2-\tau)\beta I^* > 0,\\
\Delta_2 =&a_1a_2 - a_3\\
=&(\alpha+\mu)(\delta+d+\mu)(\xi+2\mu+\eta) + \alpha(1-\tau)\beta^2 I^*(S^*+V^*)\\
&+[\xi+2\mu+\eta+(2-\tau)\beta I^*][(\alpha+\mu)^2+(1-\tau)\beta I^*(\beta I^*+\xi)+\beta I^*((1-\tau)\mu+\eta)\\
&+\mu(\beta I^*+\mu+\eta+\xi)+(\alpha+2\mu+\delta+d)(\xi+3\mu+\eta+(2-\tau)\beta I^*+\delta+d)]>0,\\
\Delta_3 =&a_3\Delta_2 - a_1^2a_4\\
=&H + (\mu-\alpha)[\eta\beta I^*(\delta+d+\mu)(\tau(\delta+d+\mu)(\delta+d+2\mu+\alpha)\\
&+(1-\tau)(\alpha+\mu)(\mu+\alpha+\delta+d))+(1-\tau)\beta^2 I^{*2}(\delta+d)^2(\xi+3\mu+\alpha+\eta+(2-\tau)\beta I^*)],\\
\Delta_4 =&a_4\Delta_3,
\end{aligned}$$

where the expression of H can be seen in Appendix. Since $a_4 > 0$, by the Routh-Hurwitz criterion, all characteristic roots of Equation (3.8) have negative real parts for $\Delta_3 > 0$.

From what has been discussed above, we have the following results.

Theorem 3.2. *For the model* $(1.2a) - (1.2d)$*, we have the following:*

(1) *If $R_0 < 1$, then the disease-free equilibrium P_0 is locally asymptotically stable; if $R_0 > 1$, P_0 is unstable.*

(2) *If $R_0 > 1$ and $\Delta_3 > 0$, then the unique positive endemic equilibrium P^* is locally asymptotically stable.*

From the proof of Theorem 3.2, one can get the following result.

Corollary 3.1. *If $R_0 > 1$ and $\mu > \alpha$, then the positive endemic equilibrium P^* is locally asymptotically stable.*

4. Global Stability

In this section, we study the global stability of the disease-free equilibrium P_0 and the endemic equilibrium P^* of the model $(1.2a) - (1.2d)$, respectively.

Theorem 4.1. *The disease-free equilibrium P_0 is globally asymptotically stable if $R_0 < 1$.*

Proof. Let $(S(t), V(t), E(t), I(t))$ be any positive solution of the model $(1.2a) - (1.2d)$ with initial conditions in R_4^+.

Since $R_0 < 1$, we can choose $\varepsilon > 0$ small enough such that

$$\frac{\alpha\beta(\Pi(\mu + (1-\tau)\xi + \eta) + (2-\tau)\varepsilon\mu(\xi + \mu + \eta))}{\mu(\alpha+\mu)(\xi+\mu+\eta)(\delta+\mu+d)} < 1. \tag{4.1}$$

By (2.2), for $\varepsilon > 0$ satisfying (4.1), there exists a $T_1 > 0$ such that if $t > T_1$,

$$S(t) \le \frac{\Pi(\mu+\eta)}{\mu(\xi+\mu+\eta)} + \varepsilon, \quad V(t) \le \frac{\xi\Pi}{\mu(\xi+\mu+\eta)} + \varepsilon.$$

From equation $(1.2c)$, it is easy to know that if $t > T_1$,

$$\dot{E}(t) \le \left(\beta\left(\frac{\Pi(\mu+\eta)}{\mu(\mu+\xi+\eta)} + \varepsilon\right) + (1-\tau)\beta\left(\frac{\xi\Pi}{\mu(\xi+\mu+\eta)} + \varepsilon\right)\right) I - (\alpha+\mu)E.$$

Consider the following auxiliary system

$$\begin{aligned} \dot{u_1}(t) &= \left(\beta\left(\frac{\Pi(\mu+\eta)}{\mu(\xi+\mu+\eta)} + \varepsilon\right) + (1-\tau)\beta\left(\frac{\xi\Pi}{\mu(\xi+\mu+\eta)} + \varepsilon\right)\right) u_2 - (\alpha+\mu)u_1, \\ \dot{u_2}(t) &= \alpha u_1 - (\delta + d + \mu)u_2. \end{aligned} \tag{4.2}$$

It is easy to prove that the equilibrium $(0,0)$ of system (4.2) is globally asymptotically stable for (4.1). By comparison, it follows that

$$\lim_{t\to\infty} E(t) = 0, \quad \lim_{t\to\infty} I(t) = 0. \tag{4.3}$$

Hence, for arbitrary $\varepsilon > 0$, there exists a $T_2 > 0$ such that if $t > T_2$, $E(t) < \varepsilon, I(t) < \varepsilon$. From Equations $(1.2a) - (1.2b)$, it is easy to know that if $t > T_2$,

$$\begin{aligned}\dot{S}(t) \geq& \Pi - (\beta\varepsilon + \xi + \mu)S + \eta V,\\ \dot{V}(t) \geq& \xi S - ((1-\tau)\beta\varepsilon + \mu + \eta)V.\end{aligned}$$

Consider the following auxiliary system

$$\begin{aligned}\dot{v_1}(t) =& \Pi - (\beta\varepsilon + \xi + \mu)v_1 + \eta v_2,\\ \dot{v_2}(t) =& \xi v_1 - ((1-\tau)\beta\varepsilon + \mu + \eta)v_2.\end{aligned} \tag{4.4}$$

It is easy to prove that the positive equilibrium $v^*(v_1^*, v_2^*)$ of system (4.4) is globally asymptotically stable, where

$$\begin{aligned}v_1^* &= \frac{\Pi((1-\tau)\beta\varepsilon + \mu + \eta)}{(\beta\varepsilon + \mu)((1-\tau)\beta\varepsilon + \mu + \eta) + \xi((1-\tau)\beta\varepsilon + \mu)},\\ v_2^* &= \frac{\xi\Pi}{(\beta\varepsilon + \mu)((1-\tau)\beta\varepsilon + \mu + \eta) + \xi((1-\tau)\beta\varepsilon + \mu)}.\end{aligned}$$

By comparison, it follows that

$$\liminf_{t\to+\infty} S(t) \geq v_1^*, \quad \liminf_{t\to+\infty} V(t) \geq v_2^*. \tag{4.5}$$

Since (4.5) holds for arbitrary $\varepsilon > 0$ sufficiently small, it follows that

$$\liminf_{t\to+\infty} S(t) \geq \frac{\Pi(\mu+\eta)}{\mu(\xi+\mu+\eta)}, \quad \liminf_{t\to+\infty} V(t) \geq \frac{\xi\Pi}{\mu(\xi+\mu+\eta)}. \tag{4.6}$$

By (4.6) and (2.2), it follows that

$$\lim_{t\to\infty} S(t) = \frac{\Pi(\mu+\eta)}{\mu(\xi+\mu+\eta)}, \quad \lim_{t\to\infty} V(t) = \frac{\xi\Pi}{\mu(\xi+\mu+\eta)}. \tag{4.7}$$

By (4.3) and (4.7), P_0 is globally asymptotically stable when $R_0 < 1$. The proof is complete. □

In the following, the method developed in [8, 10] is used to discuss the global stability of the endemic equilibrium P^*. First, we introduce this method briefly.

Let $G \subset R^n$ be an open set. Consider the differential equation:

$$\dot{x} = f(x), \tag{4.8}$$

where the function $f : x \to f(x) \in R^n, x \in G$ is continuous on G.

Denote E be an $n \times n$ matrix and $E^{[2]}$ be the second additive compound matrix of E. Let $Q(x)$ be a $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function that is continuous on G and consider

$$A = Q_f Q^{-1} + QJ^{[2]}Q^{-1},$$

where the matrix Q_f is the derivative of Q in the direction of the vector field f in system (4.8), and $J^{[2]}$ is the second additive compound matrix of the Jacobian matrix of system (4.8). Consider the *Lozinskiĭ* measure $\bar{\mu}$ of A with respect to a vector norm in $R^{\binom{n}{2}}$, that is

$$\bar{\mu}(A) = \lim_{h \to 0^+} \frac{\|I + hA\|}{h}.$$

Lemma 4.1. [10] *If G_1 is a compact absorbing subset in the interior of G, and there exist $\gamma > 0$ and a Lozinskiĭ measure $\bar{\mu}(A) \leq -\gamma$ for all $x \in G_1$, then every omega limit point of system (4.8) in the interior of G is an equilibrium in G_1.*

The Lozinskiĭ measure in Lemma 4.1 can be evaluated as [13]:

$$\bar{\mu}(A) = \inf\{\bar{k} : D_+\|z\| \leq \bar{k}\|z\|, \quad for \text{ all solutions of } z' = Az\}, \tag{4.9}$$

where D_+ is the right-hand derivative.

Next, we prove the global stability of the equilibrium P^* by Lemma 4.1.

Theorem 4.2. *If $R_0 > 1$, then the endemic equilibrium P^* is globally asymptotically stable provided that:*

$$u > \max\{\alpha,\ \eta\}. \tag{4.10}$$

Proof. Let $(S(t), V(t), E(t), I(t))$ be any positive solution of the model (1.2a) – (1.2d) with initial conditions in R_4^+. By Corollary 3.1, we see that

the endemic equilibrium P^* is locally asymptotically stable provided $R_0 > 1$ and $u > \max\{\alpha, \eta\}$.

The second additive compound matrix $J^{[2]}$ associated with the solution $(S(t), V(t), E(t), I(t))$ is

$$J^{[2]} = -diag\begin{pmatrix} \beta I + \xi + 2u + (1-\tau)\beta I + \eta \\ \beta I + \xi + 2u + \alpha \\ \beta I + \xi + 2u + \delta + d \\ (1-\tau)\beta I + 2u + \alpha + \eta \\ (1-\tau)\beta I + 2u + \delta + d + \eta \\ 2u + \delta + d + \alpha \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & -(1-\tau)\beta V & 0 & \beta S & 0 \\ (1-\tau)\beta I & 0 & \beta S + (1-\tau)\beta V & \eta & 0 & \beta S \\ 0 & \alpha & 0 & 0 & \eta & 0 \\ -\beta I & \xi & 0 & 0 & \beta S + (1-\tau)\beta V & (1-\tau)\beta V \\ 0 & 0 & \xi & \alpha & 0 & 0 \\ 0 & 0 & \beta I & 0 & (1-\tau)\beta I & 0 \end{pmatrix}.$$

Define

$$Q = \begin{pmatrix} 1/E & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/E & 0 & 0 \\ 0 & 0 & 1/I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/I & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/I \end{pmatrix},$$

then

$$A = Q_f Q^{-1} + Q J^{[2]} Q^{-1}$$
$$= -diag\begin{pmatrix} \beta I + (1-\tau)\beta I + \frac{\beta SI + (1-\tau)\beta VI}{E} + \xi + \mu + \eta - \alpha \\ \beta I + \frac{\beta SI + (1-\tau)\beta VI}{E} + \xi + \mu \\ (1-\tau)\beta I + \frac{\beta SI + (1-\tau)\beta VI}{E} + \mu + \eta \\ \beta I + \frac{\alpha E}{I} + \xi + \mu \\ (1-\tau)\beta I + \frac{\alpha E}{I} + \mu + \eta \\ \frac{\alpha E}{I} + \mu + \alpha \end{pmatrix}$$

$$+\begin{pmatrix} 0 & 0 & 0 & -\frac{(1-\tau)\beta VI}{E} & \frac{\beta SI}{E} & 0 \\ (1-\tau)\beta I & 0 & \eta & \frac{\beta SI+(1-\tau)\beta VI}{E} & 0 & \frac{\beta SI}{E} \\ -\beta I & \xi & 0 & 0 & \frac{\beta SI+(1-\tau)\beta VI}{E} & \frac{(1-\tau)\beta VI}{E} \\ 0 & \frac{\alpha E}{I} & 0 & 0 & \eta & 0 \\ 0 & 0 & \frac{\alpha E}{I} & \xi & 0 & 0 \\ 0 & 0 & 0 & \beta I & (1-\tau)\beta I & 0 \end{pmatrix}.$$

Define the norm on R^6[1]:

$$\|z\| = \max\{U_1, U_2\},$$

where $z \in R^6$ with component $z_i, i = 1, 2, \cdots, 6$, and

$$U_1(z_1,z_2,z_3)=\begin{cases} \max\{|z_1|, |z_2|+|z_3|\} & \text{if sgn}(z_1)=\text{sgn}(z_2)=\text{sgn}(z_3) \\ \max\{|z_2|, |z_1|+|z_3|\} & \text{if sgn}(z_1)=\text{sgn}(z_2)=-\text{sgn}(z_3) \\ \max\{|z_1|, |z_2|, |z_3|\} & \text{if sgn}(z_1)=-\text{sgn}(z_2)=\text{sgn}(z_3) \\ \max\{|z_1|+|z_3|, |z_2|+|z_3|\} & \text{if}-\text{sgn}(z_1)=\text{sgn}(z_2)\ =\text{sgn}(z_3), \end{cases}$$

$$U_2(z_4,z_5,z_6)=\begin{cases} |z_4|+|z_5|+|z_6| & \text{if sgn}(z_4)=\text{sgn}(z_5)=\text{sgn}(z_6) \\ \max\{|z_4|+|z_5|, |z_4|+|z_6|\} & \text{if sgn}(z_4)=\text{sgn}(z_5)=-\ \text{sgn}(z_6) \\ \max\{|z_5|, |z_4|+|z_6|\} & \text{if sgn}(z_4)=-\ \text{sgn}(z_5)=\text{sgn}(z_6) \\ \max\{|z_4|+|z_6|, |z_5|+|z_6|\} & \text{if}-\text{sgn}(z_4)=\text{sgn}(z_5)=\text{sgn}(z_6). \end{cases}$$

Clearly,

$$|z_2| \le U_1(z), |z_3| \le U_1(z), |z_2 + z_3| \le U_1(z),$$

and

$$|z_i| \le U_2(z), |z_i + z_j| \le U_2(z), |z_4 + z_5 + z_6| \le U_2(z), \quad i = 4, 5, 6, i \ne j,$$

for all $z = (z_1, z_2, z_3, z_4, z_5, z_6) \in R^6$.

Case 1: $U_1(z) > U_2(z)$, $\text{sgn}(z_1) = \text{sgn}(z_2) = \text{sgn}(z_3)$ and $|z_1| > |z_2| + |z_3|$. Then $\|z\| = |z_1|$, so that

$$\begin{aligned} D_+\|z\| =& D_+(|z_1|) \\ \le& \left(-\beta I-(1-\tau)\beta I-\frac{\beta SI+(1-\tau)\beta VI}{E}-\xi-\mu-\eta+\alpha\right)|z_1| \\ &+\frac{(1-\tau)\beta VI}{E}|z_4|+\frac{\beta SI}{E}|z_5|. \end{aligned}$$

Since $|z_4|, |z_5| \le U_2(z) < U_1(z) \le |z_1|$, we have

$$D_+\|z\| \le (\alpha - \xi - \mu - \eta)|z_1|.$$

Thus,

$$D_+\|z\| \le (\alpha - \xi - \mu - \eta)\|z\|. \tag{4.11}$$

Case 2: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_2| + |z_3| > |z_1|$. Then $\|z\| = |z_2| + |z_3|$, so that

$$\begin{aligned} D_+\|z\| =& D_+(|z_2| + |z_3|) \\ \le& \left(-\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu\right)|z_2| + \frac{\beta SI + (1-\tau)\beta VI}{E}|z_4 + z_5 + z_6| \\ &+ \left(-(1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu\right)|z_3| - \tau\beta I|z_1|. \end{aligned}$$

Since $|z_4 + z_5 + z_6| \le U_2(z) < U_1(z) \le |z_2| + |z_3|$ and $-\tau\beta I|z_1| < 0$, we have

$$D_+\|z\| \le (-\beta I - \mu)|z_2| + (-(1-\tau)\beta I - \mu)|z_3| \le -\mu(|z_2| + |z_3|).$$

Thus,

$$D_+\|z\| \le -\mu\|z\|. \tag{4.12}$$

Case 3: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2) = -\operatorname{sgn}(z_3)$ and $|z_2| > |z_1| + |z_3|$. Then $\|z\| = |z_2|$, so that

$$\begin{aligned} D_+\|z\| =& D_+(|z_2|) \\ \le& (1-\tau)\beta I|z_1| + \left(-\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \xi - \mu\right)|z_2| \\ &- \eta|z_3| + \frac{\beta SI}{E}|z_4 + z_6| + \frac{(1-\tau)\beta VI}{E}|z_4|. \end{aligned}$$

Since $|z_4|, |z_4 + z_6| \le U_2(z) < U_1(z) \le |z_2|$, $|z_1| < |z_2|$ and $-\eta|z_3| < 0$, we have

$$D_+\|z\| \le (-\tau\beta I - \xi - \mu)|z_2| \le (-\xi - \mu)|z_2|.$$

Thus,

$$D_+\|z\| \le (-\xi - \mu)\|z\|. \tag{4.13}$$

Case 4: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2) = -\operatorname{sgn}(z_3)$ and $|z_1| + |z_3| > |z_2|$. Then $\|z\| = |z_1| + |z_3|$, so that

$$\begin{aligned} D_+\|z\| =& D_+(|z_1| + |z_3|) \\ \le& \left(-(1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \xi - \mu + \alpha\right)|z_1| - \xi|z_2| \\ &+ \left(-(1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu - \eta\right)|z_3| + \frac{(1-\tau)\beta VI}{E}|z_4 + z_5 + z_6|. \end{aligned}$$

Since $|z_4+z_5+z_6| \le U_2(z) < U_1(z) \le |z_1|+|z_3|$ and $-\xi|z_2| < 0$, we have

$$\begin{aligned} D_+\|z\| \le & \left(-(1-\tau)\beta I - \frac{\beta SI}{E} - \xi - \mu + \alpha\right)|z_1| + \left(-(1-\tau)\beta I - \frac{\beta SI}{E} - \mu - \eta\right)|z_3| \\ \le & (\alpha-\mu)(|z_1|+|z_3|). \end{aligned}$$

Thus,

$$D_+\|z\| \le (\alpha-\mu)\|z\|. \tag{4.14}$$

Case 5: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = -\operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_1| > |z_2|, |z_1| > |z_3|$. Then $\|z\| = |z_1|$, so that

$$\begin{aligned} D_+\|z\| = & D_+(|z_1|) \\ \le & \left(-\beta I - (1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \xi - \mu - \eta + \alpha\right)|z_1| \\ & + \frac{(1-\tau)\beta VI}{E}|z_4| + \frac{\beta SI}{E}|z_5|. \end{aligned}$$

Since $|z_4|, |z_5| \le U_2(z) < U_1(z) \le |z_1|$, we have

$$D_+\|z\| \le (-\beta I - (1-\tau)\beta I - \xi - \mu - \eta + \alpha)|z_1| \le (\alpha - \xi - \mu - \eta)|z_1|.$$

Thus,

$$D_+\|z\| \le (\alpha - \xi - \mu - \eta)\|z\|. \tag{4.15}$$

Case 6: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = -\operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_2| > |z_1|, |z_2| > |z_3|$. Then $\|z\| = |z_2|$, so that

$$\begin{aligned} D_+\|z\| = & D_+(|z_2|) \\ \le & -(1-\tau)\beta I|z_1| + \left(-\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \xi - \mu\right)|z_2| \\ & - \eta|z_3| + \frac{\beta SI}{E}|z_4+z_6| + \frac{(1-\tau)\beta VI}{E}|z_4|. \end{aligned}$$

Since $|z_4|, |z_4+z_6| \le U_2(z) < U_1(z) \le |z_2|$, $-(1-\tau)\beta I|z_1| < 0$ and $-\eta|z_3| < 0$, we have

$$D_+\|z\| \le (-\beta I - \xi - \mu)|z_2| \le (-\xi - \mu)|z_2|.$$

Thus,

$$D_+\|z\| \le (-\xi - \mu)\|z\|. \tag{4.16}$$

Case 7: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = -\operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_3| > |z_1|, |z_3| > |z_2|$. Then $\|z\| = |z_3|$, so that

$$\begin{aligned} D_+\|z\| =& D_+(|z_3|) \\ \leq& -\beta I|z_1| - \xi|z_2| + \left(-(1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu - \eta\right)|z_3| \\ &+ \frac{\beta SI}{E}|z_5| + \frac{(1-\tau)\beta VI}{E}|z_5 + z_6|. \end{aligned}$$

Since $|z_5|, |z_5 + z_6| \leq U_2(z) < U_1(z) \leq |z_3|$, $-\beta I|z_1| < 0$, and $-\xi|z_2| < 0$, we have

$$D_+\|z\| \leq (-(1-\tau)\beta I - \mu - \eta)|z_3| \leq (-\mu - \eta)|z_3|.$$

Thus,

$$D_+\|z\| \leq (-\mu - \eta)\|z\|. \tag{4.17}$$

Case 8: $U_1(z) > U_2(z)$, $-\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_1| + |z_3| > |z_2| + |z_3|$. Then $\|z\| = |z_1| + |z_3|$, so that

$$\begin{aligned} D_+\|z\| =& D_+(|z_1| + |z_3|) \\ \leq& \left(-(1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \xi - \mu - \eta + \alpha\right)|z_1| + \xi|z_2| \\ &+ \left(-(1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu - \eta\right)|z_3| + \frac{(1-\tau)\beta VI}{E}|z_4 + z_5 + z_6|. \end{aligned}$$

Since $|z_4 + z_5 + z_6| \leq U_2(z) < U_1(z) \leq |z_1| + |z_3|$ and $|z_2| < |z_1|$, we have

$$\begin{aligned} D_+\|z\| \leq& \left(-(1-\tau)\beta I - \frac{\beta SI}{E} - \mu - \eta + \alpha\right)|z_1| + \left(-(1-\tau)\beta I - \frac{\beta SI}{E} - \mu - \eta\right)|z_3| \\ \leq& (\alpha - \mu - \eta)(|z_1| + |z_3|). \end{aligned}$$

Thus,

$$D_+\|z\| \leq (\alpha - \mu - \eta)\|z\|. \tag{4.18}$$

Case 9: $U_1(z) > U_2(z)$, $-\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_2| + |z_3| > |z_1| + |z_3|$. Then $\|z\| = |z_2| + |z_3|$, so that

$$\begin{aligned} D_+\|z\| =& D_+(|z_2| + |z_3|) \\ \leq& \tau\beta I|z_1| + \left(-(1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu\right)|z_3| \\ &+ \left(-\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu\right)|z_2| + \frac{\beta SI + (1-\tau)\beta VI}{E}|z_4 + z_5 + z_6|. \end{aligned}$$

Since $|z_4+z_5+z_6| \le U_2(z) < U_1(z) \le |z_2|+|z_3|$ and $|z_1| < |z_2|$, we have

$$D_+\|z\| \le (-(1-\tau)\beta I-\mu)|z_2|+(-(1-\tau)\beta I-\mu)|z_3| \le -\mu(|z_2|+|z_3|).$$

Thus,

$$D_+\|z\| \le -\mu\|z\|. \tag{4.19}$$

Case 10: $U_2(z) > U_1(z)$, $\operatorname{sgn}(z_4) = \operatorname{sgn}(z_5) = \operatorname{sgn}(z_6)$. Then $\|z\| = |z_4| + |z_5| + |z_6|$, so that

$$\begin{aligned} D_+\|z\| &= D_+(|z_4|+|z_5|+|z_6|) \\ &\le \left(-\mu-\frac{\alpha E}{I}\right)|z_4|+\left(-\mu-\frac{\alpha E}{I}\right)|z_5|+\left(-\alpha-\mu-\frac{\alpha E}{I}\right)|z_6|+\frac{\alpha E}{I}|z_2+z_3|. \end{aligned}$$

Since $|z_2+z_3| \le U_1(z) < U_2(z) \le |z_4|+|z_5|+|z_6|$, we have

$$D_+\|z\| \le -\mu|z_4|-\mu|z_5|+(-\alpha-\mu)|z_6| \le -\mu(|z_4|+|z_5|+|z_6|).$$

Thus,

$$D_+\|z\| \le -\mu\|z\|. \tag{4.20}$$

Case 11: $U_2(z) > U_1(z)$, $\operatorname{sgn}(z_4) = \operatorname{sgn}(z_5) = -\operatorname{sgn}(z_6)$ and $|z_4|+|z_5| > |z_4|+|z_6|$. Then $\|z\| = |z_4|+|z_5|$, so that

$$\begin{aligned} D_+\|z\| &= D_+(|z_4|+|z_5|) \\ &\le \left(-\beta I-\mu-\frac{\alpha E}{I}\right)|z_4|+\left(-(1-\tau)\beta I-\mu-\frac{\alpha E}{I}\right)|z_5|+\frac{\alpha E}{I}|z_2+z_3|. \end{aligned}$$

Since $|z_2+z_3| \le U_1(z) < U_2(z) \le |z_4|+|z_5|$, we have

$$D_+\|z\| \le (-\beta I-\mu)|z_4|+(-(1-\tau)\beta I-\mu)|z_5| \le -\mu(|z_4|+|z_5|).$$

Thus,

$$D_+\|z\| \le -\mu\|z\|. \tag{4.21}$$

Case 12: $U_2(z) > U_1(z)$, $\operatorname{sgn}(z_4) = \operatorname{sgn}(z_5) = -\operatorname{sgn}(z_6)$ and $|z_4|+|z_6| > |z_4|+|z_5|$. Then $\|z\| = |z_4|+|z_6|$, so that

$$\begin{aligned} D_+\|z\| &= D_+(|z_4|+|z_6|) \\ &\le \left(-2\beta I-\xi-\mu-\frac{\alpha E}{I}\right)|z_4|+(\eta-(1-\tau)\beta I)|z_5|+\left(-\alpha-\mu-\frac{\alpha E}{I}\right)|z_6|+\frac{\alpha E}{I}|z_2|. \end{aligned}$$

Since $|z_2| \le U_1(z) < U_2(z) \le |z_4| + |z_6|$, $|z_5| < |z_6|$ and $-(1-\tau)\beta I|z_5| < 0$, we have

$$D_+\|z\| \le (-2\beta I - \xi - \mu)|z_4| + (\eta - \alpha - \mu)|z_6| \le (\eta - \mu)(|z_4| + |z_6|).$$

Thus,

$$D_+\|z\| \le (\eta - \mu)\|z\|. \tag{4.22}$$

Case 13: $U_2(z) > U_1(z)$, $\operatorname{sgn}(z_4) = -\operatorname{sgn}(z_5) = \operatorname{sgn}(z_6)$ and $|z_5| > |z_4| + |z_6|$. Then $\|z\| = |z_5|$, so that

$$\begin{aligned} D_+\|z\| =& D_+(|z_5|) \\ \le& -\xi|z_4| + \left(-(1-\tau)\beta I - \frac{\alpha E}{I} - \mu - \eta\right)|z_5| + \frac{\alpha E}{I}|z_3|. \end{aligned}$$

Since $|z_3| \le U_1(z) < U_2(z) \le |z_5|$ and $-\xi|z_4| < 0$, we have

$$D_+\|z\| \le (-(1-\tau)\beta I - \mu - \eta)|z_5| \le (-\mu - \eta)|z_5|.$$

Thus,

$$D_+\|z\| \le (-\mu - \eta)\|z\|. \tag{4.23}$$

Case 14: $U_2(z) > U_1(z)$, $\operatorname{sgn}(z_4) = -\operatorname{sgn}(z_5) = \operatorname{sgn}(z_6)$ and $|z_4| + |z_6| > |z_5|$. Then $\|z\| = |z_4| + |z_6|$, so that

$$\begin{aligned} &D_+\|z\| = D_+(|z_4| + |z_6|) \\ \le& \left(-\xi - \mu - \frac{\alpha E}{I}\right)|z_4| + (-\eta - (1-\tau)\beta I)|z_5| + \left(-\alpha - \mu - \frac{\alpha E}{I}\right)|z_6| + \frac{\alpha E}{I}|z_2|. \end{aligned}$$

Since $|z_2| \le U_1(z) < U_2(z) \le |z_4| + |z_6|$ and $(-\eta - (1-\tau)\beta I)|z_5| < 0$, we have

$$D_+\|z\| \le (-\xi - \mu)|z_4| + (-\alpha - \mu)|z_6| \le -\mu(|z_4| + |z_6|).$$

Thus,

$$D_+\|z\| \le -\mu\|z\|. \tag{4.24}$$

Case 15: $U_2(z) > U_1(z)$, $-\operatorname{sgn}(z_4) = \operatorname{sgn}(z_5) = \operatorname{sgn}(z_6)$ and $|z_4| + |z_6| > |z_5| + |z_6|$. Then $\|z\| = |z_4| + |z_6|$, so that

$$\begin{aligned} &D_+\|z\| = D_+(|z_4| + |z_6|) \\ \le& \left(-2\beta I - \xi - \mu - \frac{\alpha E}{I}\right)|z_4| + ((1-\tau)\beta I - \eta)|z_5| + \left(-\alpha - \mu - \frac{\alpha E}{I}\right)|z_6| + \frac{\alpha E}{I}|z_2|. \end{aligned}$$

Since $|z_2| \le U_1(z) < U_2(z) \le |z_4| + |z_6|$, $|z_5| < |z_4|$ and $-\eta|z_5| < 0$, we have

$$D_+\|z\| \le (-(1+\tau)\beta I - \xi - \mu)|z_4| + (-\alpha - \mu)|z_6| \le -\mu(|z_4| + |z_6|).$$

Thus,

$$D_+\|z\| \le -\mu\|z\|. \tag{4.25}$$

Case 16: $U_2(z) > U_1(z)$, $-\operatorname{sgn}(z_4) = \operatorname{sgn}(z_5) = \operatorname{sgn}(z_6)$ and $|z_5| + |z_6| > |z_4| + |z_6|$. Then $\|z\| = |z_5| + |z_6|$, so that

$$\begin{aligned} &D_+\|z\| = D_+(|z_5| + |z_6|) \\ \le &-(\xi + \beta I)|z_4| + \left(-\mu - \eta - \frac{\alpha E}{I}\right)|z_5| + \left(-\alpha - \mu - \frac{\alpha E}{I}\right)|z_6| + \frac{\alpha E}{I}|z_3|. \end{aligned}$$

Since $|z_3| \le U_1(z) < U_2(z) \le |z_5| + |z_6|$ and $-(\xi + \beta I)|z_4| < 0$, we have

$$D_+\|z\| \le (-\mu - \eta)|z_5| + (-\alpha - \mu)|z_6| \le -\mu(|z_5| + |z_6|).$$

Thus,

$$D_+\|z\| \le -\mu\|z\|. \tag{4.26}$$

Combing the results of (4.11) – (4.26), we obtain

$$D_+\|z\| \le \max\{-\mu + \alpha,\ -\mu + \eta\}\|z\|.$$

By (4.9),

$$\bar{\mu}(A) \le \max\{-\mu + \alpha,\ -\mu + \eta\}.$$

Thus, $\bar{\mu}(A) < 0$ by (4.10).

In Section 3, it has shown that when $R_0 > 1$, the model $(1.2a) - (1.2d)$ has a unique endemic equilibrium P^* and the disease-free equilibrium P_0 is unstable. The instability of P_0, together with $P_0 \in \partial D$, implies the uniform persistence of the state variables. This can be seen by using the same arguments from Theorem 4.3 in [3] and Proposition 3.3 in [7]. The uniform persistence, together with boundedness of D, is equivalent to the existence of a compact absorbing set in D. By Lemma 4.1, the endemic equilibrium is globally asymptotically stable in the interior of D. The proof is complete. □

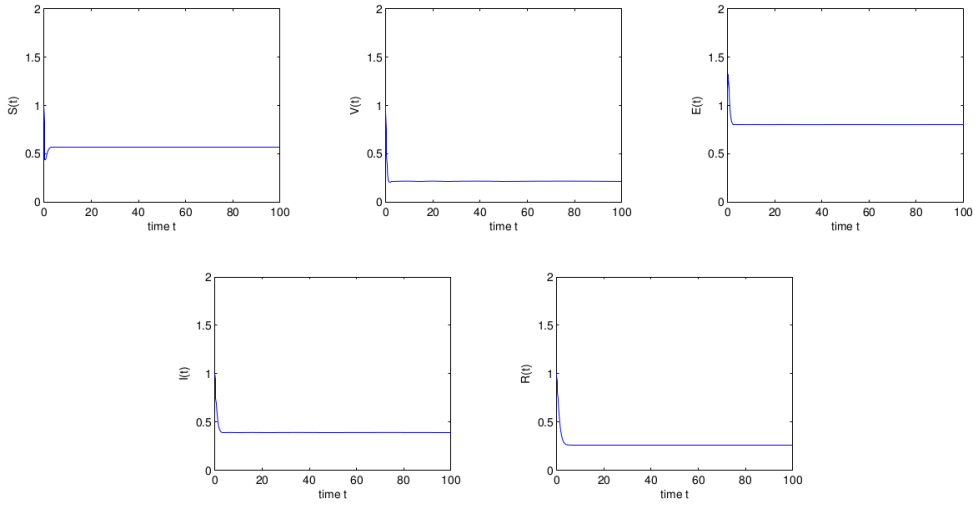

Fig. 1: The temporal solution found by numerical integration of the model $(1.2a) - (1.2e)$ with $\mu = 1.4671, \beta = 3.4180, \Pi = 3.4748, \xi = 1.0062, d = 0.5059, \delta = 0.9750, \alpha = 1.4399, \tau = 0.9308, \eta = 1.0664$ and initial conditions $S(0) = 1, V(0) = 1, E(0) = 1, I(0) = 1, R(0) = 1$.

5. Numerical Simulations

In this section, we show the feasibility of the conditions of Theorem 4.2.

Example. In $(1.2a) - (1.2e)$, let $\mu = 1.4671, \beta = 3.4180, \Pi = 3.4748, \xi = 1.0062, d = 0.5059, \delta = 0.9750, \alpha = 1.4399, \tau = 0.9308, \eta = 1.0664$. The model $(1.2a) - (1.2e)$ with above coefficients has an endemic equilibrium

$$P^*(0.5665, 0.2171, 0.7999, 0.3907, 0.2596).$$

A direct calculation show that $R_0 = 1.0003 > 1, \eta - \mu = -0.4007 < 0, \alpha - \mu = -0.0272 < 0$. By Theorem 4.2, we see that the endemic equilibrium P^* is globally asymptotically stable. The numerical simulation illustrates our result (see Figure 1).

6. Conclusion

In this paper, the dynamics of a SEIR epidemic model with a waning preventive vaccines is investigated. We have shown that the dynamics of the system are

almost completely determined by the basic reproductive number R_0. If $R_0 < 1$, the disease free equilibrium is globally asymptotically stable while the endemic equilibrium is not feasible. In this case, the disease dies out. If $R_0 > 1$, the endemic equilibrium is globally asymptotically stable provided $u > \max\{\alpha, \eta\}$. To control the disease, a strategy should reduce the basic reproduction number to below unity. From the expression of R_0, we see that R_0 is an increasing function of η. Thus, it is necessary and important for public health management to control an epidemic by increasing the duration of the loss of immunity induced by vaccination $(1/\eta)$, which reduces the the basic reproduction number.

Appendix

The expression of H in Δ_3 is defined in the following.

$$
\begin{aligned}
H &= \alpha\tau\beta^2 S^* I^* \{(\alpha+\mu)(\delta+d+\mu)(\xi+\eta) + \alpha(1-\tau)\beta^2 I^*(S^*+V^*) \\
&+ \mu[\alpha(\mu+\delta+d) + (\xi+\eta+(2-\tau)\beta I^*)(\alpha+\mu+\delta+d)] \\
&+ (\xi+\mu+\eta+(2-\tau)\beta I^*)[(\alpha+\mu+\delta+d)(\xi+\mu+\eta+(2-\tau)\beta I^*+\delta+d) \\
&+ (\alpha+\mu)^2] + (\xi+2\mu+\eta+(2-\tau)\beta I^*)[(1-\tau)\beta I^*(\beta I^*+\mu+\xi+\eta) \\
&+ \tau\eta\beta I^* + \mu(\beta I^* + \eta + \xi + \delta + d)]\} \\
&+ \alpha(1-\tau)\beta^2 I^*(S^*+V^*)\{(\alpha+\mu)(\delta+d+\mu)(1-\tau)\beta I^* \\
&+ (\alpha+2\mu+\delta+d)[(\beta I^* + \mu + \xi + \eta)(\mu + (1-\tau)\beta I^*) + \tau\eta\beta I^*]\} \\
&+ (1-\tau)\beta I^*(\alpha+\mu)(\delta+d+\mu)\{(\alpha+\mu)[\mu(\xi+2\mu+\eta+\delta+d)+\alpha(\mu+(1-\tau)\beta I^*)] \\
&+ (\mu+\delta+d)[2\mu(\xi+2u+\eta+(2-\tau)\beta I^*)+(\mu+(1-\tau)\beta I^*)(\alpha+\delta+d)] \\
&+ (\xi+2\mu+\eta+(2-\tau)\beta I^*)[\mu(\beta I^* + \alpha + 2\mu + \eta + \xi) \\
&+ (\xi+2\mu+\eta+(2-\tau)\beta I^*)(\alpha+2\mu+\delta+d)]\} \\
&+ (1-\tau)\beta I^*(\mu+\delta+d)^3(\beta I^* + \mu + \xi + \eta)(\xi+2\mu+\eta+(2-\tau)\beta I^*) \\
&+ (1-\tau)\beta^2 I^{*2}[\mu(\xi+2\mu+\eta+(2-\tau)\beta I^*)^2(\mu+2(\delta+d)) \\
&+ (\delta+d)^2(\xi+3\mu+\eta+(2-\tau)\beta I^*+\alpha)(\xi+\eta+(2-\tau)\beta I^*)] \\
&+ [(1-\tau)\beta I^*(\beta I^*+\mu+\xi+\eta)+\tau\eta\beta I^*]\{(\alpha+2\mu+\delta+d)[(\alpha+\mu)(\delta+d+\mu)(\xi+\eta) \\
&+ (\xi+2\mu+\eta+(2-\tau)\beta I^*)(\beta I^* + \mu + \eta + \xi)(2\mu + (1-\tau)\beta I^*) \\
&+ (\xi+2\mu+\eta+(2-\tau)\beta I^*)\tau\eta\beta I^*]
\end{aligned}
$$

$$
\begin{aligned}
&+(\xi+2\mu+\eta+(2-\tau)\beta I^*)(\alpha+\mu)[\alpha(\alpha+2\mu)+\beta I^*(\delta+d+\mu)\\
&+(\alpha+2\mu+\delta+d)(\xi+2\mu+\eta+(2-\tau)\beta I^*)+\mu^2]\}\\
&+\mu(\xi+2\mu+\eta+(2-\tau)\beta I^*)(\alpha+2\mu+\delta+d)(\beta I^*+\mu+\eta+\xi)[\mu(\beta I^*+\mu+\eta+\xi)\\
&+(\xi+2\mu+\eta+(2-\tau)\beta I^*)(\alpha+2\mu+\delta+d)]\\
&+(\xi+2\mu+\eta+(2-\tau)\beta I^*)^2(\mu+\delta+d)^2[\beta I^*\eta+(1-\tau)(\mu+\xi)\beta I^*]\\
&+\mu(\alpha+\mu)(\delta+d+\mu)[(1-\tau)(\alpha+\delta+d)\eta\beta I^*+(\alpha+\delta+d)\beta I^*(\xi+2\mu)\\
&+2\mu(\beta I^*+\mu+\eta+\xi)(\xi+2\mu+\eta)+(\alpha+\delta+d)(\mu+\eta+\xi)(\xi+2\mu+\eta)]\\
&+\mu(\alpha+\mu)^2[(1-\tau)\beta I^*(\delta+d)(\mu+\eta+\xi)+(1-\tau)\beta I^*(\alpha+2\mu))\\
&+(\beta I^*+\mu+\eta+\xi)((\xi+2\mu+\eta+\beta I^*)(\alpha+2\mu+\delta+d)]\\
&+\mu(\mu+\delta+d)(\alpha+2\mu+\delta+d)^2[\eta(\mu+\eta+\xi)\\
&+(\xi+2\mu+(2-\tau)\beta I^*)(\beta I+\mu+\eta+\xi)+]\\
&+(1-\tau)\beta I^*\mu\eta(\mu+\delta+d)[(\alpha+\mu)(\alpha+2\mu+\delta+d)+(\mu+\delta+d)^2]\\
&+\tau\eta\beta I^*(\delta+d+\mu)^3(\xi+\eta+(2-\tau)\beta I^*)
\end{aligned}
$$

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