

A FOUR-PARAMETER INTEREST RATES MODEL INCORPORATING AVERAGE OF PAST SHORT RATES

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Abstract: We consider a four-parameter short rate model for the evolution of interest rates. In this model we take *normal level* of the short rate as an exponentially weighted average of the past short rates as suggested by Malkiel [14]. We first derive the differential equation for the price of a zero-coupon bond under this model. A complete explicit solution of the partial differential equation is obtained under a suitable simplifying assumption to obtain an explicit model for the evolution of interest rates. The relevant yield curve is also obtained. We then consider a most desirable feature of an interest rates model, incorporating current term structure of interest rates into the model. Again, a complete explicit solution is obtained together with the yield curve and its asymptotic behaviour. A well-known previously given model is included and discussed in the context of our present interest rates model.

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1. Introduction

For the evolution of interest rates, Vasicek [15] was the first to propose a three

constant-parameter model for the short rate:

$$dr = a(b - r)dt + \sigma dX, \quad (1.1)$$

where σ is volatility of interest rate and dX is a Wiener process drawn from a normal distribution with mean zero and variance dt . While Vasicek model incorporates the property of mean reversion for the predicted interest rates, it suffers the possibility of giving undesirable negative rates. Cox, Ingersoll and Ross [5] proposed a short rate model by the stochastic differential equation:

$$dr = a(b - r)dt + \sigma\sqrt{r}dX, \quad (1.2)$$

which avoids negative or zero rates provided $\sigma^2 \leq 2ab$. Several other short rate models have been proposed and studied. Hull and White [10] extended Vasicek model to include time-dependent parameters. A special case of Vasicek model was first developed by Dothan [6].

It has been desired for an interest rates model to be able to incorporate into it current term structure of interest rates. For the purpose, Hull and White [10] consider a short rate model with time-dependent parameters to incorporate the current term structure of interest rates. Hull and White [11] proposed another variation of the short rate model:

$$dr = [\theta(t) - ar]dt + \sigma dX, \quad (1.3)$$

where a is a constant, to fit the current term structure of interest rates in the model. For more discussion of interest rate models and their solutions, see Black, Derman and Toy [1], Chawla [2], Cox, Ingersoll and Ross [5], Duffie and Kan [7], Heath, Jarrow and Morton [8], Hughston [9], Klugman [12].

A four-parameter short rate model for the evaluation of interest rates was studied by Klugman and Wilmott [13]. As in Wilmott et al. [16], consider a four-parameter random walk model for the short rate of interest described by the stochastic differential equation:

$$dr = u(r, t)dt + w(r, t)dX, \quad (1.4)$$

where

$$w(r, t) = \sqrt{\alpha r - \beta}, \quad u(r, t) = (\eta - \gamma r) + \lambda w(r, t).$$

Complete explicit solutions of the bond pricing partial differential equation, giving the price of a zero-coupon bond under the short rate model (1.4), have been given by Chawla [2]. A generalization of an interest rates model in which the *normal level* of the short rate is an exponentially weighted average of past

short rates was suggested by Malkiel [14]. Recently, Chawla [3] gave a generalization of the Vasicek model with the suggestion of Malkiel incorporated, called a generalized Vasicek-Malkiel model for the pricing of zero-coupon bonds.

In the present paper we consider a modification of the four-parameter short rate model (1.4), introducing a suitable time-dependent parameter to incorporate Malkiel's idea of mean level short rate as being the average of past short rates. We derive the relevant partial differential equation giving the price of a zero-coupon bond under this modified short rate model. A complete explicit solution of the partial differential equation is obtained under a suitable simplifying assumption to obtain an explicit model for the evolution of interest rates. The relevant yield curve is also obtained. We then consider a most desirable feature of an interest rates model, incorporating current term structure of interest rates into the model. Again, a complete explicit solution is obtained together with the yield curve and its asymptotic behaviour. A well-known previously given model is included and discussed in the context of our present interest rates model.

2. A Four-Parameter Interest Rates Model

2.1. Incorporating Average of Past Short Rates

In order to incorporate Malkiel's idea of mean level short rate being the average of past short rates, we consider a suitable modification of the short rate model (1.4) as follows. We consider a model for the short rate described by the stochastic differential equation:

$$dr = u(r, t) dt + w(r, t) dX, \quad (2.1)$$

where we take

$$u(r, t) = \eta + \gamma \{\theta(t) - r\}, \quad w(r, t) = \sqrt{\alpha r - \beta},$$

with α , β , γ and η constants and where $\theta(t)$ is an exponentially weighted average of the past short rates:

$$\theta(t) = \mu \int_{-\infty}^t e^{-\mu(t-s)} r(s) ds. \quad (2.2)$$

Here, μ is a parameter, $\mu > 0$, and the differential equation form for (2.2) is

$$d\theta = \mu(r - \theta) dt = v(r, \theta) dt, \quad \text{say.} \quad (2.3)$$

Let $V(t, T; r, \theta)$, or simply $V(t, T)$, denote the price of a zero-coupon bond at time t with maturity T , $t < T$, and final value $V(T, T) = Z$. A measure of future values of interest rates is the *yield curve*. With

$$Y(t, T) = -\frac{1}{\tau} \ln \left(\frac{V(t, T)}{V(T, T)} \right), \quad (2.4)$$

where $\tau = T - t$ is time to maturity, plot of Y against τ is called the *term structure of interest rates*. From (2.4) it is clear that, for fixed t , *interest rate implied by the yield curve* is:

$$r(T) = r(t, T) = \frac{d}{dT} [(T - t) Y(t, T)] = -\frac{1}{V(t, T)} \frac{\partial}{\partial T} V(t, T). \quad (2.5)$$

Whenever a function $F(t, T)$ is a function of τ only we shall write $F(t, T) = F(\tau)$.

Following arguments given in Wilmott et al. [16] extended to the present case and using no-arbitrage arguments, for the value of a zero-coupon bond $V(t, T)$,

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + v \frac{\partial V}{\partial \theta} - rV}{\partial V / \partial r},$$

must be independent of maturity T , and we set

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + v \frac{\partial V}{\partial \theta} - rV}{\partial V / \partial r} = -u(r, t),$$

and so

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + u \frac{\partial V}{\partial r} + v \frac{\partial V}{\partial \theta} - rV = 0.$$

Thus, the bond pricing equation providing the value of a zero-coupon bond at time t with maturity T , for our four-parameter short rate model (2.1), is

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}(\alpha r - \beta) \frac{\partial^2 V}{\partial r^2} + [\eta + \gamma(\theta - r)] \frac{\partial V}{\partial r} \\ + \mu(r - \theta) \frac{\partial V}{\partial \theta} - rV = 0. \end{aligned} \quad (2.6)$$

We now seek a solution of the bond pricing equation (2.6) in the form:

$$V(t, T) = Z e^{A(\tau) + rB(\tau) + \theta C(\tau)}, \quad (2.7)$$

for suitable $A(\tau)$, $B(\tau)$ and $C(\tau)$ as functions of τ . Substituting (2.7) in (2.6) we have, in terms of τ ,

$$\begin{aligned} & \left(-\frac{\partial A}{\partial \tau} - \frac{1}{2}\beta B^2 + \eta B \right) + r \left(-\frac{\partial B}{\partial \tau} + \frac{1}{2}\alpha B^2 - \gamma B + \mu C - 1 \right) \\ & + \theta \left(-\frac{\partial C}{\partial \tau} + \gamma B - \mu C \right) = 0. \end{aligned} \quad (2.8)$$

Since the coefficients of 1, r and θ must separately be zero, we obtain the following simultaneous set of three ordinary differential equations for the determination of $A(\tau)$, $B(\tau)$ and $C(\tau)$:

$$\frac{\partial A}{\partial \tau} = \eta B - \frac{1}{2}\beta B^2, \quad (2.9)$$

$$\frac{\partial B}{\partial \tau} = \frac{1}{2}\alpha B^2 - \gamma B + \mu C - 1, \quad (2.10)$$

$$\frac{\partial C}{\partial \tau} = \gamma B - \mu C. \quad (2.11)$$

Note that, for $\tau = 0$, $A(0) = B(0) = C(0) = 0$.

Now, in order to be able to find a solution of the above differential equation system, we make a simplifying assumption that

$$C(\tau) = m(B(\tau))^2, \quad (2.12)$$

for a suitable constant m . Substituting from (2.12) in (2.11) we have

$$\frac{\partial B}{\partial \tau} + \frac{\mu}{2}B = \frac{\gamma}{2m}, \quad (2.13)$$

and substituting from (2.12) in (2.10) and setting

$$m = -\frac{\alpha}{2\mu}, \quad (2.14)$$

we get

$$\frac{\partial B}{\partial \tau} + \gamma B = -1. \quad (2.15)$$

For a consistent solution $B(\tau)$ of (2.13) and (2.15) we must have

$$\gamma = \frac{\mu}{2}, \quad \frac{\gamma}{2m} = -1. \quad (2.16)$$

The solution of (2.14) and (2.16) in terms of α is

$$\mu = \sqrt{2\alpha}, \quad \gamma = \sqrt{\frac{\alpha}{2}}, \quad m = -\frac{1}{2}\sqrt{\frac{\alpha}{2}}.$$

For finding $B(\tau)$, equation (2.15) is a first-order linear ordinary differential equation whose solution is given by

$$B(\tau) = -\frac{1}{\gamma} (1 - e^{-\gamma\tau}). \quad (2.17)$$

To find $A(\tau)$, from (2.9) we have

$$A(\tau) = \eta I_1(\tau) - \frac{1}{2}\beta I_2(\tau), \quad (2.18)$$

where we have set

$$I_1(\tau) = \int_0^\tau B(s) ds, \quad I_2(\tau) = \int_0^\tau (B(s))^2 ds.$$

Since

$$\int B(s) ds = -\frac{1}{\gamma} \int (1 - e^{-\gamma s}) ds = -\frac{1}{\gamma} \left(s + \frac{e^{-\gamma s}}{\gamma} \right),$$

therefore

$$I_1(\tau) = -\frac{1}{\gamma} (\tau + B(\tau)).$$

Again, since

$$\begin{aligned} \int (B(s))^2 ds &= \frac{1}{\gamma^2} \int (1 - 2e^{-\gamma s} + e^{-2\gamma s}) ds \\ &= \frac{1}{2\gamma^3} (2\gamma s + 4e^{-\gamma s} - e^{-2\gamma s}), \end{aligned}$$

therefore

$$I_2(\tau) = \frac{1}{\gamma^2} \left(\tau + B(\tau) - \frac{\gamma}{2} (B(\tau))^2 \right).$$

With these results, from (2.18) we obtain

$$A(\tau) = -\frac{1}{\gamma} \left(\eta + \frac{\beta}{2\gamma} \right) (\tau + B(\tau)) + \frac{\beta}{4\gamma} (B(\tau))^2. \quad (2.19)$$

Thus, the price of a zero-coupon bond by our present four-parameter short rate model (2.1) is given by (2.7) where $A(\tau)$ is given by (2.19), $B(\tau)$ is given by (2.17) and $C(\tau)$ is given by (2.12).

The yield curve for the present bond pricing model, from (2.4) and (2.7), is

$$Y(t, T) = -\frac{1}{\tau} [A(\tau) + rB(\tau) + \theta C(\tau)].$$

Substituting for $A(\tau)$, $B(\tau)$ and $C(\tau)$, we obtain the yield curve given by

$$Y(t, T) = \frac{1}{\tau} \left[\frac{1}{\gamma} \left(\eta + \frac{\beta}{2\gamma} \right) (\tau + B(\tau)) + rB(\tau) + \left(\theta m - \frac{\beta}{4\gamma} \right) (B(\tau))^2 \right]. \quad (2.20)$$

Since, for $\tau \rightarrow \infty$, $B(\tau) \sim -\frac{1}{\gamma}$, $\frac{B(\tau)}{\tau} \sim 0$, the asymptotic behaviour of the yield curve (2.20), for $\tau \rightarrow \infty$, is

$$Y(t, T) \sim \frac{1}{\gamma} \left(\eta + \frac{\beta}{2\gamma} \right). \quad (2.21)$$

We next consider the following particular case.

2.2. The Case $\beta = 0$, $\alpha = \sigma^2$

This is a well-known special case of the four-parameter random walk for the short rate (1.4) considered by Cox, Ingersoll and Ross [5]. For this case, in our case with the random walk (2.1) for the short rate,

$$\gamma = \frac{1}{\sqrt{2}}\sigma, \quad m = -\frac{1}{2\sqrt{2}}\sigma,$$

and, while $B(\tau)$ and $C(\tau)$ remain the same as above, $A(\tau)$ simplifies to the following:

$$A(\tau) = -\frac{\eta}{\gamma} (\tau + B(\tau)).$$

The yield curve is now given by

$$Y(t, T) = \frac{1}{\tau} \left[\frac{\eta}{\gamma} (\tau + B(\tau)) + rB(\tau) + \theta m (B(\tau))^2 \right],$$

with the asymptotic behaviour,

$$Y(t, T) \sim \frac{\eta}{\gamma}.$$

3. Incorporating Current Yield Curve Into Present Interest Rates Model

A most important feature of an interest rates model is that it should be consistent with current term structure of interest rates in the market. For the purpose of incorporating current yield curve into our present four-parameter interest rates model, we now regard η as a function of time, $\eta = \eta(t)$, and use it for the purpose.

Treating $\eta = \eta(t)$, from (2.9) we have

$$\frac{\partial A}{\partial t} = -\eta(t) B(T-t) + \frac{1}{2}\beta(B(T-t))^2.$$

Since $A(0) = 0$, integrating from T to t ,

$$A(t, T) = \int_t^T \eta(s) B(T-s) ds - \frac{1}{2}\beta I(t, T), \quad (3.1)$$

where have set

$$I(t, T) = \int_t^T (B(T-s))^2 ds.$$

To fit the yield curve at time $t = 0$, with the notation that for any function $\phi(t, T)$: $\phi_0(T) = \phi(0, T)$, the initial yield curve is

$$Y_0(T) = -\frac{1}{T} [A_0(T) + r_0 B_0(T) + \theta_0 C_0(T)], \quad (3.2)$$

where $r_0 = r(0)$, $\theta_0 = \theta(0)$. Substituting for $B(T-s)$ from (2.17) and for $A_0(T)$ from (3.2) in (3.1), we get

$$\int_0^T \eta(s) (1 - e^{-\gamma(T-s)}) ds = \gamma F_0(T), \quad (3.3)$$

where we have set

$$F_0(T) = Y_0(T) T + r_0 B_0(T) + \theta_0 C_0(T) - \frac{1}{2}\beta I_0(T). \quad (3.4)$$

Equation (3.3) is an integral equation for the determination of $\eta(s)$. To solve it, we differentiate (3.3) with respect to T and using the rule:

$$\frac{d}{dx} \int_a^x f(s, x) ds = \int_a^x \frac{\partial}{\partial x} f(s, x) ds + f(x, x), \quad (3.5)$$

we get

$$\int_0^T \eta(s) e^{-\gamma(T-s)} ds = F'_0(T). \quad (3.6)$$

Adding (3.3) and (3.6) we have

$$\int_0^T \eta(s) ds = \gamma F_0(T) + F'_0(T). \quad (3.7)$$

Differentiating (3.7) with respect to T and again using the rule (3.5), we finally get

$$\eta(T) = \gamma F'_0(T) + F''_0(T). \quad (3.8)$$

This determines $\eta(T)$.

To find corresponding $A(t, T)$, we go back to (3.1), substitute for $\eta(s)$ from (3.8) to have

$$A(t, T) = J(t, T) - \frac{\beta}{2} I(t, T), \quad (3.9)$$

where we have set

$$J(t, T) = \int_t^T \eta(s) B(T-s) ds.$$

To calculate $J(t, T)$, substituting for $\eta(s)$ from (3.8) we have

$$J(t, T) = \gamma \int_t^T F'_0(s) B(T-s) ds + \int_t^T F''_0(s) B(T-s) ds.$$

Since

$$\frac{\partial}{\partial s} B(T-s) = e^{-\gamma(T-s)},$$

integrating the second integral by parts we obtain

$$\begin{aligned} J(t, T) &= \int_t^T F'_0(s) \left[\gamma B(T-s) - e^{-\gamma(T-s)} \right] ds - F'_0(t) B(\tau) \\ &= - \int_t^T F'_0(s) ds - F'_0(t) B(\tau). \end{aligned}$$

Thus,

$$J(t, T) = -[F_0(T) - F_0(t)] - F'_0(t) B(\tau). \quad (3.10)$$

With (3.10), from (3.9) we get

$$A(t, T) = -[F_0(T) - F_0(t)] - F'_0(t) B(\tau) - \frac{\beta}{2} I(t, T). \quad (3.11)$$

We now calculate

$$I(t, T) = \int_t^T (B(T-s))^2 ds = \frac{1}{\gamma^2} \int_t^T \left(1 - 2e^{-\gamma(T-s)} + e^{-2\gamma(T-s)}\right) ds.$$

Performing the integration and rearranging the terms we obtain

$$I(t, T) = \frac{1}{\gamma^2} \left[\tau + B(\tau) - \frac{\gamma}{2} (B(\tau))^2 \right]. \quad (3.12)$$

We define *forward yield* at time $t = 0$ by

$$f(t, T) = \frac{1}{\tau} [Y_0(T)T - Y_0(t)t].$$

From (3.4) we calculate

$$F_0(T) - F_0(t) = f(t, T)\tau + r_0 B^*(t, T) + \theta_0 C^*(t, T) - \frac{1}{2} \beta I^*(t, T), \quad (3.13)$$

where we have set

$$\begin{aligned} B^*(t, T) &= B_0(T) - B_0(t), \quad C^*(t, T) = C_0(T) - C_0(t), \\ I^*(t, T) &= I_0(T) - I_0(t). \end{aligned}$$

With (2.17) it is easy to see that

$$B^*(t, T) = e^{-\gamma t} B(\tau).$$

With (2.12) and (2.17) we calculate

$$\begin{aligned} C^*(t, T) &= m \left[(B_0(T))^2 - (B_0(t))^2 \right] \\ &= \frac{m}{\gamma} e^{-\gamma t} B(\tau) \left[-2(1 - e^{-\gamma t}) + \gamma e^{-\gamma t} B(\tau) \right]. \end{aligned}$$

Again, with (3.12) we calculate

$$I^*(t, T) = \frac{1}{\gamma^2} \left[\tau + e^{-\gamma t} (2 - e^{-\gamma t}) B(\tau) - \frac{\gamma}{2} (e^{-\gamma t} B(\tau))^2 \right].$$

With the above results, from (3.13) we obtain

$$\begin{aligned} F_0(T) - F_0(t) &= \left[f(t, T) - \frac{\beta}{2\gamma^2} \right] \tau + \left[r_0 - \frac{2m\theta_0}{\gamma} (1 - e^{-\gamma t}) \right. \\ &\quad \left. - \frac{\beta}{2\gamma^2} (2 - e^{-\gamma t}) \right] e^{-\gamma t} B(\tau) + \left(m\theta_0 + \frac{\beta}{4\gamma} \right) (e^{-\gamma t} B(\tau))^2. \end{aligned} \quad (3.14)$$

To calculate $F'_0(t)$, from (3.4) we have

$$F'_0(T) = \frac{d}{dT} [Y_0(T) T] + r_0 B'_0(T) + \theta_0 C'_0(T) - \frac{1}{2} \beta I'_0(T).$$

But from (2.5),

$$\frac{d}{dT} [Y_0(T) T] = r(0, T),$$

from (2.17),

$$B'_0(T) = -e^{-\gamma T},$$

from (2.12) and (2.17),

$$\begin{aligned} C'_0(T) &= \frac{d}{dT} [m(B_0(T))^2] = 2mB_0(T) B'_0(T) \\ &= -2me^{-\gamma T} B_0(T), \end{aligned}$$

and from (3.12),

$$\begin{aligned} I'_0(T) &= \frac{1}{\gamma^2} \frac{d}{dT} \left[T + B_0(T) - \frac{\gamma}{2} (B_0(T))^2 \right] \\ &= (B_0(T))^2. \end{aligned}$$

Therefore,

$$F'_0(t) = r(0, t) - r_0 e^{-\gamma t} - 2m\theta_0 e^{-\gamma t} B_0(t) - \frac{\beta}{2} (B_0(t))^2. \quad (3.15)$$

With (3.12), (3.14) and (3.15), from (3.11) we finally have

$$\begin{aligned} A(t, T) &= -f(t, T) \tau - r(0, t) B(\tau) \\ &\quad - \left[m\theta_0 e^{-2\gamma t} - \frac{\beta}{4\gamma} (1 - e^{-2\gamma t}) \right] (B(\tau))^2. \end{aligned} \quad (3.16)$$

With this $A(t, T)$, the value of a zero-coupon bond by our present four-parameter interest rate model (2.1), with the initial term structure of interest rates incorporated at time $t = 0$, is given by (2.7) where the values of $B(t, T)$ and $C(t, T)$ are as given before.

Now, from (2.4) the yield curve for the present case is

$$Y(t, T) = -\frac{1}{\tau} [A(\tau) + rB(\tau) + \theta C(\tau)].$$

Substituting for $A(t, T)$ from (3.16) and for $B(t, T)$ and $C(t, T)$ as given before, the yield curve for the present case is

$$Y(t, T) = f(t, T) + [r(0, t) - r] \frac{B(\tau)}{\tau} - \left[\frac{\beta}{4\gamma} (1 - e^{-2\gamma t}) + m(\theta - \theta_0 e^{-2\gamma t}) \right] \frac{(B(\tau))^2}{\tau}. \quad (3.17)$$

For asymptotic behaviour of the yield curve for $\tau \rightarrow \infty$, since

$$B(\tau) \sim -\frac{1}{\gamma}, \quad \frac{B(\tau)}{\tau} \sim 0,$$

therefore

$$Y(t, T) \sim f(t, T). \quad (3.18)$$

Also, for $t = 0$, from (3.16) we have

$$\begin{aligned} A(0, T) &= -f(0, T)T - r_0 B_0(T) - m\theta_0 (B_0(T))^2 \\ &= -Y_0(T)T - r_0 B_0(T) - \theta_0 C_0(T), \end{aligned}$$

giving

$$Y_0(T) = -\frac{1}{T} [A_0(T) + r_0 B_0(T) + \theta_0 C_0(T)], \quad (3.19)$$

as it should so that our present interest rates model produces and is consistent with the initial term structure of interest rates observed in the market.

We again consider the following particular case.

4. The Case $\beta = 0, \alpha = \sigma^2$

As noted above, this is the well-known case of the four-parameter random walk (1.4) for the short rate considered by Cox, Ingersoll and Ross [5]. For this case, in our case with the random walk (2.1) for the short rate, while $B(\tau)$ and $C(\tau)$ remain the same as given before, $A(t, T)$ simplifies to

$$A(t, T) = -f(t, T)\tau - r(0, t)B(\tau) - m\theta_0 e^{-2\gamma t} (B(\tau))^2. \quad (3.20)$$

The corresponding yield curve from (3.17) simplifies to

$$Y(t, T) = f(t, T) + [r(0, t) - r] \frac{B(\tau)}{\tau} - m(\theta - \theta_0 e^{-2\gamma t}) \frac{(B(\tau))^2}{\tau}, \quad (3.21)$$

while the asymptotic behaviour of the yield curve remains the same as in (3.18).

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