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SECOND HANKEL DETERMINANT FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract: In the present investigation, we introduce some new subclasses of analytic-univalent functions and determine the sharp upper bounds of the second Hankel determinant for the functions belonging to such classes.

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Key Words: analytic functions, starlike functions, convex functions, close to convex functions, starlike functions with respect to symmetric points, close-to-convex functions with respect to symmetric points, Hankel determinant

1. Introduction, Definitions and Preliminaries

We let $\mathcal{A}$ to denote the class of functions analytic in $\mathbb{U}$ and having the power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the unit disc $\mathbb{U} = \{z : |z| < 1\}$. Let $\mathcal{S}$ be the class of functions $f(z) \in A$ and univalent in $\mathbb{U}$.

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The q^{th} Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [27, 28] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has been considered by several authors in the literature, see [24]. For example, Noor [25] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions in $\mathbb{U}$ with bounded boundary. Later, Ehrenborg [7] considered the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some its properties were discussed by thoroughly by Layman in [15].

Also, the Hankel determinant was studied by various authors including Hayman [12] and Pommerenke [29]. Easily, one can observe that the Fekete-Szegö functional is $H_2(1)$. Then Fekete-Szegö further generalized the estimate $\left|a_3 - \mu a_2^2\right|$, where μ is real and $f \in \mathbb{U}$. Ali [2] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional. For the discussion in this paper, the Hankel determinant for the case $q = 2$ and $n = 2$ are being considered

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

Janteng, Halim and Darus [14] have determined the functional $\left|a_2a_4 - a_3^2\right|$ and found a sharp bound for the functions f in the subclass RT of $\mathbb{U}$, consisting of functions whose derivative has a positive real part studied by Mac Gregor [18]. In their work, they have shown that if $f \in RT$ then $\left|a_2a_4 - a_3^2\right| \leq \frac{4}{9}$.The same authors[12] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses namely, starlike and convex denoted by ST and CV of $\mathbb{U}$ and have shown that $\left|a_2a_4 - a_3^2\right| \leq 1$ and $\left|a_2a_4 - a_3^2\right| \leq \frac{1}{8}$, respectively. Mishra and Gochhayat [21] have obtained sharp bound to the non-linear functional $\left|a_2a_4 - a_3^2\right|$ for the class of analytic functions denoted by $R_\lambda(\alpha, \rho)$ $\left(0 \leq \rho \leq 1, 0 \leq \lambda < 1, |\alpha| < \frac{\pi}{2}\right)$.

Similarly, the same coefficient inequality was calculated for certain subclasses of analytic functions by many authors, see e.g. [1], [3], [4], [9-11], [21], [22], [29], [31-39].

Motivated by the above mentioned results obtained by different authors in this direction, we seek upper bound of the function $\left|a_3 - \mu a_2^2\right|$ for functions belonging to the defined classes.

A function $f(z) \in \mathcal{A}$ is said to be in the class RST(α) $(\alpha \geq 0)$ in $\mathbb{U}$, if it satisfies the condition:

$$Re\left[\alpha f'(z) + (1-\alpha)\frac{zf'(z)}{f(z)}\right] > 0, \quad \forall z \in \mathbb{U}. \tag{2}$$

This class was studied by D. Vamshee Krishna and T. Ramreddy. It is observed that for $\alpha = 0$ and $\alpha = 1$ in (2), we respectively get $RST(0) = ST$ and $RST(1) = RT$.

$C(\alpha)$ denotes the subclass of functions $f(z) \in A$ satisfying the condition

$$Re\left[\alpha f'(z) + (1-\alpha)\frac{zf'(z)}{g(z)}\right] > 0, \tag{3}$$

where

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*. \tag{4}$$

In particular,

1. $C(1) \equiv RT$,
2. $C(0) \equiv CC$, the class of close-to-convex functions.

Let $C'_*(\alpha)$ be the subclass of functions $f(z) \in A$, satisfying the condition

$$Re\left[\alpha f'(z) + (1-\alpha)\frac{zf'(z)}{h(z)}\right] > 0, \tag{5}$$

where

$$h(z) = z + \sum_{n=2}^{\infty} d_n z^n \in K. \tag{6}$$

We have the following obvious observations:

1. $C'_*(1) \equiv RT$,
2. $C'_*(0) \equiv C'$.

$C_s^{*(\alpha)}$ denote the subclass of functions $f(z) \in A$ satisfying the condition

$$Re\left[\alpha f'(z) + (1-\alpha)\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)\right] > 0. \tag{7}$$

The following observations are obvious:

1. $C_s^{*(1)} \equiv RT$,

2. $C_s^{*(0)} \equiv S_s^*$, the class of starlike functions with respect to symmetric points introduced by Sakaguchi [33].

Let C_s^{α} be the subclass of functions $f(z) \in A$, satisfying the condition

$$Re\left[\alpha f'(z) + (1-\alpha)\left(\frac{2zf'(z)}{g(z)-g(-z)}\right)\right] > 0, \tag{8}$$

where

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S_s^*. \tag{9}$$

In particular,

1. $C_s^1 \equiv RT$,

2. $C_s^0 \equiv C_s$, the class of close-to-convex functions with respect to symmetric points introduced by Das and Singh [6].

Let $C_{1(s)}^{\alpha}$ be the subclass of functions $f(z) \in A$, satisfying the condition

$$Re\left[\alpha f'(z) + (1-\alpha)\left(\frac{2zf'(z)}{h(z)-h(-z)}\right)\right] > 0, \tag{10}$$

where

$$h(z) = z + \sum_{n=2}^{\infty} d_n z^n \in K_s. \tag{11}$$

We have the following obvious observations:

1. $C_{1(s)}^{(1)} \equiv RT$,

2. $C_{1(s)}^{(0)} \equiv C_s'$.

2. Preliminary Results

Let P be the family of all functions p analytic in $\mathbb{U}$ for which $Re\left(p\left(z\right)\right) > 0$ and

$$p\left(z\right) = 1 + p_1 z + p_2 z^2 + \cdots, \quad \forall z \in \mathbb{U}. \tag{12}$$

Lemma 2.1. ([26],[30]) $|p_k| \leq 2$, $(k = 1, 2, 3, \ldots)$.

Lemma 2.2. *If $p \in P$, then*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right) x,$$

$$4p_3 = p_1^3 + 2p_1\left(4 - p_1^2\right) x - p_1\left(4 - p_1^2\right) x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right) z,$$

for some x and z satisfying $|x| \leq 1$ and $p_1 \in [0, 2]$.

This result was proved by Libera and Zlotkiewiez [15,16].

3. Main Results

Theorem 3.1. *If $f\left(z\right) \in C(\alpha)$, then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{\left(3 - \alpha\right)^2}{9}. \tag{13}$$

Proof. Since $C_s^{(\alpha)}$ denotes the subclass of functions $f\left(z\right) \in A$, satisfying the condition (8), so

$$\alpha f'(z) + (1 - \alpha) \frac{z f'(z)}{g(z)} = p\left(z\right), \tag{14}$$

where

$$g\left(z\right) = z + \sum_{n=2}^{\infty} b_n z^n \in S_s^*. \tag{15}$$

On equating the coefficients of z, z^2 and z^3 in the expansion of (14), we have

$$\begin{cases} a_2 = \dfrac{p_1}{2} + \dfrac{b_2(1-\alpha)}{2} \\ a_3 = \dfrac{p_2}{2} + \dfrac{(1-\alpha)\left(2a_2b_2 + b_3 - b_2^2\right)}{2} \\ a_4 = \dfrac{p_3}{4} + \dfrac{(1-\alpha)\left(3a_3b_2 + 2a_2b_3 + b_4 - 2a_2b_2^2 - 2b_2b_3 + b_2^3\right)}{4} \end{cases}. \tag{16}$$

From (15), we can easily verify that

$$b_2 = p_1, b_3 = \frac{p_2 + p_1^2}{2},\ b_4 = \frac{p_3}{3} + \frac{p_1p_2}{2} + \frac{p_1^3}{6}.$$

So (16) yields

$$\begin{cases} a_2 = \dfrac{(2-\alpha)p_1}{2} \\ a_3 = \dfrac{(3-\alpha)p_2}{6} + \dfrac{(1-\alpha)(3-2\alpha)p_1^2}{6} \\ a_4 = \dfrac{(4-\alpha)p_3}{12} + \dfrac{(1-\alpha)(2-\alpha)p_1p_2}{4} + \dfrac{(1-\alpha)\left(3\alpha^2 - 6\alpha + 2\right)p_1^3}{12} \end{cases}. \tag{17}$$

Using (17) yields

$$a_2a_4 - a_3^2$$

$$= \frac{1}{72}\left[\begin{array}{l} 3(2-\alpha)(4-\alpha)p_1p_3 \\ + \left[9(1-\alpha)(2-\alpha)^2 - 4(1-\alpha)(3-\alpha)(3-2\alpha)\right]p_1^2p_2 \\ + \left[3(1-\alpha)(2-\alpha)\left(3\alpha^2 - 6\alpha + 2\right) - 2(1-\alpha)^2(3-2\alpha)^2\right]p_1^4 \\ -2(3-\alpha)^2p_2^2 \end{array}\right]. \tag{18}$$

Using Lemma 2.1 and Lemma 2.2 in (18), we have

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{288}\Big[\Big[3(2-\alpha)(4-\alpha) + 2\alpha^2(1-\alpha)$$

$$-4(1-\alpha)\left(\alpha^3 - 4\alpha^2 + 6\right) - 2(3-\alpha)^2\Big]p_1^4$$

$$- \left[-6(2-\alpha)(4-\alpha) - 2\alpha^2(1-\alpha) - 4(3-\alpha)^2 \right] p_1^2 \left(4-p_1^2\right) x$$
$$- \left[3(2-\alpha)(4-\alpha) p_1^2 + 2(3-\alpha)^2 \left(4-p_1^2\right) \right] \left(4-p_1^2\right) x^2$$
$$+6(2-\alpha)(4-\alpha) p_1 \left(4-p_1^2\right) \left(1-|x|^2\right) z \Big].$$

Assume that $p_1 = p$ and $p \in [0,2]$, using the triangular inequality and $|z| \leq 1$, we have

$$\left|a_2a_4 - a_3^2\right|$$
$$\leq \frac{1}{288} \Big[\left[3(2-\alpha)(4-\alpha) + 2\alpha^2(1-\alpha) - 4(1-\alpha)\left(\alpha^3 - 4\alpha^2 + 6\right) \right.$$
$$\left. -2(3-\alpha)^2 \right] p^4 + \left[-6(2-\alpha)(4-\alpha) - 2\alpha^2(1-\alpha) \right.$$
$$\left. -4(3-\alpha)^2 \right] p^2 \left(4-p^2\right) |x|$$
$$+ \left[3(2-\alpha)(4-\alpha) p^2 + 2(3-\alpha)^2 \left(4-p^2\right) \right] \left(4-p^2\right) |x|^2$$
$$+6(2-\alpha)(4-\alpha)\left(4-p^2\right) p \left(1-|x|^2\right) \Big],$$

$$\left|a_2a_4 - a_3^2\right|$$
$$\leq \frac{1}{288} \Big[\left[3(2-\alpha)(4-\alpha) + 2\alpha^2(1-\alpha) - 4(1-\alpha)\left(\alpha^3 - 4\alpha^2 + 6\right) \right.$$
$$\left. -2(3-\alpha)^2 \right] p^4 + 6(2-\alpha)(4-\alpha) p \left(4-p^2\right)$$
$$+ \left[-6(2-\alpha)(4-\alpha) - 2\alpha^2(1-\alpha) - 4(3-\alpha)^2 \right] p^2 \left(4-p^2\right) \delta$$
$$+ \left[3(2-\alpha)(4-\alpha) p^2 - 6(2-\alpha)(4-\alpha) p \right.$$
$$\left. +2(3-\alpha)^2 \left(4-p^2\right) \right] \left(4-p^2\right) \delta^2 \Big].$$

Therefore

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{288} F(\delta),$$

where $\delta = |x| \leq 1$ and

$$F(\delta) = \left[3(2-\alpha)(4-\alpha) + 2\alpha^2(1-\alpha) - 4(1-\alpha)\left(\alpha^3 - 4\alpha^2 + 6\right) \right.$$
$$\left. -2(3-\alpha)^2 \right] p^4 + 6(2-\alpha)(4-\alpha) p \left(4-p^2\right)$$
$$+ \left[-6(2-\alpha)(4-\alpha) - 2\alpha^2(1-\alpha) - 4(3-\alpha)^2 \right] p^2 \left(4-p^2\right) \delta$$

$$+\left[3(2-\alpha)(4-\alpha)p^2-6(2-\alpha)(4-\alpha)p\right.$$
$$\left.+2(3-\alpha)^2\left(4-p^2\right)\left(4-p^2\right)\delta^2\right]$$

is an increasing function. Therefore $\text{Max}F(\delta)=F(1)$.

Consequently,

$$\left|a_2a_4-a_3^2\right|\leq\frac{1}{288}G(p), \tag{19}$$

where $G(p)=F(1)$. So,

$$G(p)=A(\alpha)p^4-4B(\alpha)p^2+32(3-\alpha)^2,$$

where

$$A(\alpha)=2\left(2\alpha^4-12\alpha^3+15\alpha^2-18\alpha+30\right),$$

and

$$B(\alpha)=\left(-2\alpha^3+13\alpha^2-66\alpha+96\right),$$

Now

$$G'(p)=4A(\alpha)p^3-8B(\alpha)p$$

and

$$G''(p)=12A(\alpha)p^2-8B(\alpha)$$

then $G'(p)=0$ gives

$$p\left[4A(\alpha)p^2-8B(\alpha)\right]=0.$$

$G''(p)$ is negative at

$$p=\sqrt{\frac{96-66\alpha+13\alpha^2-2\alpha^3}{30-18\alpha+15\alpha^2-12\alpha^3+2\alpha^4}}=p'.$$

So $\text{Max}G(p)=G(p')$. Hence from (19), we obtain (3.1). □

The result is sharp for $p_1=p', p_2=p_1^2-2$ and $p_3=p_1\left(p_1^2-3\right)$.

For $\alpha=1$ and $\alpha=0$ respectively, we obtain the following results:

Corollary 3.2. *If $g(z)\in RT$, then*

$$\left|a_2a_4-a_3^2\right|\leq\frac{4}{9}.$$

Remark 3.3. For the choice of $\alpha=1$, the result coincides with those of A. Janteng, S.A. Halim and M. Darus ([12], [13]).

Corollary 3.3. *If $g(z) \in CC$, then*

$$\left|a_2a_4 - a_3^2\right| \leq 1.$$

Theorem 3.4. *If $f \in C'_*(\alpha)$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{(7-\alpha)^2}{324}. \tag{20}$$

The result is sharp for $p_1 = p', p_2 = p_1^2 - 2$ and $p_3 = p_1\left(p_1^2 - 3\right)$.
For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 3.5. *If $h(z) \in RT$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{9}.$$

Corollary 3.6. *If $h(z) \in C'$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{49}{324}.$$

4. Functions with Respect to Symmetric Points

Theorem 4.1. *If $f \in C_s^{*(\alpha)}$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{9}. \tag{21}$$

The result is sharp for $p_1 = 0, p_2 = -2$ and $p_3 = 0$.
For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 4.2. *If $f(z) \in RT$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{9}.$$

Corollary 4.3. *If $f(z) \in S_s^*$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{9}.$$

Theorem 4.4. *If C_s^{α}, then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{(3-\alpha)^2}{9}. \tag{22}$$

The result is sharp for $p_1 = 0, p_2 = -2$ and $p_3 = 0$.
For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 4.5. *If $g(z) \in RT$, then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{4}{9}.$$

Corollary 4.6. *If $g(z) \in S_s^*$, then*

$$\left|a_2 a_4 - a_3^2\right| \leq 1.$$

Theorem 4.7. *If $C_{1(s)}^{\alpha}$, then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{(7-\alpha)^2}{81}. \tag{23}$$

The result is sharp for $p_1 = 0, p_2 = -2$ and $p_3 = 0$.
For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 4.8. *If $h(z) \in RT$, then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{4}{9}.$$

Corollary 4.9. *If $h(z) \in C_s'$, then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{49}{81}.$$

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