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A GENERALIZATION OF CONVEX FUNCTIONS

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Abstract: The object of this paper is to obtain sharp results involving growth and distortion properties for the classes V_k and C_k^* of analytic functions in the unit disk.

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1. Introduction and Definitions

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic and univalent in the open unit disk $E = \{z : |z| < 1\}$.

Let C denote the class of convex functions:

$$f(z) \in C \text{ if and only if for } z \in E, \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0.$$

For $k \geq 2$, denote by V_k the class of normalized functions of bounded boundary rotation at most $k\pi$. Thus $g(z) \in V_k$ if and only if $g(z)$ is analytic in E, $g'(z) \neq 0$, $g(0) = g'(0) - 1 = 0$ and for $z \in E$:

$$\int_0^{2\pi} \left| \Re \frac{(zg'(z))'}{g'(z)} \right| d\theta \leq k\pi. \tag{1.1}$$

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It is known [1] that for $2 \le k \le 4$, V_k consists only of univalent functions (in fact, close-to-convex functions). This is another definition of the class V_k.

For fixed k, let V_k denote the class of functions $g(z)$ normalized so that

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

which are analytic in E and have an integral representation of the form

$$g'(z) = \exp\left\{ \frac{1}{\pi} \int_0^{2\pi} \log\left(1 - ze^{-it}\right)^{-1} d\mu(t) \right\}, \tag{1.2}$$

where $\mu(t)$ is real-valued and bounded variation on $[0, 2\pi]$ with

$$\begin{aligned} &\int_0^{2\pi} d\mu(t) = 2\pi, \\ &\int_0^{2\pi} |d\mu(t)| \le k\pi. \end{aligned} \tag{1.3}$$

The representation formula (1.2) together with (1.3) is due to Paatero [1] and is equivalent to the definition (1.1) for $g(z) \in V_k$.

In 1917, Lowner [2] was the first to consider functions of bounded boundary rotation. Later, Paatero [1, 3] made an exhaustive study of the class. The function $g_k(z)$ defined for $z \in E$ by

$$g_k(z) = \frac{1}{k}\left[\left(\frac{1+z}{1-z}\right)^{\frac{k}{2}} - 1\right] = \sum_{n=1}^{\infty} B_n(k) z^n \tag{1.4}$$

belongs to V_k and is extremal for many problems.

Paatero [1] proved sharp distortion theorems for $g(z) \in V_k$.

Let $f(z)$ be analytic in E, $f'(0) \ne 0$ and normalized so that

$$f(0) = 0, \quad f'(0) = 1.$$

Then, for $k \ge 2$, $f(z) \in T_k$ if there exists a function $g(z) \in V_k$, such that for $z \in E$

$$\Re \frac{f'(z)}{g'(z)} > 0. \tag{1.5}$$

Clearly $T_2 = K$, the class of close-to-convex functions.

We now define a new subclass C_k^* which has the same relationship with T_k as C has with S^* (the class of starlike functions).

Let $f(z)$ be analytic in E and normalized so that $f(0) = 0$, $f'(0) = 1$ and $f'(z) \neq 0$. Then $f(z) \in C_k^*$ $(k \geq 2)$ if there exists a function $g(z) \in V_k$ such that for $z \in E$

$$\Re \frac{(zf'(z))'}{g'(z)} > 0. \tag{1.6}$$

Clearly, $C_2^* = C^*$, the class of quasi-convex functions [5].

It follows easily from definition (1.6) that

$$f(z) \in C_k^* \text{ if and only if } zf'(z) \in T_k. \tag{1.7}$$

Let us consider the function

$$F_k(z) = \frac{1}{k+2}\left[\left(\frac{1+z}{1-z}\right)^{\frac{k}{2}+1} - 1\right] = z + \sum_{n=2}^{\infty} A_n(k)z_n. \tag{1.8}$$

It is then easy to show that [7], $F_k(z) \in T_k$.

2. Known Results

Theorem 2.1. (see [1]) *If $g(z) \in V_k$, then for $z = re^{i\theta} \in E$*

$$\frac{1}{k}\left[1 - \left(\frac{1-r}{1+r}\right)^{\frac{k}{2}}\right] \leq |g(z)| \leq \frac{1}{k}\left[\left(\frac{1+r}{1-r}\right)^{\frac{k}{2}} - 1\right]$$

$$\frac{1}{1-r^2}\left(\frac{1-r}{1+r}\right)^{\frac{k}{2}} \leq g'(z) \leq \frac{1}{1-r^2}\left(\frac{1+r}{1-r}\right)^{\frac{k}{2}}.$$

The function $g_k(z)$ defined for $z \in E$ by

$$g_k(z) = \frac{1}{k}\left[\left(\frac{1+z}{1-z}\right)^{\frac{k}{2}} - 1\right] = \sum_{n=2}^{\infty} B_n z^n$$

shows that these inequalities are sharp.

Theorem 2.2. (see [6]) *Let $g(z) \in V_k$ and $\xi \in E$. Then $F(z) \in V_k$, where $F(z)$ is given by*

$$F(z) = \frac{g\left(\frac{\xi+z}{1+\bar{\xi}z}\right) - g(\xi)}{g'(\xi)\left[1 - |\xi|^2\right]}.$$

Theorem 2.3. (see [7]) *Let $f(z) \in T_k$, then for $z = re^{i\theta} \in E$*

$$\frac{(1-r)^{\frac{k}{2}}}{(1+r)^{\frac{k}{2}+2}} \le |f'(z)| \le \frac{(1+r)^{\frac{k}{2}}}{(1-r)^{\frac{k}{2}+2}}.$$

These bounds are sharp, equality being attained for the functions $F_k(z)$ defined by

$$F_k(z) = \frac{1}{k+2}\left[\left(\frac{1+z}{1-z}\right)^{\frac{k}{2}+1} - 1\right].$$

3. Some of the Basic Properties of Functions in the Classes V_k and C_k^*

Theorem 3.1. *Let $g(z) \in V_k$, then for $z = re^{i\theta} \in E$*

$$\frac{k(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}+1}\left[1-\left(\frac{1-r}{1+r}\right)^{\frac{k}{2}}\right]} \le \left|\frac{g'(z)}{g(z)}\right| \le \frac{k(1+r)^{\frac{k}{2}-1}}{(1-r)^{\frac{k}{2}+1}\left[\left(\frac{1+r}{1-r}\right)^{\frac{k}{2}}-1\right]}.$$

The result is sharp.

Proof. Since $g(z) \in V_k$, Theorem 2.2 shows that $F(z)$ defined by

$$F(z) = \frac{g\left(\frac{\xi+z}{1+\bar{\xi}z}\right) - g(\xi)}{g'(\xi)\left[1-|\xi|^2\right]} \tag{3.1}$$

is in V_k, when ξ is any arbitrary point in E. Thus, with $z = -\xi$ we obtain

$$F(-\xi) = \frac{-g(\xi)}{g'(\xi)\left[1-|\xi|^2\right]}. \tag{3.2}$$

Now, using the distortion Theorem 2.1 for $g(z) \in V_k$, we obtain

$$\frac{1}{k}\left[1-\left(\frac{1-|\xi|}{1+|\xi|}\right)^{\frac{k}{2}}\right] \le |F(-\xi)| \le \frac{1}{k}\left[\left(\frac{1+|\xi|}{1-|\xi|}\right)^{\frac{k}{2}} - 1\right]$$

and so (3.2) gives

$$\frac{1}{k}\left[1-\left(\frac{1-|\xi|}{1+|\xi|}\right)^{\frac{k}{2}}\right] \le \left|\frac{-g(\xi)}{g'(\xi)(1-|\xi|^2)}\right| \le \frac{1}{k}\left[\left(\frac{1+|\xi|}{1-|\xi|}\right)^{\frac{k}{2}} - 1\right],$$

that is

$$(1-|\xi|^2)\frac{1}{k}\left[1-\left(\frac{1-|\xi|}{1+|\xi|}\right)^{\frac{k}{2}}\right] \le \left|\frac{g(\xi)}{g'(\xi)}\right| \le \frac{1}{k}\left[\left(\frac{1+|\xi|}{1-|\xi|}\right)^{\frac{k}{2}}-1\right](1-|\xi|^2).$$

Since ξ is an arbitrary point in E, the result follows.

Remark. We note that when $k = 2$ we obtain the classical distortion theorem for the class C of normalized convex functions.

Theorem 3.2. *Let $f(z) \in C_k$, then for $z \in re^{i\theta} \in E$ and $k \ge 2$:*

$$\frac{1}{k+2}\left[1-\left(\frac{1-r}{1+r}\right)^{\frac{k}{2}+1}\right] \le |zf'(z)| \le \frac{1}{k+2}\left[\left(\frac{1+r}{1-r}\right)^{\frac{k}{2}+1}-1\right],$$

$$\frac{1}{k+2}\int_0^r\left[1-\left(\frac{1-t}{1+t}\right)^{\frac{k}{2}+1}\right]\frac{dt}{t} \le |f(z)| \le \frac{1}{k+2}\int_0^r\left[\left(\frac{1+t}{1-t}\right)^{\frac{k}{2}+1}-1\right]\frac{dt}{t}.$$

Proof. Since $f(z) \in C_k^*$, (1.7) shows that $zf'(z) \in T_k$.
Thus, from Theorem 2.3,

$$\frac{1}{k+2}\left[1-\left(\frac{1-r}{1+r}\right)^{\frac{k}{2}+1}\right] \le |zf'(z)| \le \frac{1}{k+2}\left[\left(\frac{1+r}{1-r}\right)^{\frac{k}{2}+1}-1\right]. \qquad (3.3)$$

Integrating the right-hand inequality in (3.3) from 0 to z, we obtain

$$|f(z)| \le \int_0^{|z|}|f'(z)|d|z| \le \frac{1}{k+2}\int_0^r\left[\left(\frac{1+t}{1-t}\right)^{\frac{k}{2}+1}-1\right]\frac{dt}{t}.$$

In order to obtain a lower bound for $|f(z)|$, we proceed as follows. Let z_1 be such that $|z_1| = r$ and $|f(z_1)| \le |f(z)|$ for all z with $|z| = r$. Writing $\omega = f(z)$, it follows that the line-segment λ from $\omega = 0$ to $\omega = f(z)$ lies entirely in the image of $f(z)$. Let L be the pre-image of λ. Then

$$|f(z)| \ge |f(z_1)| = \int_\lambda |d\omega| = \int_L \left|\frac{d\omega}{dz}\right||dz| \ge \frac{1}{k+2}\int_0^r\left[1-\left(\frac{1-t}{1+t}\right)^{\frac{k}{2}+1}\right]\frac{dt}{t}.$$

Then, the theorem follows. The function Φ defined by

$$\Phi(z) = \frac{1}{k+2}\int_0^z\left[\left(\frac{1+t}{1-t}\right)^{\frac{k}{2}+1}-1\right]\frac{dt}{t}$$

shows that equality can occur.

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