

A FAST AND ACCURATE LATTICE MODEL TO EVALUATE OPTIONS UNDER THE VARIANCE GAMMA PROCESS

Massimo Costabile

Department of Economics
Statistics and Finance
University of Calabria
Cubo 3 C, Rende, 87036, ITALY

Abstract: We develop a lattice-based model to evaluate European and American plain vanilla options when the underlying asset price is driven by a variance gamma process. By applying the Lévy-Itô decomposition of the process, we obtain a compound Poisson process made up of a linear drift and the sum of the jumps taken by the process. A multinomial lattice is derived to approximate the compound Poisson process and is used as the corner stone to approximate the evolution of a certain asset price. European and American options are evaluated and, because numerical results show monotonic convergence at the rate of $1/n$, we apply a simple two-point Richardson extrapolation and obtain a fast and accurate pricing model.

AMS Subject Classification: 91G20, 91G60

Key Words: lattice models, variance gamma, Richardson extrapolation, Lévy process

1. Introduction

It is well known that the simple and appealing hypotheses of normally distributed asset returns behind the celebrated Black-Scholes-Merton model is not confirmed by several empirical analysis. Specifically, financial econometricians highlight that stock returns typically show skewness and excess kurtosis leading to empirical distributions with fatter tails than those of a normally

distributed random variate. The effort of searching for alternative models has stimulated many researchers for decades. Today, infinite activity pure jump Levy processes have gained wide popularity and represent effective models to describe financial asset returns. These models are supported by many empirical investigations and, among others, we may refer to Carr et al. [3] who found that statistical and risk-neutral returns of indexes and single equities tend to be pure jump process with infinite activity and finite variation. Within the class of pure jump processes with infinite activity and finite variation is the variance gamma (VG) model proposed by Madan and Seneta [11] and then generalized by Madan and Milne [10], Madan et al. [8] and Carr et al. [3]. The VG process, as illustrated in [8], is a three-parameter model that describes the evolution of a log stock price by generalizing a Brownian motion with constant drift and volatility. The first parameter is the volatility of the Brownian motion while the remaining two parameters control the skewness and kurtosis of the process. Madan et al. [8] build up the VG process by subordinating Brownian motion with an independent gamma process that represents a random time change. The rationale under this assumption is that calendar time must not be considered per se but only in terms of its economic relevance. As a consequence, periods of financial turmoil speed up the calendar clock while the clock slows down during periods of normal activity. The transformation between the deterministic calendar time and the random “business time” is obtained by means of an increasing gamma process.

Except for the case of plain vanilla European options, analytical formulas are difficult to derive when the log-price of the underlying asset is driven by the VG process. Hence, numerical methods play an essential role to obtain in an efficient way accurate estimates of option prices. Monte Carlo simulations and numerical methods to solve the associated PIDE are the most common approaches to solving the evaluation problem. Because Monte Carlo methods are based on simulations that go forward in time, they are very efficient when dealing with path-dependent European-style contracts. In contrast, for the evaluation of American options it is straightforward for numerical methods to solve the associated PIDE because they work backwards in time.¹

Lattice based models are very popular for evaluating and hedging derivative securities when the underlying asset is driven by a diffusion process. In fact, they are simple and transparent tools that can be easily implemented to approximate in an efficient way the dynamics of a small number of state variables. This is the reason why, after the Cox-Ross-Rubinstein models a lot of binomial and trinomial lattice based algorithms have been proposed in financial

¹For a comprehensive analysis of numerical methods for Lévy processes, see [4].

literature. To the best of author's knowledge two models based on multinomial lattices have been proposed to approximate pure jump Lévy processes with infinite activity. Kellezi and Webber [6] propose a lattice that approximates the transition density function but numerical results are not accurate in the case of American options. Maller et al. [12] (MSS) develop a multinomial tree model based on a discretization of the Lévy measure of the process and provide a proof for the convergence of the proposed approximating scheme. The main problem in constructing a lattice model to approximate a pure jump Lévy processes with infinite activity is that the process is heavy tailed and a lot of branches stemming from each node of the tree are needed to capture the probability into the tails of the distribution. As a consequence, the pricing problem becomes readily unmanageable from a computational point of view when the number of times steps increases.

A possible way to obtain fast and accurate prices when applying a discrete time approximation model is to extrapolate the desired value from those computed with a limited number of time steps. In particular, if a numerical scheme shows monotone convergence and if the rate of convergence can be calculated, Richardson extrapolation can be applied to achieve high accuracy with a small number of time steps. In the case of state variables driven by a geometric Brownian motion Richardson extrapolation has been extensively applied to compute option prices and sensitivities (see Tian [15], for example).

We propose a lattice-based model that, coupled with a simple Richardson extrapolation, furnishes a fast method to compute accurate prices for European and American options under VG dynamics. The starting point of the proposed model is the Lévy-Itô decomposition of the VG process according to which it may be decomposed as a sum of two components: the first component is deterministic and represents the drift of the process while the second component is stochastic and is given by the sum of its jumps. Because the Lévy measure of the VG process presents a singularity in the origin, we truncate it in a small interval centered at zero and approximate the small jumps of the process with their expectation. Hence, we obtain an approximating process that is a compound Poisson process with a finite Lévy measure whose intensity and jump density may be easily derived. At this point, we build up a multinomial lattice, and the jump density is approximated at each time slice in a way that is similar, in spirit, to that proposed by Amin [2] to approximate jump diffusion processes. Extensive numerical experiments show a virtually monotonic convergence with rate $1/n$. Hence, we apply a simple two-point Richardson extrapolation and obtain fast and accurate estimates of option prices both for European and American plain vanilla options.

The rest of the paper is organized as follows. In Section 2, we illustrate the lattice approximating the VG process. In Section 3, we present numerical results, while in Section 4, we draw conclusions.

2. The Approximating Multinomial Lattice

We consider a three-parameter VG process $(X_t, t \geq 0)$ with an initial value zero and the Lévy measure given by

$$\nu(dx) = \kappa(x)dx = \frac{\exp(\theta x/\sigma^2)}{v|x|} \exp\left(-\frac{1}{\sigma}\sqrt{\frac{2}{v} + \frac{\theta^2}{\sigma^2}}|x|\right) dx,$$

where $\theta \in \mathbb{R}, \sigma > 0$ and $v > 0$. The characteristic function is

$$\mathbb{E}[\exp(iuX_t)] = (1 - iu\theta v + \frac{1}{2}\sigma^2 v u^2)^{-t/v}.$$

We recall that when a pure jump Lévy process with characteristic triplet $(\gamma, 0, \nu)$ has finite variation, by applying the Lévy-Itô decomposition, it may be described in the following way:

$$X_t = bt + \int_{[0,t] \times \mathbb{R}} x J_X(ds \times dx) = bt + \sum_{s \leq t} \Delta X_s 1_{\Delta X_s \neq 0},$$

where $b = \gamma - \int_{|x| \leq 1} x \nu(dx)$, J_X is a Poisson random measure on $[0, \infty) \times \mathbb{R}$ with intensity $\nu(dx)dt$ and $1_{\mathcal{A}}$ is the indicator function on set \mathcal{A} . A VG process has finite variation and it may be easily shown that $\gamma = \int_{|x| \leq 1} x \nu(dx)$. Moreover, because the process has finite variation, its small jumps may be approximated by their expected value; hence, the process X_t may be approximated by the following compound Poisson process

$$X_t^\varepsilon = \sum_{s \leq t} \Delta X_s 1_{\varepsilon \leq |\Delta X_s|} + \mathbb{E} \left[\sum_{s \leq t} \Delta X_s 1_{|\Delta X_s| < \varepsilon} \right], \quad \varepsilon > 0. \quad (1)$$

It is convenient now to set $A = \frac{\theta}{\sigma^2}$ and $B = \frac{\sqrt{\theta^2 + 2\sigma^2/v}}{\sigma^2}$ so that the expectation in equation (1) may be computed as

$$b^\varepsilon = \mathbb{E} \left[\sum_{s \leq t} \Delta X_s 1_{|\Delta X_s| < \varepsilon} \right]$$

$$= \int_{-\varepsilon}^{+\varepsilon} x \kappa(x) dx = \frac{1}{v} \left(\frac{e^{(A-B)\varepsilon} - 1}{A - B} + \frac{e^{-(A+B)\varepsilon} - 1}{A + B} \right).$$

By substituting, we obtain an approximating compound Poisson process

$$X_t^\varepsilon = b^\varepsilon t + \sum_{s \leq t} \Delta X_s 1_{\varepsilon \leq |\Delta X_s|},$$

with finite Lévy measure

$$\nu^\varepsilon(dx) = \kappa^\varepsilon(x) dx = \left(\frac{e^{Ax-B|x|}}{v|x|} 1_{|x| \geq \varepsilon} \right) dx.$$

The intensity of the compound Poisson process X_t^ε may be computed as

$$\begin{aligned} U(\varepsilon) &= \int_{|x| > \varepsilon} \kappa^\varepsilon(x) dx = \frac{1}{v} \int_{|x| > \varepsilon} \frac{e^{Ax-B|x|}}{|x|} dx = \\ &= \frac{1}{v} \int_{\varepsilon}^{\infty} \frac{1}{x} [e^{(A-B)x} + e^{-(A+B)x}] dx = \frac{1}{v} [\text{Ei}(\varepsilon(A+B)) + \text{Ei}(-\varepsilon(A-B))], \end{aligned}$$

where $\text{Ei}(x) = \int_x^{\infty} \frac{e^{-t}}{t} dt$ is the exponential integral function evaluated at x . Moreover, the jump size distribution is $f^\varepsilon(x) = \frac{\kappa^\varepsilon(x)}{U(\varepsilon)}$.

We are in a position now to build up the multinomial lattice to approximate the compound Poisson process X_t^ε . At first, we divide the time horizon $[0, T]$ into n subintervals of equal length, $\Delta t = T/n$. Then, we denote by (i, j) a generic node of the lattice so that $(0, 0)$ represents the initial node where the process has a 0-value; $(i, 0)$ represents the lowest node at time $i\Delta t$, $(i, 1)$ is the second lowest node and so on ($i = 0, \dots, n$). Coherently, $X^\varepsilon(i, j)$ denotes the value of the discretized X_t^ε process at node (i, j) , and the difference between two consecutive nodes, $X^\varepsilon(i, j+1) - X^\varepsilon(i, j) = \Delta x$, at each time slice is constant. Then, m_u "up" branches, 1 "middle" branch and m_d "down" branches emanate from each node of the lattice. Given that the approximating process X_t^ε has value $X^\varepsilon(i, j)$ at node (i, j) , to describe the possible values at the next time step, we must distinguish the case where the process does not take jumps from that in which the process jumps up or down. In the first case, the process increases by a deterministic amount represented by the local drift; hence, it assumes the value $X^\varepsilon(i, j) + b^\varepsilon \Delta t$. If the process jumps up, it may take one of the m_u possible values $X^\varepsilon(i, j) + b^\varepsilon \Delta t + l\Delta x$, $l = 1, \dots, m_u$, while if the process jumps downward, it may assume one of the m_d possible values $X^\varepsilon(i, j) + b^\varepsilon \Delta t + l\Delta x$, $l = -1, \dots, -m_d$. In Figure 1, we depict a sketch of the multinomial lattice.

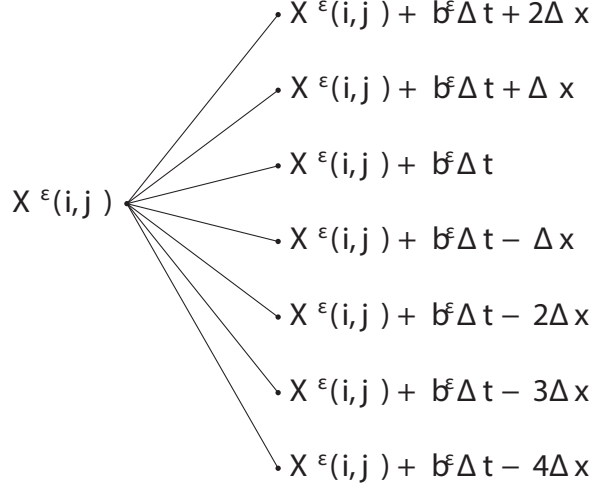


Figure 1: The multinomial lattice with $m_u = 2$ and $m_d = 4$.

Transition probabilities are defined in a similar way, in spirit, to those derived by Amin [2] in the case of jump-diffusion processes. We observe, at first, that for a compound Poisson process, the jump probability in an interval of length Δt is proportional to the jump intensity plus a higher order term, i.e., $U(\varepsilon)\Delta t + o(\Delta t)$. The probability of multiple jumps during the time period Δt is $o(\Delta t)$. Hence, the probability of a single jump is $U(\varepsilon)\Delta t$. We assume that, in each interval, no multiple jumps occur.

The next step is to approximate the jump size distribution. To do this, we partition the interval of the jump density $(-\infty, -\varepsilon] \cup [\varepsilon, +\infty)$ into $m_d + m_u$ subintervals. All but the first and the last subinterval have equal length Δx and are centered around a possible node of the lattice, so they are of the form $((l - 1/2)\Delta x, (l + 1/2)\Delta x)$ for $l = -m_d + 1, \dots, -1, 1, \dots, m_u - 1$, and to guarantee that there is no overlap between such subintervals and the truncation interval $(-\varepsilon, +\varepsilon)$, we set $\varepsilon = \Delta x/2$. To cover the whole interval where the jump density is defined, we define the first subinterval as $(-\infty, -(m_d - 1/2)\Delta x)$ and the last subinterval as $((m_u - 1/2)\Delta x, +\infty)$. Because we know the density of the jump size, $f^\varepsilon(x)$, the probability associated with each subinterval may be easily computed.

We are in a position now to define the transition probabilities along the lattice. Given that a jump has occurred in the time interval Δt , the probability that the process jumps from $X^\varepsilon(i, j)$ to $X^\varepsilon(i, j) + b^\varepsilon \Delta t + l\Delta x$ ($l = -m_d + 1, \dots, -1, 1, \dots, m_u - 1$) is defined as the probability that the jump size belongs

to the interval $(l - 1/2)\Delta x, (l + 1/2)\Delta x$. The probability to jump from $X^\varepsilon(i, j)$ to $X^\varepsilon(i, j) + b^\varepsilon \Delta t + m_u \Delta x$ is defined as the probability that the jump size falls into the interval $((m_u - 1/2)\Delta x, +\infty)$, and similarly, the probability of reaching $X^\varepsilon(i, j) + b^\varepsilon \Delta t - m_d \Delta x$ is defined as the probability that the jump size falls into the interval $(-\infty, -(m_d - 1/2)\Delta x)$. The probability that the process does not jump, i.e., it goes from $X^\varepsilon(i, j)$ to $X^\varepsilon(i, j) + b^\varepsilon \Delta t$, is $1 - U(\varepsilon)\Delta t$.

Hence, denoting by $q_{i,j}^l$ the probability that the process goes from $X^\varepsilon(i, j)$ to $X^\varepsilon(i, j) + b^\varepsilon \Delta t + l\Delta x, l = -m_d, \dots, m_u$, the results are the following:

$$q_{i,j}^l = \begin{cases} U(\varepsilon)\Delta t \int_{-\infty}^{(-m_d + \frac{1}{2})\Delta x} f^\varepsilon(x)dx, & \text{for } l = -m_d \\ U(\varepsilon)\Delta t \int_{(l - \frac{1}{2})\Delta x}^{(l + \frac{1}{2})\Delta x} f^\varepsilon(x)dx, & \text{for } l = -m_d + 1, \dots, -1, \\ 1 - U(\varepsilon)\Delta t, & \text{for } l = 0, \\ U(\varepsilon)\Delta t \int_{(l - \frac{1}{2})\Delta x}^{(l + \frac{1}{2})\Delta x} f^\varepsilon(x)dx, & \text{for } l = 1, \dots, m_u - 1 \\ U(\varepsilon)\Delta t \int_{(m_u - \frac{1}{2})\Delta x}^{\infty} f^\varepsilon(x)dx, & \text{for } l = m_u. \end{cases}$$

With a little algebra, one can show that transition probabilities may be written as

$$q_{i,j}^l = \begin{cases} \frac{1}{v} \text{Ei}((A + B)(m_d - \frac{1}{2})\Delta x) & \text{for } l = -m_d \\ \frac{1}{v} [\text{Ei}((A + B)(-l - \frac{1}{2})\Delta x) + \\ -\text{Ei}((A + B)(-l + \frac{1}{2})\Delta x)]\Delta t, & \text{for } l = -m_d + 1, \dots, -1, \\ 1 - U(\varepsilon)\Delta t, & \text{for } l = 0, \\ \frac{1}{v} [\text{Ei}((B - A)(l - \frac{1}{2})\Delta x) + \\ -\text{Ei}((B - A)(l + \frac{1}{2})\Delta x)]\Delta t, & \text{for } l = 1, \dots, m_u - 1 \\ \frac{1}{v} \text{Ei}((B - A)(m_u - \frac{1}{2})\Delta x) & \text{for } l = m_u. \end{cases}$$

Because the truncation level ε is proportional to Δx , the proposed multinomial model is such that when the number of time steps increases, the width of the truncation interval tends to zero so that the convergence of the multinomial lattice model toward the corresponding target VG process is guaranteed (see [12] for a proof).

In order to build up an evaluation model based on the discrete approximation of the VG process described above, we have to develop a discrete approximation of the random evolution of a certain risky stock which influences the payoff of a generic contingent claim.

Under some risk-neutral probability measure Q , the stock value at a generic time $t > 0$ is

$$S_t = S_0 \exp((r - \delta)t + X_t - \omega t),$$

where S_0 is the stock price at time $t = 0$, r is the continuously compounded risk-free interest rate, δ is a continuous dividend yield, and ω is the compensator that makes the stock price process a martingale, i.e., $\mathbb{E}^Q(S_t) = S_0 \exp((r - \delta)t)$. The compensator is determined by imposing $\mathbb{E}^Q(\exp(X_t)) = \exp(\omega t)$, which implies $\omega = -\frac{1}{v} \ln(1 - \theta v - \sigma^2 v/2)$.

The multinomial lattice approximating the evolution of the stock price has the same structure of the multinomial lattice that discretizes the approximating process X_t^ε . Hence, the stock value at a generic node (i, j) will be $S(i, j) = S(0, 0) \exp((r - \delta)i\Delta t + X^\varepsilon(i, j) - \tilde{\omega}i\Delta t)$ with $S(0, 0) = S_0$. The discrete compensator $\tilde{\omega}$ makes the discrete process approximating the underlying asset price evolution a martingale under the risk-neutral probability measure and is equal to $r + \frac{1}{T} \log(V_0)$, where V_0 is the price at time $t = 0$ of a contingent claim with payoff, at maturity $t = T$, equal to $\exp(X^\varepsilon(n, j))$ for $j = 0, \dots, n(m_d + m_u) + 1$.

In this framework, the evaluation problem of a derivative security may be addressed straightforwardly by applying the usual backward induction. Denoting by $C(i, j)$ the value of a European-style contingent claim at a node (i, j) of the multinomial lattice, it can be computed as follows:

$$C(i, j) = e^{-r\Delta t} \sum_{k=j}^{j+m_d+m_u} q_{i,j}^{k-j-m_d} C(i+1, k). \quad (2)$$

Clearly, an American-style contingent claim may be computed by setting its value at node (i, j) equal to the maximum between the continuation value as derived in the right-hand side of equation (2) and the contingent claim payoff in the case of early exercise.

3. Numerical Results

To assess the goodness of the proposed approximation model, at first we evaluate plain vanilla European put options. The model parameters are the parameters estimated by Madan et al. [8], i.e., $\theta = -0.14$, $\sigma = 0.12$ and $v = 0.2$, while the risk-free interest rate is $r = 0.1$, the dividend yield is $\delta = 0$, and the initial underlying asset price is $S_0 = 100$. Being the approximation of the jump size density central in the construction of the multinomial lattice, we illustrate in Figure 2 two examples for the truncation level $\varepsilon = 0.001$ (left plot) and $\varepsilon = 0.005$ (right plot). Clearly, because the jump density is obtained by dividing the Lévy measure of the process by the jump intensity, narrowing the truncation interval determines a jump density with thinner tails.

To implement the multinomial lattice, we have to choose the size of Δx , the difference between two consecutive nodes at a given time period $i\Delta t$. A small Δx results in a better approximation of the jump density but implies a higher number of branches and requires a heavier computational effort for evaluation purposes. As in [12], we propose to set $\Delta x = \alpha\sqrt{\Delta t}$ with α a positive constant that makes the lattice construction more flexible so that a suitable number of branches can be readily chosen. In the numerical experiments reported in the present paper the parameter α is set equal to 0.05.

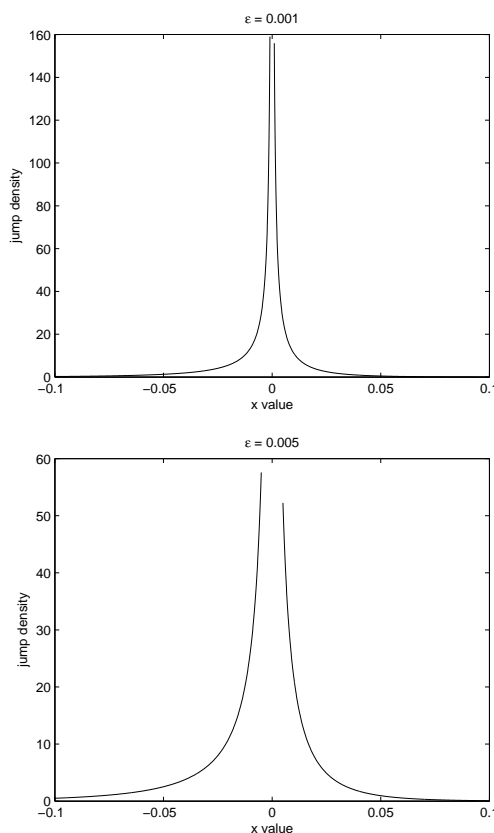


Figure 2: The two graphs present the jump density of the VG process with the parameters as estimated by Madan et al. [8], i.e., $\theta = -0.14$, $\sigma = 0.12$ and $\nu = 0.2$. The graph on the left has a truncation level $\varepsilon = 0.001$, while for the graph on the right, $\varepsilon = 0.005$.

Figure 3 illustrates the pricing error, $e(n) = EP(n) - P_{AN}$, defined as

the difference between the prices of European put options computed with the multinomial lattice and the corresponding prices obtained through the analytical formula proposed by Madan et al. [8]. The plot on the left is relative to a time to maturity $T = 0.25$ years, while the plot on the right corresponds to $T = 1$ years. The number of time steps is $n = 10, 15, 20, \dots, 200$ when the time to maturity is $T = 0.25$, while in the case $T = 1$, $n = 10$ is not considered because the corresponding jump probability, $U(\varepsilon)\Delta t$, falls outside of $[0, 1]$. Three different cases are analyzed relative to strike prices $K = 90, 100, 110$. The number of lower branches, m_d , has been selected as the maximum positive integer such that the probability of the jump size belonging to the fictitious interval $((m_d - 1/2)\Delta x, (m_d + 1/2)\Delta x)$ is greater than 10^{-8} . The number of upper branches, m_u , is defined analogously. Extensive numerical experiments have highlighted that even moving further into the tails of the jump density by adding intervals with lower probability mass does not significantly improve the precision of the proposed evaluation model. Figure 3 shows a convergent pattern of the prices computed with the multinomial lattice. At $n = 200$, the relative percentage error is approximately 0.5% in the worst case ($K = 90$) when $T = 0.25$ and 1% in the worst case ($K = 90$) when $T = 1$. Even if the proposed multinomial lattice model furnishes accurate prices, particularly in the short maturity case, its computational cost is relevant and may represent a serious disincentive to applying the multinomial model as a practical tool to compute option prices under the VG process.

Nevertheless, Figure 3 shows a key feature of the convergence pattern: it is virtually monotonic. This monotonicity is very important because it is well known that when an approximating model exhibits smoothness of convergence, extrapolation techniques can be applied to enhance the rate of convergence. Specifically, to apply the simple two-point Richardson extrapolation method, it must be that the prices computed with the discrete time model converge at a rate $1/n$ and that the convergent pattern exhibits smoothness.² An empirical analysis of the error ratio can help us to assess the rate of convergence and the smoothness of the option prices computed with the proposed model. In fact, if the convergence of the multinomial lattice model is smooth at the rate $1/n$, the error ratio $\rho_e(n) = e(n)/e(2n)$ converges to $\rho = 2$. In Table 1, we illustrate the error ratio relative to the same plain vanilla European put options illustrated above for a number of time steps $n = 10, 20, 40, 80, 160$ ($\rho(10)$ is not considered

²It would be desirable to derive a rigorous proof to assess the rate of convergence of the proposed multinomial model but this is a hard task and, to the best of our knowledge, there is no contribution available that deals with the rate of convergence of lattice based models approximating Lévy processes with jumps.

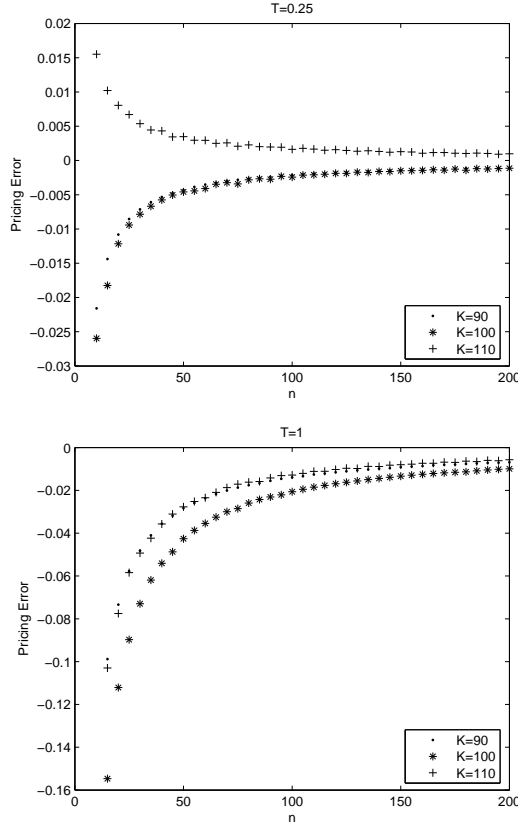


Figure 3: The two graphs present the convergence of European put options under the VG process with parameters $\theta = -0.14$, $\sigma = 0.12$ and $v = 0.2$, while the risk-free interest rate is $r = 0.1$, the underlying asset value at inception is $S_0 = 100$, and three different levels of the strike price are considered, $K = 90, 100, 110$. The graph on the left illustrates the case with time to maturity $T = 0.25$ while the graph on the right is relative to the case $T = 1$.

n	$T = 0.25$			$T = 1$		
	K=90	K=100	K=110	K=90	K=100	K=110
10	1.9983	2.1352	1.9271	-	-	-
20	2.0270	2.1183	1.8625	2.0618	2.0746	2.1737
40	2.0209	2.0462	1.8909	2.0260	2.0866	2.2126
80	2.0102	1.9591	2.1092	2.0333	2.0753	2.2050
160	1.9778	1.9971	1.8334	2.0579	2.0573	2.2252

Table 1: This table reports the error ratios, $\rho_e(n)$, of plain vanilla European put options for a number of time steps $n = 10, 20, 40, 80$. The underlying asset price at inception is $S_0 = 100$, and the risk-free interest rate is $r = 0.1$. Three different strikes, $K = 90, 100, 110$, and two different maturities, $T = 0.25, 1$ are considered ($\rho(10)$ is not considered in the case of $T = 1$ because the price $EP(10)$ is meaningless). The parameters of the VG process are $\sigma = 0.12$, $\theta = -0.14$ and $v = 0.2$.

in the case $T = 1$ because the price $EP(10)$ is meaningless).

As it is evident by looking at Table 1 and is confirmed by extensive additional numerical experiments, the error ratio computed with the multinomial model is close to 2. This value means that a simple two-point Richardson extrapolation can be applied to enhance the rate of convergence. In Table 2, we report the prices of the European put options described above obtained with a two-point Richardson extrapolation. To consider a wider set of option prices we also evaluated contracts with strike price $K = 95$ and $K = 105$. Each price, $RE(n)$, is equal to $2EP(2n) - EP(n)$, where $EP(n)$ is the put option price computed by the multinomial lattice with n time steps, and $n = 10, 20, 40, 80$ ($n = 10$ is not considered in the case $T = 1$ because the jump probability, $U(\varepsilon)\Delta t$, falls outside the interval $[0, 1]$). The rows labeled AN illustrate the prices obtained with the analytical formula of Madan et al. [8], while the rows MSS report the prices obtained with the lattice model as implemented by MSS [12].³

To assess the computational cost of the proposed multinomial model, in Table 3, we report the number of upper and lower branches, m_u and m_d , respectively, emanating from each node of the lattice when the number of time

³MSS implemented the multinomial lattice model by setting $\Delta x = \alpha\sqrt{\Delta t}$ with $\alpha = \sqrt{\theta^2 v + \sigma^2}$, the standard deviation of the VG process, and $n = 200$, $m_d = 200$, $m_u = 200$.

$T = 0.25$					
	K=90	K=95	K=100	K=105	K=110
RE(10)	0.2304	0.6233	1.5724	3.6977	7.5578
RE(20)	0.2305	0.6221	1.5715	3.6930	7.5578
RE(40)	0.2305	0.6216	1.5709	3.6929	7.5574
RE(80)	0.2304	0.6219	1.5707	3.6932	7.5571
RE(160)	0.2304	0.6217	1.5708	3.6928	7.5574
AN	0.2304	0.6218	1.5708	3.6925	7.5572
MSS	0.2298	0.6212	1.5720	3.6974	7.5594

$T = 1$					
	K=90	K=95	K=100	K=105	K=110
RE(10)	-	-	-	-	-
RE(20)	0.5369	1.0334	1.8578	3.1308	4.9679
RE(40)	0.5352	1.0315	1.8560	3.1318	4.9651
RE(80)	0.5350	1.0306	1.8547	3.1283	4.9632
RE(160)	0.5349	1.0301	1.8541	3.1296	4.9624
AN	0.5347	1.0300	1.8538	3.1277	4.9617
MSS	0.5319	1.0274	1.8532	3.1321	4.9697

Table 2: This table reports the European put option prices, $RE(n) = 2P(2n) - P(n)$, obtained with the two-point Richardson extrapolation for $n = 10, 20, 40, 80, 160$ ($n = 20, 40, 80, 160$ in the case $T = 1$). The underlying asset price at inception is $S_0 = 100$, and the risk-free interest rate is $r = 0.1$. Five different strikes, $K = 90, 95, 100, 105, 110$, and two different maturities, $T = 0.25, 1$ years, are considered. The parameters of the VG process are $\sigma = 0.12$, $\theta = -0.14$ and $v = 0.2$. The rows AN report the corresponding prices computed with the analytical formula of Madan et al., [8] while the rows MSS illustrate the prices computed with the multinomial lattice as implemented by MSS [12].

$T = 0.25$					
n	m_d	m_u	N	$N_{RE}(\text{time})$	N_{bin}
10	87	44	7216	44617 (< 1)	298
20	118	60	37401	237522(1.1)	688
40	161	83	200121	1279122(4.2)	1598
80	220	113	1079001	6926682(12.9)	3721
160	300	154	5847681	37588482(38.2)	8669
320	408	210	31740801	-	-
MSS(200)	200	200	8040201	-	4009

$T = 1$					
n	m_d	m_u	N	$N_{RE}(\text{time})$	N_{bin}
10	-	-	-	-	-
20	64	32	20181	127642(< 1)	504
40	87	44	107461	684262(2.6)	1168
80	118	60	576801	3719682(9.1)	2726
160	161	83	3142881	20246082(26.8)	6362
320	220	113	17103201	-	-
MSS(200)	200	200	8040201	-	4009

Table 3: This table reports the number of branches, m_d and m_u , and the total number of nodes, N , of the proposed multinomial lattice while the column N_{RE} contains the number of nodes needed to compute the extrapolated option prices together with the computational time, in seconds, needed to compute the option price (reported in brackets). The column labeled N_{bin} reports the number of steps needed for a binomial recombining lattice to generate a number of nodes corresponding to N_{RE} .

steps represents a geometric sequence with the initial value 10 (20 in the case $T = 1$) and the common ratio 2. The column labeled N reports the total number of nodes contained in each lattice. The column N_{RE} contains the number of nodes needed to compute the extrapolated option prices. The CPU time, in seconds, of each extrapolated option price is reported in brackets (all the computations were performed on a 3 GHz computer with 1 GB of RAM, running Windows XP). Finally, to further clarify the computational cost of the multinomial lattice with Richardson extrapolation, in the column labeled N_{bin} we report the number of steps needed for a binomial recombining lattice to generate a number of nodes corresponding to N_{RE} .

By looking at Tables 2 and 3, some interesting considerations arise. First, we note that the proposed multinomial model coupled with Richardson extrapolation furnishes highly accurate option prices with a relatively small number of time steps. In fact, in the short maturity case, $T = 0.25$ years, the extrapolated prices, $RE(10)$, obtained with 10 and 20 time steps are affected by a maximum error of 0.2% (when $K = 95$), and the pricing error reduces further when $n = 20, 40, 80$ or 160. In the case with maturity $T = 1$, the extrapolated prices, $RE(20)$, are affected by a maximum error of 0.4% (when $K = 90$), which reduces further when a higher number of time steps is considered.

Second, to give an idea of the computational effort needed to compute option prices, it is worth mentioning that 44617 nodes are needed to obtain the prices $RE(10)$. It may be interesting to observe that this number of nodes corresponds to the number of nodes of a binomial recombining lattice with approximately 298 time steps. The CPU time needed to compute such prices is less than one second. In the case with maturity $T = 1$ year, 127642 nodes are needed to compute the prices $RE(20)$, and this number corresponds to the number of nodes generated by a binomial reconnecting tree with 504 time steps. Also in this case the CPU time is less than 1 second. Hence, in both cases, the application of the Richardson extrapolation allows us to reach higher precision and considerably reduces the computational cost of the multinomial lattice model.

In the case of American options, the above analysis may not be extended straightforwardly because of the presence of the unknown optimal exercise boundary. To investigate on an empirical ground the convergence of the proposed multinomial model, we have conducted several numerical experiments. We observed the same monotonic pattern already shown in the case of plain vanilla European options. As an example, in Figure 4 we illustrate the prices of American put options for a number of time steps $n = 10, 15, 20, \dots, 200$. The contractual parameters are those reported in [5], i.e., the underlying asset

price at inception is $S_0 = 1369.41$, the risk-free interest rate is $r = 0.0541$, the continuous dividend yield is $\delta = 0.012$, and the maturity is $T = 0.56164$ years. Four different strikes, $K = 1200, 1260, 1320, 1380$, are considered while the parameters of the VG process are $\theta = -0.22898, \sigma = 0.20722$ and $v = 0.50215$.

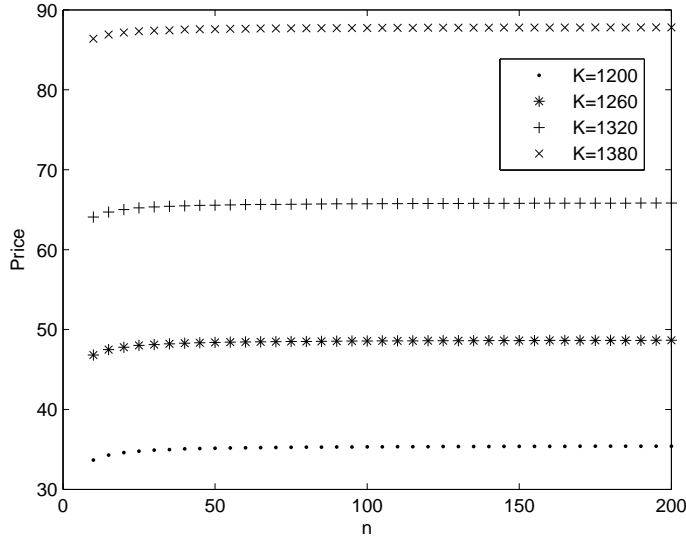


Figure 4: The graph presents the convergence of American put options under the VG process with parameters $\theta = -0.22898, \sigma = 0.20722$ and $v = 0.50215$. The risk-free interest rate is $r = 0.0541$, the continuous dividend yield is $\delta = 0.012$, the underlying asset value at inception is $S_0 = 1369.41$, the maturity is $T = 0.56164$ years, and four different levels of the strike price are considered, $K = 1200, 1260, 1320, 1380$.

Next, to investigate the order of convergence of the multinomial lattice model, we consider the difference ratio, $\rho_d(n) = (AP(n) - AP(2n)) / (AP(2n) - AP(4n))$, where $AP(n)$ stands for the price of an American put option after n time steps. It is easy to show that, if the error ratio, $\rho_e(n)$, converges to a constant, then the difference ratio, $\rho_d(n)$, converges to the same constant. It follows that if $\rho_d(n)$ converges to 2, the proposed lattice model has a convergence of order $1/n$ also in the case of plain vanilla American options, and simple two-point Richardson extrapolation can be used to enhance the rate of convergence. On this basis, we conducted extensive numerical experiments highlighting that

n	K=1200	K=1260	K=1320	K=1380
10	2.0460	2.0106	2.0958	2.0345
20	1.9919	2.0804	1.9913	2.0936
40	2.0692	1.9643	2.0254	2.0245
80	1.9645	2.0872	2.0283	1.9832

Table 4: This table reports the difference ratios, $\rho_d(n) = (AP(n) - AP(2n))/(AP(2n) - AP(4n))$, of plain vanilla American put options for a number of time steps $n = 10, 20, 40, 80$. The underlying asset price at inception is $S_0 = 1369.41$, the risk-free interest rate is $r = 0.0541$, the continuous dividend yield is $\delta = 0.012$, and the maturity is $T = 0.56164$ years. Four different strikes, $K = 1200, 1260, 1320, 1380$ are considered. The parameters of the VG process are $\sigma = 0.20722$, $\theta = -0.22898$ and $\nu = 0.50215$.

the difference ratio of American put options is very close to 2, and extrapolations may be applied to obtain fast and accurate estimates of option prices. As an example, in Table 4, we report the difference ratios for the American put options described above for $n = 10, 20, 40, 80$.

We are in a position now to compute American option prices by applying the two-point Richardson extrapolation. Let $RE_A(n)$ denote the price of an American put option computed using the two-point Richardson extrapolation, i.e., $RE_A(n) = 2AP(2n) - AP(n)$. In Table 5, we report such prices for the same options described above. In the row labeled HM, we report the prices obtained by Hirs and Madan [5] by applying a finite difference scheme. The last row, labeled H, shows the prices obtained by Whitley [16], who implemented the same finite difference scheme proposed in [5]. The percentage difference between the prices presented in [5] and in [16] is approximately 0.1%, which may be due, according to Whitley, to a different choice of the minimum and of the maximum underlying asset price in the grid used for price calculations. By looking at Table 5, it emerges that the prices computed by the proposed multinomial lattice coupled with Richardson extrapolation for $n = 10$ steps have a relative error smaller than 0.1% with respect to the prices obtained both by Hirs and Madan and by Whitley. Increasing the number of time steps, the prices obtained through the lattice based algorithm are very close to those obtained by Whitley.

In Table 6, we compare the option prices obtained by the proposed multinomial lattice model coupled with Richardson extrapolation with those computed

	K=1200	K=1260	K=1320	K=1380
$RE_A(10)$	35.5196	48.7658	65.9589	87.9113
$RE_A(20)$	35.4977	48.7603	65.9117	87.8961
$RE_A(40)$	35.4996	48.7403	65.9138	87.8765
$RE_A(80)$	35.4916	48.7448	65.9107	87.8739
$RE_A(160)$	35.4937	48.7395	65.9090	87.8748
HM	35.5301	48.7976	65.9908	87.9911
W	35.484	48.734	65.906	87.880

Table 5: This table reports the prices for American put options, computed with a two-point Richardson extrapolation, $RE_A(n)$ for $n = 10, 20, 40, 80, 160$. The underlying asset price at inception is $S_0 = 1369.41$, the risk-free interest rate is $r = 0.0541$, the continuous dividend yield is $\delta = 0.012$, and the maturity is $T = 0.56164$ years. Four different strikes, $K = 1200, 1260, 1320, 1380$ are considered. The parameters of the VG process are $\sigma = 0.20722$, $\theta = -0.22898$ and $\nu = 0.50215$. The row labeled HM reports the prices obtained with the finite difference scheme proposed by Hirta and Madan [5], while the last row, labeled W, reports the prices obtained with the same finite difference scheme implemented by Whitley [16].

by the finite difference scheme proposed by Levendorskiĭ et al. [7], reported in the last row labeled LKZ. In the rows labeled HM are reported the prices computed by the numerical scheme of Hirs and Madan [5] as implemented by Levendorskiĭ et al. It emerges that, already with $n = 10$ time steps, the relative error of the prices computed by the multinomial model is at most approximately 0.1% with respect to both the values obtained with the Hirs and Madan model and those computed with the Levendorskiĭ et al. method. The CPU time needed to compute such prices with the lattice based model is less than one second, while the finite difference method of Levendorskiĭ et al. requires 7 seconds and that of Hirs and Madan requires 9 seconds when the coarsest grid considered in [7] is adopted (Levendorskiĭ et al. worked on a PC with 1.8 GHz, 256 MB under Windows XP).

In Table 7, we illustrate the prices of American put options computed with the multinomial lattice coupled with Richardson extrapolation. The prices computed with $n = 10$ time steps are not considered because the jump probability, $U(\varepsilon)\Delta t$, falls outside the interval $[0, 1]$. The prices in the row labeled Exact are computed with the CONV method proposed by Lord et al. [9] as reported in the paper of Hilber et al. [4]. In all the cases considered the prices computed with the proposed lattice-based method are highly accurate.

4. Conclusions

We have considered the problem of evaluating European and American plain vanilla options when the log price of the underlying asset is driven by a VG process. Invoking the Lévy-Ito decomposition, a VG process may be described as the sum of a deterministic drift plus its jumps. A problem arises because the Lévy measure of the process has a singularity in the origin, but because it has finite variation, we may truncate the process in a small interval around zero and approximate the small jumps with their expected value. As a result, a compound Poisson process may be easily derived that converges to the target VG process as the truncation interval shrinks. We have developed a lattice model to approximate the compound Poisson process and have used it as the corner stone to build up a lattice that describes the dynamics of a certain stock price. Then, we have evaluated European and American plain vanilla options by applying the usual backward induction. Numerical results have shown a virtually monotonic convergent pattern, and on an empirical ground, we can assert that the rate of convergence is of order $1/n$. This result allows us to apply a simple two-point Richardson extrapolation to enhance the rate of convergence.

	S_0				
	1176.29	1200.05	1234.13	1259.06	1274.26
$RE_A(10)$	134.6704	120.7061	103.5139	92.7628	86.8814
$RE_A(20)$	134.6875	120.6766	103.4877	92.7616	86.8407
$RE_A(40)$	134.6893	120.6710	103.4957	92.7344	86.8245
$RE_A(80)$	134.6814	120.6677	103.4859	92.7378	86.8289
$RE_A(160)$	134.6832	120.6667	103.4877	92.7366	86.8278
HM	134.822	120.775	103.567	92.8015	86.8861
LKZ	134.794	120.756	103.546	92.7745	86.8546

	S_0				
	1300	1326.26	1353.05	1402.65	1436.72
$RE_A(10)$	77.8571	69.7509	62.4989	51.2508	44.9094
$RE_A(20)$	77.8153	69.7183	62.4621	51.2409	44.8874
$RE_A(40)$	77.8069	69.7118	62.4505	51.2219	44.8785
$RE_A(80)$	77.8016	69.7066	62.4479	51.2228	44.8750
$RE_A(160)$	77.8004	69.7038	62.4480	51.2204	44.8736
HM	77.8496	69.7482	62.4849	51.2532	44.9042
LKZ	77.8103	69.7011	62.4309	51.1894	44.8358

Table 6: This table reports the prices for American put options, computed with a two-point Richardson extrapolation, $RE_A(n)$ for $n = 10, 20, 40, 80$. The strike price is $K = 1300$, the risk-free interest rate is $r = 0.0541$, the continuous dividend yield is $\delta = 0$, and the maturity is $T = 0.56164$ years. Ten different value of the asset price at inception are considered. The parameters of the VG process are $\sigma = 0.20722$, $\theta = -0.22898$ and $v = 0.50215$. The rows labeled HM report the prices obtained with the finite difference scheme proposed by Hirs and Madan [5], while the rows labeled LKZ report the prices obtained with the finite difference scheme proposed by Levendorskiĭ et al. [7].

	$S_0 = 8$	$S_0 = 9$	$S_0 = 10$	$S_0 = 11$	$S_0 = 12$
$RE_A(20)$	2.060171	1.326301	0.800681	0.471244	0.279438
$RE_A(40)$	2.060182	1.326370	0.800636	0.471236	0.279393
$RE_A(80)$	2.060176	1.326392	0.800605	0.471228	0.299393
$RE_A(160)$	2.060162	1.326358	0.800583	0.471212	0.279393
Exact	2.060179	1.326384	0.800601	0.471222	0.279391

Table 7: This table reports the prices for American put options, computed with a two-point Richardson extrapolation, $RE_A(n)$ for $n = 10, 20, 40, 80, 160$. The strike price is $K = 10$, the risk-free interest rate is $r = 0.06$, the continuous dividend yield is $\delta = 0$, and the maturity is $T = 0.5$ years. Five different asset price, $S_0 = 8, 9, 10, 11, 12$ are considered. The parameters of the VG process are $\sigma = \sqrt{1/8}$, $\theta = 0$ and $v = 0.25$. The row labeled Exact reports the prices calculated using the CONV method described in Lord et al. [9].

As a result, fast and accurate estimates of the option prices have been obtained. The proposed lattice-based model is flexible enough to be applied also in the case of pure jump Lévy processes with infinite variation such as the normal inverse Gaussian process and this will be a goal for future research.

References

- [1] A. Almendral, C.W. Oosterlee, On American options under the variance gamma process, *Appl. Math. Financ.*, **14** (2007), 131-152.
- [2] K.I. Amin, Jump diffusion option valuation in discrete time, *J. Financ.*, **48** (1993), 1833-1863.
- [3] P. Carr, H. Geman, D.B. Madan, M. Yor, The fine structure of asset returns: An empirical investigation, *J. Bus.*, **75** (2002), 305-332.
- [4] N. Hilber, N. Reich, C. Schwab, C. Winter, Numerical methods for Lévy processes, *Financ. Stoch.*, **13** (2009), 471-500.
- [5] A. Hirs, D. B. Madan, Pricing american options under variance gamma, *J. Comput. Financ.*, **7** (2003), 63-80.

- [6] E. Kellezi, N. Webber, Valuing bermudan options when asset returns are Lévy processes, *Quant. Financ.*, **4** (2004), 87-100.
- [7] S. Levendorskii, O. Kudryavtsev, V. Zherder, Pricing of american options under Lévy processes, *J. Comput. Financ.*, **9** (2006), 69-97.
- [8] D.B. Madan, P. Carr, E.C. Chang, The variance gamma process and option pricing, *European Finance Review*, **2** (1998), 79-105.
- [9] R. Lord, F. Fang, F. Bervoets, C.W. Oosterlee, A fast and accurate FFT based method for pricing early-exercise options under Lévy processes. *SIAM J. Sci. Comput.*, **30** (2008), 1678-1705.
- [10] D.B. Madan, F. Milne, Option pricing with VG martingale components, *Math. Financ.*, **1** (1991), 39-55.
- [11] D.B. Madan, E. Seneta, The variance gamma (V.G.) model for share market returns, *J. Bus.*, **63** (1990), 511-524.
- [12] R.A. Maller, D.H. Solomon, A. Szimayer, A multinomial approximation for american option prices in Lévy process models, *Math. Financ.*, **16** (2006), 613-633.
- [13] N. Rambeerich, D.Y. Tangman, M. Bhuruth, Numerical pricing of American options under infinite activity Lévy processes, *J. Fut. Mark.*, **31** (2011), 809-829.
- [14] I. Wang, J. Wan, P. Forsyth, Robust numerical valuation of European and American options under the CGMY process, *J. Comput. Financ.*, **10** (2007), 31-69.
- [15] Y.S. Tian, A flexible binomial option pricing model, *J. Fut. Mark.*, **19** (1999), 817-843.
- [16] A. Whitley, *Pricing of European, Bermudan and American Options under the Exponential Variance Gamma Process*. Master's Thesis, Mathematical Institute, University of Oxford (2009). Available online at <http://eprints.maths.ox.ac.uk/786/>.