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ON OSCILLATORY MOTION OF A GENERALIZED MHD OLDROYD-B FLUID

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Abstract: We present an analysis for the flow of an incompressible generalized magnetohydrodynamic (MHD) Oldroyd-B fluid with constant pressure gradient. Fractional derivative is used in the governing equation. Exact analytic solutions are obtained for the velocity field and shear stress in series form in terms of Fox H-functions by means of the Fourier sine transform and discrete Laplace transform. All the imposed initial and boundary conditions are satisfied by the obtained solutions.

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Key Words: generalized Oldroyd-B fluid, fractional derivatives, exact solutions, Fox H-functions, discrete Laplace transform

1. Introduction

The study of fluid motions due to spinning or oscillating bodies have received much attention due to its importance not only to the field of academics but also to the industry. Such motions have many applications in many industrial and biological processes such as food industry, oil exploitation, the periodicity

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of blood flow in the cardiovascular system [6], chemistry and bio-engineering, soap and cellulose solutions etc. Unsteady flows of viscoelastic fluids due to the oscillations of a rigid flat plate is of considerable interest. One of rate type viscoelastic fluids is the Oldroyd-B model which seems to be amenable to analysis and more important to experiments, sometime used as a test to check the performance of numerical methods for the computation of different flows.

Recently, the fractional derivative approach (see [8]) is proving to be an important tool for describing the behaviors of such types of fluids. Many researchers studied different problems related to such fluids. In their works, the time derivatives of an integer order in the constitutive equations for generalized Oldroyd-B fluids are replaced by the so-called Riemann-Liouville fractional derivatives. Qi and Xu [7] investigated the Stokes problem for a viscoelastic fluid with a generalized Oldroyd-B model. Hyder discussed the flows of generalized Oldroyd-B fluid between two side walls perpendicular to the plate. Fetecau et al. [2, 3, 4, 5] investigated some accelerated flows of a generalized Oldroyd-B fluid. Khan et al. [12] studied the flow of generalized Oldroyd-B fluid between two side walls. Moreover, MHD flows have wide converged on the development of energy generation and in astrophysical and geophysical fluid dynamics. Recently, the theory of MHD has received much attention, see [10, 11] and reference therein. In this paper, we consider the MHD flow of an incompressible generalized Oldroyd-B fluid. Exact solutions for the velocity field and shear stress are obtained by using the Fourier sine transform and Laplace transform technique for the fractional calculus operators. The obtained solutions satisfy all the imposed initial and boundary conditions.

2. Governing Equations

For an incompressible and unsteady generalized Oldroyd-B fluid the constitutive equation is given as [5]:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}; \qquad (1 + \lambda \frac{D^\alpha}{Dt^\alpha})\mathbf{S} = \mu(1 + \theta \frac{D^\beta}{Dt^\beta})\mathbf{A}, \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{S}$ is the extra stress tensor, p$\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Ericksen tensor, μ is the dynamic viscosity, λ and θ are relaxation and retardation times, α and β are the fractional calculus parameters such that $0 \leq \alpha \leq \beta \leq 1$, and

$$\frac{D^\alpha \mathbf{S}}{Dt^\alpha} = D_t^\alpha + (\mathbf{V}.\nabla)\mathbf{S} - \mathbf{LS} - \mathbf{SL},$$

$$\frac{D^\beta \mathbf{A}}{Dt^\beta} = D_t^\beta + (\mathbf{V}.\nabla)\mathbf{A} - \mathbf{LA} - \mathbf{AL}. \tag{2}$$

In the above relations $\mathbf{V}$ is the velocity, ∇ is the gradient operator, D_t^α and D_t^β are the fractional differentiation operators of order α and β based on the Riemann-Liouville definition, defined as (see e.g. [8])

$$D_t^p[f(t)] = \frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^p}d\tau, \qquad 0 \le p \le 1, \tag{3}$$

where $\Gamma(\cdot)$ is the Gamma function. The model reduces to the ordinary Oldroyd-B model when $\alpha = \beta = 1$. In the following we shall determine a velocity field and an extra stress of the form

$$\mathbf{V} = u(y,t)\mathbf{i},\ \mathbf{S} = S(y,t), \tag{4}$$

where u is the velocity and $\mathbf{i}$ is the unit vectors in the x-direction. Substituting Eq.(4) into Eq.(1) and taking account of the initial condition

$$S(y,0) = 0,\ y > 0, \tag{5}$$

the fluid being at rest up to the time $t = 0$, we get

$$(1 + \lambda D_t^\alpha)S_{xy} = \mu(1 + \theta D_t^\beta)\partial_y u(y,t), \tag{6}$$

where $S_{yy} = S_{zz} = S_{xz} = S_{yz} = 0$, and $S_{xy} = S_{yx}$. The fluid is permeated by an imposed magnetic field B_0 which acts in the positive y-coordinate. In the low-magnetic Reynolds number approximation, the magnetic body force is represented by $\sigma B_0^2 u$. Then, the equation of motion yields the following scalar equation:

$$\partial_y S_{xy} - \partial_x p - \sigma B_0^2 u = \rho \partial_t u, \qquad \partial_y p = \partial_z p = 0, \tag{7}$$

where ρ is the constant density of the fluid and $\partial_x p$ is the pressure gradient along x-axis. Eliminating S_{xy} between Eqs. (6) and (7), we find the governing equation under the form

$$(1+\lambda D_t^\alpha)\partial_t u(y,t) = \nu(1+\theta D_t^\beta)\partial_y^2 u(y,t) - M(1+\lambda D_t^\alpha)u(y,t) + \frac{1}{\rho}(1+\lambda D_t^\alpha)\partial_x P, \tag{8}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid and $M = \sigma B_0^2 u$.

3. Statement of the Problem

Supposed that a generalized Oldroyd-B fluid is occupying the space above a flat plate. Initially the fluid as well as the plate are at rest, and at time $t = 0^+$ the plate oscillate in its plane with the velocity $V\cos(wt)$ or $V\sin(wt)$ (V is a constant). Due to the shear, the fluid is moved gradually. Accordingly, the initial and boundary conditions of velocity field are:

$$u(y,0) = \partial_t u(y,0) = 0 \quad \text{for} \quad y > 0, \tag{9}$$

$$u(0,t) = V\sin(wt) \quad \text{or} \quad V\cos(wt) \quad \text{for} \quad t > 0. \tag{10}$$

Also, the natural conditions have to be satisfied,

$$u(y,t), \quad \partial_y u(y,t) \to 0 \quad \text{as} \quad y \to \infty, \quad \text{and} \quad t > 0. \tag{11}$$

4. Calculation of Velocity Field

Let us employing non-dimensional quantities

$$u^* = \frac{u}{V},\ y^* = \frac{yV}{\nu},\ t^* = \frac{tV}{\nu}\lambda^* = \lambda(\frac{V}{\nu})^{\alpha},\ \theta^* = \theta(\frac{V^2}{\nu})^{\beta},\ M^* = \frac{M\nu}{V^2}.$$

The dimensionless mark * is omitted next for simplicity. Thus, the equation of dimensionless motion becomes

$$(1+\lambda D_t^{\alpha})\partial_t u(y,t) = \nu(1+\theta D_t^{\beta})\partial_y^2 u(y,t) - M(1+\lambda D_t^{\alpha})u(y,t) + \frac{1}{\rho}(1+\lambda D_t^{\alpha})\partial_x P, \tag{12}$$

with the given conditions defined as in Eqs. (9), (10) and (11). In order to solve the above problem first we multiply both sides of Eq. (12) by $\sin(\xi y)$, and integrating the result with respect to y from 0 to ∞ and taking the corresponding initial and boundary conditions, we attain the differential equations

$$\begin{aligned}(1+\lambda D_t^{\alpha})\partial_t u_s(\xi,t) &= \nu(1+\theta D_t^{\beta})(\xi y\sin(wt) - (\xi y)^2 u_s(\xi,t)) \\ &\quad -M(1+\lambda D_t^{\alpha})u_s(\xi,t) - A\frac{1}{\xi}(1+\lambda\frac{t^{-\alpha}}{\Gamma(1-\alpha)})(1-(-1)^n),\end{aligned} \tag{13}$$

where $u_s(\xi,t)$ is the Fourier sine transform of $u(y,t)$ satisfying the initial conditions

$$u_s(\xi,0) = \partial_t u_s(\xi,0) = 0, \quad \xi > 0. \tag{14}$$

Applying the Laplace transform for sequential fractional derivatives to Eq. (13) and using the initial condition Eq. (14), we get

$$\bar{u}_s(\xi,s)=\frac{(S+M)^{-1}}{((1+\lambda s^{\alpha}))+\nu\xi^2(1+\theta s^{\beta})}[\frac{\nu(1+\theta s^{\beta})\xi w}{s^2+w^2}-\frac{A(1+\lambda D_t^{\alpha})}{\xi S}(1-(-1)^n)]. \tag{15}$$

To get the solution for velocity field, first we write Eq. (15) in series form as

$$\begin{aligned}\bar{u}_s(\xi,s)=&\frac{w}{(s^2+w^2)}\sum_{i=0}^{\infty}\sum_{o=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+o+k+l}\xi^{-(2i+1)}\nu^{-i}\lambda^k\theta^l M^o\Gamma(o-i)}{o!k!l!\Gamma(-1)\Gamma(i)\Gamma(i)\Gamma(-i)}\\ &-\frac{\Gamma(k-i)\Gamma(l+i)A(1-(-1)^n)}{s^{-i+o-\alpha k-\beta l}}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{j+m+n+p}\xi^{2j-1}\nu^j}{m!n!p!\Gamma(-1)\Gamma(-j)}\\ &\times\frac{\lambda^n\theta^m M^p\Gamma(m-j)\Gamma(n+j)\Gamma(p+j+1)}{\Gamma(j)\Gamma(-j-1)s^{1+p-\alpha n-\beta m}}.\end{aligned} \tag{16}$$

Now apply the discrete inverse Laplace transform and inverse finite Fourier sine transform (see e.g. [9]) to Eq. (16), we get

$$\begin{aligned}u_s(y,t)=&\frac{2}{\pi}\sum_{\xi=1}^{\infty}\sin(\xi y)\sin(wt)\sum_{i=0}^{\infty}\sum_{o=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{i+o+k+l}\xi^{-(2i+1)}\nu^{-i}\lambda^k\theta^l M^o}{o!k!l!\Gamma(-1)\Gamma(i)\Gamma(i)\Gamma(-i)}\\ &\times\frac{\Gamma(k-i)\Gamma(l+i)t^{-i+o-\alpha k-\beta l-1}}{\Gamma(o-i)\Gamma(-i+o-\alpha k-\beta l)}-\frac{2}{\pi}\sum_{\xi=1}^{\infty}\sin(\xi y)A(1-(-1)^n)\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\\ &\sum_{p=0}^{\infty}\frac{(-1)^{j+m+n+p}\xi^{2j-1}\nu^j\lambda^n\theta^m M^p\Gamma(m-j)\Gamma(n+j)\Gamma(p+j+1)}{t^{p-\alpha n-\beta m}m!n!p!\Gamma(-1)\Gamma(-j)\Gamma(j)\Gamma(-j-1)\Gamma(1+p-\alpha n-\beta m)}.\end{aligned} \tag{17}$$

To get Eq. (17) in a more compact form we use Fox H-function [1],

$$\begin{aligned}u_s(y,t)=&\frac{2}{\pi}\sum_{\xi=1}^{\infty}\sin(\xi y)\sin(wt)\sum_{i=0}^{\infty}\sum_{o=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+o+k}\xi^{-(2i+1)}\nu^{-i}\lambda^k\theta^l M^o}{t^{i-o+\alpha(k)}o!k!}\\ &\times H^{1,3}_{3,6}\left[\frac{\theta}{t^{\beta}}\middle|\begin{matrix}(1-o+i,0),(1-k+i,0),(1-i,1)\\(2,0),(1-i,0),(1-i,0),(1+i,0),(0,1),(1+i-o+\alpha k,-\beta)\end{matrix}\right]\\ &-A(1-(-1)^n)\frac{2}{\pi}\sum_{\xi=1}^{\infty}\sin(\xi y)\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+m+n}\xi^{2j-1}\nu^j\lambda^n\theta^m M^p t^{-\alpha n-\beta m}}{m!n!}\\ &\times H^{1,3}_{3,6}\left[Mt\middle|\begin{matrix}(1-m+j,0),(1-n-j,0),(-j,1)\\(2,0),(1+j,0),(1-j,0),(2+j,0),(0,1),(\alpha n+\beta m,1)\end{matrix}\right].\end{aligned} \tag{18}$$

We use the following property of Fox H-function for obtaining Eq. (18):

$$H^{1,s}_{s,t+1}\left[\sigma \left| \begin{array}{c} (1-a_1,A_1),...,(1-a_s,A_s) \\ (1-b_1,B_1),...,(1-b_t,B_t) \end{array} \right.\right] = \sum_{r=0}^{\infty} \frac{\Gamma(a_1+A_1r)...\Gamma(a_s+A_sr)}{r!\Gamma(b_1+B_1r)...\Gamma(b_t+B_tr)}. \quad (19)$$

5. Calculation of Shear Stress

Taking the Laplace transform of Eq. (6), we get

$$\bar{S}_{xy} = \frac{\mu(1+\theta s^{\beta})}{(1+\lambda s^{\alpha})} \frac{\partial \bar{u}(y,s)}{\partial y}. \quad (20)$$

The image function $\bar{u}(y,s)$ of $u(y,t)$ can be easily obtained from Eq. (19). Introducing then $\bar{u}(y,s)$ into eq. (20), we get

$$\bar{S}_{xy} = \frac{2\mu}{\pi} \frac{(1+\theta s^{\beta})}{1+\lambda s^{\alpha}} \sum_{\xi=1}^{\infty} \xi \cos(\xi y) \frac{1}{(S+M)((1+\lambda s^{\alpha})) + \nu\xi^2(1+\theta s^{\beta})}$$
$$\times \left[\frac{\nu(1+\theta s^{\beta})\xi w}{s^2+w^2} - \frac{A(1+\lambda s^{\alpha})}{\xi S}(1-(-1)^n)\right]. \quad (21)$$

To get a more compact form we write eq. (21) in series form and then take the inverse Laplace transform, the obtained result expressed in Fox H-function is given by

$$S_{xy} = \frac{2\mu}{\pi} \sum_{\xi=1}^{\infty} \cos(\xi y) \sin(wt) \sum_{i=0}^{\infty} \sum_{o=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+o+k} \xi^{-2i} \nu^{-i} \lambda^k M^o t^{-i+o-\alpha k}}{o!k!}$$
$$\times H^{1,3}_{3,6}\left[\frac{\theta}{t^{\beta}} \left| \begin{array}{c} (1-o+i,0),(1-k+i,0),(1-i,1) \\ (2,0),(1-i,0),(1-i,0),(1+i,0),(0,1),(1+i-o+\alpha k,-\beta) \end{array} \right.\right]$$
$$- A(1-(-1)^n) \frac{2\mu}{\pi} \sum_{\xi=1}^{\infty} \cos(\xi y) \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{j+m+n} \xi^{2j-1} \nu^j \lambda^n \theta^m t^{-\alpha n-\beta m}}{m!n!}$$
$$\times H^{1,3}_{3,6}\left[Mt \left| \begin{array}{c} (1-m+j,0),(1-n-j,0),(-j,1) \\ (2,0),(1+j,0),(1-j,0),(2+j,0),(0,1),(\alpha n+\beta m,1) \end{array} \right.\right]. \quad (22)$$

References

[1] A.M. Mathai, R.K. Saxena and H.J. Haubold, *The H-functions: Theory and Applications.* Springer, New York (2010).

[2] C. Fetecau, S.C. Prasad and K.R. Rajagopal, A note on the flow induced by a constantly accelerating plate in an Oldroyd-B fluid, *Applied Mathematical Modeling*, **31** (2007), 647-654.

[3] C. Fetecau, C. Fetecau, M. Kamran and D. Vieru, Exact solutions for the flow of a generalized Oldroyd-B fluid induced by a constantly accelerating plate between two side walls perpendicular to the plate, *J. Non-Newtonian Fluid Mech*, **156** (2009), 189-201.

[4] C. Fetecau, M. Athar and C. Fetecau, Unsteady flow of a generalized Maxwell fluid with fractional derivative due to a constantly accelerating plate, *Computers and Mathematics with Applications*, **57** (2009), 596-603.

[5] D. Vieru, C. Fetecau and C. Fetecau, Flow of a generalized Oldroyd-B fluid due to a constantly accelerating plate, *Applied Mathematics and Computation*, **201** (2008), 834-842.

[6] G. Zaman, S. Islam, Y.H. Kang and I.H. Jung, Blood flow of an Oldroyd-B fluid in a blood vessel incorporating a Brownian stress. *Sci. Chi. Phy. Mech & Ast.*, **55** (2012), 125-131.

[7] H.T. Qi and M.Y. Xu, Stokes first problem for a viscoelastic fluid with the generalized Oldroyd-B model, *Acta Mech. Sin.*, **23** (2007), 463-469.

[8] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York (1999).

[9] I.N. Sneddon, *Fourier Transforms*, McGraw-Hill, New York (1951).

[10] K.D. Singh, B.P. Garg, Exact solution of an oscillatory free convective MHD flow in a rotating porous channel with radiative heat. *Proc. Nat. Acad. Sci. India*, **80** (2010), 81-89.

[11] K.D. Singh and P. Reena, An analysis of an oscillatory rotating MHD Poiseuille flow with injection/suction and Hall currents. *Proc. Indian Natn. Sci. Acad.*, **76** (2010), 201-207.

[12] M. Khan, T. Hayat, S. Asghar, Exact solution for MHD flow of a generalized Oldroyd-B fluid with modified Darcy law, *International Journal of Engineering Science*, **44** (2006), 333-339.