International Journal of Applied Mathematics

Volume 27 No. 5 2014, 437-453

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

doi: http://dx.doi.org/10.12732/ijam.v27i5.2

SOLUTION OF CERTAIN FRACTIONAL KINETIC EQUATIONS

Ahmad Shafee[1], Leda Galué[2], Shyam Kalla[3 §]

[1]Public Authority of Applied Education & Training
KUWAIT

[2]Centro de Investigación de Matemática Aplicada
Facultad de Ingeniería, Universidad del Zulia
VENEZUELA

[3]Vyas Institutes of Higher Education
Jodhpur, INDIA

[3]Visiting Professor (Nov.-Dec. 2013) at Department of Mathematics
Kuwait, University, KUWAIT

Abstract: Several authors have treated fractional kinetic equations and obtained their solutions, using different methods. Recently Saxena and Kalla have obtained solution of fractional kinetic equation, using Laplace transform. In this paper a correct version of their solution and examples are given. Further, a more general form of fractional kinetic equation and its solution are introduced, and an alternative method to obtain solution of such equations is provided. The results presented are in compact form and convenient for numerical computation. Furthermore some special cases and examples are obtained, and various figures to illustrate the behavior of the solution are given.

AMS Subject Classification: 26A33, 33C65

Key Words: fractional differential equations, Laplace transform, convolution, generalized hypergeometric function, Mittag-Leffler function

Received: July 12, 2014

§Correspondence author

1. Introduction

Fractional kinetic equations are studied to determine certain physical phenomena governing diffusion in porous media, reaction and relaxation in complex systems, anomalous diffusion etc. In this connection, one can refer to the books by Hilfer [5], Kiryakova [7], Podlubny [12], and other works in this field. Fractional kinetic equations are studied by Hille and Tamarkin [6], Glöckle and Nonnenmacher [3], Saichev and Zaslavsky [13], Saxena et al. [15-17], and Zaslavsky [23], among others for their importance in the solution of certain applied problems.

Saxena et al. [18-19] have investigated the solution of certain fractional differ-integral equations, related to reaction diffusion equations. Transform technique is used and solutions are expressed in terms of the Mittag-Leffler and H-functions [10, 8].

Saxena and Kalla [14] have obtained solution of a generalized fractional kinetic equation and gave some examples. Here we give a correct version of their solution and examples given by them. We introduce a more general form of fractional kinetic equation and obtain its solution using Laplace transform. An alternative method of deriving solutions of such equations is also given.

2. The Generalized Hypergeometric Function

The generalized hypergeometric function ${}_qF_q$ is defined by [21, p. 19-20]

$$ {}_pF_q\left(A_1, \cdots, A_p; B_1, \cdots, B_q; z\right) = \sum_{k=0}^{\infty} \frac{(A_1)_k \cdots (A_p)_k}{(B_1)_k \cdots (B_q)_k} \frac{z^k}{k!}, \tag{1} $$

where

$$ (a)_k = \begin{cases} 1, & \text{if } k = 0, \\ \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)(a+2)\cdots(a+k-1), & \text{if } k = 1, 2, 3, \cdots. \end{cases} $$

p and q are positive integer or zero, the variable z, the numerator parameters $A_1, \cdots, A_p$, and the denominator parameters $B_1, \cdots, B_q$ take on complex values, provided that $B_j \neq 0, -1, -2, \cdots;\ j = 1, \cdots, q$.

The ${}_pF_q$ series:

(i) converges for $|z| < \infty$ if $p \leq q$,

(ii) converges for $|z| < 1$ if $p = q + 1$, and

(iii) diverges for all z, $z \neq 0$, if $p > q + 1$.

As particular cases of (1) we have,

$$e^z = {}_0F_0(-;-;z). \tag{2}$$

$$(1-z)^{-a} = {}_1F_0(a;-;z). \tag{3}$$

3. Mittag-Leffler Function

In 1903, the Swedish mathematician Gosta Mittag-Leffler [10] defined and studied a function, now called as Mittag-Leffler function and defined as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \mathrm{Re}(\alpha) > 0. \tag{4}$$

In 1905, this function was generalized by Wiman [22] as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0. \tag{5}$$

The integral representation of this function is as follows:

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(s)\Gamma(1-s)}{F(\beta - \alpha s)} (-z)^{-s} ds, \tag{6}$$

where the path of integral separates all poles of $\Gamma(s)$ and $\Gamma(1-s)$. This representation can be used to express the Mittag-Leffler function in terms of H-function.

4. Mellin-Ross Function

It is defined as [9],

$$E_t(\nu, a) = t^\nu \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k + \nu + 1)} = \frac{t^\nu}{\Gamma(\nu + 1)} \ {}_1F_1(1; \nu + 1; at). \tag{7}$$

5. Riemann-Liouville Operator

Many authors have studied operators of fractional calculus, due to their applications in solving integral and differintegral equations and fractional order models.

The Riemann-Liouville fractional integral operator is defined by [9], [11], [12]:

$$ {}_0D_x^{-\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} f(t)\, dt, \quad \mathrm{Re}(\gamma) > 0,\ x > 0, \tag{8} $$

whereas the Riemann-Liouville fractional differential operator is defined by

$$ {}_0D_x^{\mu} f(x) = D^n \left[{}_0D_x^{\mu-n} f(x) \right], \quad \mathrm{Re}(\mu) \geq 0, x > 0, n = [\mu] + 1. \tag{9} $$

For $f(t) = t^\delta$ we have ([9], [11], [12])

$$ {}_0D_t^{-\gamma} t^\delta = \frac{\Gamma(\delta+1)}{\Gamma(\delta+\gamma+1)} t^{\gamma+\delta}, \quad \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > -1, t > 0 \tag{10} $$

and

$$ {}_0D_t^{\mu} t^\delta = \frac{\Gamma(\delta+1)}{\Gamma(\delta-\mu+1)} t^{\delta-\mu}, \quad \mathrm{Re}(\mu) \geq 0, \mathrm{Re}(\delta) > -1, t > 0. \tag{11} $$

Now, we establish some results which will be used next in the paper.

From (1) and (8), for $\mathrm{Re}(\gamma) > 0$

$$ {}_0D_t^{-\gamma} \left\{ t^{\delta-1}\ {}_pF_q\left(A_1, \cdots, A_p; B_1, \cdots, B_q; at\right) \right\} = \frac{t^{\gamma-1}}{\Gamma(\gamma)} \times \sum_{k=0}^{\infty} \frac{(A_1)_k \cdots (A_p)_k}{(B_1)_k \cdots (B_q)_k} \frac{a^k}{k!} \int_0^t \left(1 - \frac{u}{t}\right)^{\gamma-1} u^{k+\delta-1} du, $$

and now, making a simple change of variable and evaluating the resulting integral, we obtain

$$ {}_0D_t^{-\gamma} \left\{ t^{\delta-1}\ {}_pF_q\left(A_1, \cdots, A_p; B_1, \cdots, B_q; at\right) \right\} = t^{\gamma+\delta-1} \times \sum_{k=0}^{\infty} \frac{(A_1)_k \cdots (A_p)_k\ \Gamma(\delta+k)}{(B_1)_k \cdots (B_q)_k\ \Gamma(\delta+\gamma+k)} \frac{a^k t^k}{k!}, \ \mathrm{Re}(\delta) > 0, $$

that is,

$$ {}_0D_t^{-\gamma} \left\{ t^{\delta-1}\ {}_pF_q\left(A_1, \cdots, A_p; B_1, \cdots, B_q; at\right) \right\} = \frac{\Gamma(\delta)}{\Gamma(\delta+\gamma)} t^{\delta+\gamma-1} \times {}_{p+1}F_{q+1}\left(A_1, \cdots, A_p, \delta; B_1, \cdots, B_q, \delta+\gamma; at\right), \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0. \tag{12} $$

Then, from (9) and (12), for $\mathrm{Re}(\mu) \geq 0$:

$$
{}_0D_t^{\mu}\left\{t^{\delta-1}\ {}_pF_q\left(A_1,\cdots,A_p;B_1,\cdots,B_q;at\right)\ \right\} =
\frac{d^n}{dt^n}\left\{\frac{\Gamma(\delta)}{\Gamma(\delta+n-\mu)}t^{\delta+n-\mu-1}\ {}_{p+1}F_{q+1}\left(A_1,\cdots,A_p,\delta;B_1,\cdots,B_q,\delta+n-\mu;at\right)\ \right\},
$$

with $\mathrm{Re}(n-\mu) > 0$.

Using (1), we get

$$
\begin{aligned}
{}_0D_t^{\mu}\left\{t^{\delta-1}\ {}_pF_q\left(A_1,\cdots,A_p;B_1,\cdots,B_q;at\right)\ \right\} &= \frac{\Gamma(\delta)}{\Gamma(\delta+n-\mu)} \\
&\times\sum_{k=0}^{\infty}\frac{(A_1)_k\cdots(A_p)_k(\delta)_k}{(B_1)_k\cdots(B_q)_k(\delta+n-\mu)_k}\frac{a^k}{k!}\frac{d^n}{dt^n}\left\{t^{k+\delta+n-\mu-1}\right\} \\
&= \frac{\Gamma(\delta)}{\Gamma(\delta-\mu)}t^{\delta-\mu-1}_{p+1}F_{q+1}\left(A_1,\cdots,A_p,\delta;B_1,\cdots,B_q,\delta-\mu;at\right), \\
&\qquad \mathrm{Re}(\mu)\geq 0, \mathrm{Re}(\delta)>0. \quad (13)
\end{aligned}
$$

6. Kinetic Equations

If we integrate the standard kinetic equation

$$
D_tN_i(t) = \frac{d}{dt}N_i(t) = -c_iN_i(t), \qquad c_i > 0, \tag{14}
$$

we obtain [4, p. 58]

$$
N_i(t) - N_i(0) = -c_iD_t^{-1}N_i(t), \tag{15}
$$

where D_t^{-1} is the Riemann-Liouville integral operator (8). Back to Haubold and Mathai [7], i is the number of density of species, $N_i = N_i(t)$ and $N_i(0) = N_0$ is the number density of that species at time $t = 0$.

We have [2, p. 182]

$$
\mathcal{L}\left\{D_t^{-\nu}\varphi(t)\right\} = s^{-\nu}\Phi(s), \tag{16}
$$

where

$$
\mathcal{L}\left\{\varphi(t)\right\} = \Phi(s) = \int_0^{\infty} e^{-st}\varphi(t)dt, \quad \mathrm{Re}(s) > 0. \tag{17}
$$

In 2008 Saxena and Kalla [14] considered the following equation

$$
N(t) - N_0f(t) = -c^{\nu}D_t^{-\nu}N(t), \quad \mathrm{Re}(\nu) > 0,\ c > 0
$$

and obtained its solution as:

$$N(t) = N_0 \sum_{k=0}^{\infty} (-1)^k \frac{c^{k\nu}}{\Gamma(k\nu)} t^{k\nu-1} * f(t), \tag{18}$$

where

$$t^{k\nu-1} * f(t) = \int_0^t (t-u)^{k\nu-1} f(u)du.$$

Result (18) can be proved by using the convolution theorem for Laplace Transform, for more details we can refer to Saxena et al. [16, p. 661].

7. A Generalized Fractional Kinetic Equation

Here we consider the following more general fractional kinetic equation

$$c^{\mu} D_t^{-\mu} N(t) - N_0 f(t) = -c^{\nu} D_t^{-\nu} N(t), \tag{19}$$

with $\mathrm{Re}(\mu) \geq 0, \mathrm{Re}(\nu) > 0, c > 0,\ N_0 > 0$ is a constant.

It follows from (16) that

$$c^{\mu} s^{-\mu} \mathcal{L}\{N(t)\} - N_0 F(s) = -c^{\nu} s^{-\nu} \mathcal{L}\{N(t)\},$$

which implies

$$\mathcal{L}\{N(t)\} = \frac{N_0}{c^{\mu}} \frac{F(s)}{s^{-\mu}\left[1 + c^{\nu-\mu} s^{\mu-\nu}\right]}$$

provided $s > c$, this leads to

$$\begin{aligned}
\mathcal{L}\{N(t)\} &= \frac{N_0}{c^{\mu}} \left[\sum_{k=0}^{\infty} (-1)^k c^{(\nu-\mu)k} s^{(\mu-\nu)k}\right] s^{\mu} F(s) \\
&= \frac{N_0}{c^{\mu}} \left[s^{\mu} F(s) + \sum_{k=1}^{\infty} (-1)^k c^{(\nu-\mu)k} s^{(\mu-\nu)k} s^{\mu} F(s)\right].
\end{aligned} \tag{20}$$

Now, it is necessary to consider $\mathcal{L}\{D_t^{\mu} f(t)\}$:

According to (9)

$$\begin{aligned}
\mathcal{L}\{D_t^{\mu} f(t)\} &= \mathcal{L}\left\{D^n \left[D_t^{\mu-n} f(t)\right]\right\} = \mathcal{L}\left\{D^n \left[D_t^{-(n-\mu)} f(t)\right]\right\} \\
&= \mathcal{L}\{D^n g(t)\},
\end{aligned}$$

where

$$g(t) = D_t^{-(n-\mu)} f(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t (t-u)^{n-\mu-1} f(u)du, \tag{21}$$

$$\mathrm{Re}(n-\mu)>0.$$

It is well known that for $n=1,2,3,\cdots$

$$\mathcal{L}\left\{f^{(n)}(t)\right\}=\mathcal{L}\left\{D^n f(t)\right\}=s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0),$$

therefore,

$$\mathcal{L}\left\{D_t^\mu f(t)\right\}=s^n\mathcal{L}\left\{g(t)\right\}-s^{n-1}g(0)-\cdots-g^{(n-1)}(0). \tag{22}$$

Also,

$$\mathcal{L}\left\{g(t)\right\}=\mathcal{L}\left\{D_t^{-(n-\mu)}f(t)\right\}=s^{-(n-\mu)}F(s),$$

where we have used (16).

Replacing the previous result in (22) yields

$$\mathcal{L}\left\{D_t^\mu f(t)\right\}=s^\mu F(s)-s^{n-1}g(0)-\cdots-g^{(n-1)}(0),$$

and assuming that $g(t)$ satisfies the condition

$$g(0)=g'(0)=g''(0)=\cdots=g^{(n-1)}(0)=0, \tag{23}$$

we get that

$$\mathcal{L}\left\{D_t^\mu f(t)\right\}=s^\mu F(s). \tag{24}$$

Now, applying the inverse Laplace transform to (20) and using (24), we obtain

$$N(t)=\frac{N_0}{c^\mu}\left[D_t^\mu f(t)+\sum_{k=1}^{\infty}(-1)^k c^{(\nu-\mu)k}\frac{t^{(\nu-\mu)k-1}}{\Gamma((\nu-\mu)k)}*D_t^\mu f(t)\right]$$

with

$$\frac{t^{(\nu-\mu)k-1}}{\Gamma((\nu-\mu)k)}*D_t^\mu f(t)=\frac{1}{\Gamma((\nu-\mu)k)}\int_0^t(t-u)^{(\nu-\mu)k-1}D_u^\mu f(u)\,du$$

$$=D_t^{-(\nu-\mu)k}\left\{D_t^\mu f(t)\right\},\ \mathrm{Re}(\nu-\mu)>0, k=1,2,3,\cdots\ .$$

Then the solution of (19) has the representation

$$N(t)=\frac{N_0}{c^\mu}\left[D_t^\mu f(t)+\sum_{k=1}^{\infty}(-1)^k c^{(\nu-\mu)k}D_t^{-(\nu-\mu)k}\left\{D_t^\mu f(t)\right\}\right], \tag{25}$$

$$\mathrm{Re}(\mu)\geq 0, \mathrm{Re}(\nu-\mu)>0.$$

For $\mu = 0$, we have

$$N(t) = N_0 \left[f(t) + \sum_{k=1}^{\infty} (-1)^k c^{\nu k} D_t^{-\nu k} f(t) \right], \quad \mathrm{Re}(\nu) > 0, \tag{26}$$

and this is the correct form of the solution of Saxena and Kalla [14].

8. Special Cases

Some special cases follow from (25):

If $f(t) = t^{\delta-1}_p F_q\left(A_1, \cdots, A_p; B_1, \cdots, B_q; at\right)$, from (21) and (12):

$$g(t) = \frac{\Gamma(\delta)\ t^{\delta+n-\mu-1}}{\Gamma(\delta+n-\mu)}\ {}_{p+1}F_{q+1}\left(A_1, \cdots, A_p, \delta; B_1, \cdots, B_q, \delta+n-\mu; at\right),$$

therefore

$$g'(t) = \frac{\Gamma(\delta)\ t^{\delta+n-\mu-2}}{\Gamma(\delta+n-\mu-1)}{}_{p+1}F_{q+1}\left(A_1, \cdots, A_p, \delta; B_1, \cdots, B_q, \delta+n-\mu-1; at\right),$$

$$g''(t) = \frac{\Gamma(\delta)\ t^{\delta+n-\mu-3}}{\Gamma(\delta+n-\mu-2)}{}_{p+1}F_{q+1}\left(A_1, \cdots, A_p, \delta; B_1, \cdots, B_q, \delta+n-\mu-2; at\right),$$

$$g^{(n-1)}(t) = \frac{\Gamma(\delta)\ t^{\delta-\mu}}{\Gamma(\delta-\mu+1)}{}_{p+1}F_{q+1}\left(A_1, \cdots, A_p, \delta; B_1, \cdots, B_q, \delta-\mu+1; at\right),$$

and taking $\mathrm{Re}(\delta-\mu) > 0$, the condition (23) is satisfied.

Applying successively (13) and (12)

$$D_t^{-(\nu-\mu)k} D_t^{\mu} \left\{ t^{\delta-1}\ {}_pF_q\left(A_1, \cdots, A_p; B_1, \cdots, B_q; at\right) \right\} =$$

$$\frac{\Gamma(\delta)}{\Gamma(\delta-\mu)} D_t^{-(\nu-\mu)k} \left\{ t^{\delta-\mu-1}\ {}_{p+1}F_{q+1}\left(A_1, \cdots, A_p, \delta; B_1, \cdots, B_q, \delta-\mu; at\right) \right\} =$$

$$\frac{\Gamma(\delta)\ t^{(\nu-\mu)k+\delta-\mu-1}}{\Gamma((\nu-\mu)k+\delta-\mu)}\ {}_{p+1}F_{q+1}\left(A_1, \cdots, A_p, \delta; B_1, \cdots, B_q, (\nu-\mu)k+\delta-\mu; at\right),$$

$$\mathrm{Re}(\mu) \geq 0, \mathrm{Re}(\delta-\mu) > 0, \mathrm{Re}(\nu-\mu) > 0, \mathrm{Re}(\delta) > 0, k = 1, 2, 3, \cdots\ .$$

Then, substituting this expression in (25) and using (13), we get

$$N(t) = \frac{N_0}{c^\mu}\Gamma(\delta)t^{\delta-\mu-1}\left[\frac{1}{\Gamma(\delta-\mu)}\ {}_{p+1}F_{q+1}\left(A_1,\cdots,A_p,\delta;B_1,\cdots,B_q,\delta-\mu;at\right)+\right.$$

$$\left.\sum_{k=1}^{\infty}\frac{(-1)^k(ct)^{(\nu-\mu)k}}{\Gamma((\nu-\mu)k+\delta-\mu)}\ {}_{p+1}F_{q+1}\left(A_1,\cdots,A_p,\delta;B_1,\cdots,B_q,(\nu-\mu)k+\delta-\mu;at\right),\right.$$

that is,

$$N(t) = \frac{N_0}{c^\mu}\Gamma(\delta)t^{\delta-\mu-1}\left[\frac{1}{\Gamma(\delta-\mu)}\ {}_{p+1}F_{q+1}\left(A_1,\cdots,A_p,\delta;B_1,\cdots,B_q,\delta-\mu;at\right)-\right.$$

$$(ct)^{\nu-\mu}\sum_{k=0}^{\infty}\frac{(-1)^k(ct)^{(\nu-\mu)k}}{\Gamma((\nu-\mu)k+\nu+\delta-2\mu)}\quad\times$$

$$\left.{}_{p+1}F_{q+1}\left(A_1,\cdots,A_p,\delta;B_1,\cdots,B_q,(\nu-\mu)k+\nu+\delta-2\mu;at\right)\ \right], \tag{27}$$

$$\mathrm{Re}(\mu)\geq 0, \mathrm{Re}(\delta-\mu)>0, \mathrm{Re}(\nu-\mu)>0, \mathrm{Re}(\delta)>0.$$

Particular Cases of (27):

(I) If $A_j = 0$ for any $j = 1,\cdots,p$, then for $f(t) = t^{\delta-1}$:

$$\begin{aligned} N(t) &= \frac{N_0}{c^\mu}\Gamma(\delta)t^{\delta-\mu-1}\left[\frac{1}{\Gamma(\delta-\mu)}-\right. \\ &\left.(ct)^{\nu-\mu}\sum_{k=0}^{\infty}\frac{(-1)^k(ct)^{(\nu-\mu)k}}{\Gamma((\nu-\mu)k+\nu+\delta-2\mu)}\right] \\ &= \frac{N_0}{c^\mu}\Gamma(\delta)t^{\delta-\mu-1}\left[\frac{1}{\Gamma(\delta-\mu)}-\right. \\ &\left.(ct)^{\nu-\mu}E_{\nu-\mu,\nu+\delta-2\mu}(-c^{\nu-\mu}t^{\nu-\mu})\right], \end{aligned} \tag{28}$$

$$\mathrm{Re}(\mu)\geq 0, \mathrm{Re}(\delta-\mu)>0, \mathrm{Re}(\nu-\mu)>0, \mathrm{Re}(\delta)>0,$$

and for $\mu = 0$, we obtain

$$N(t) = N_0\Gamma(\delta)t^{\delta-1}\left[\frac{1}{\Gamma(\delta)}-(ct)^\nu E_{\nu,\nu+\delta}(-c^\nu t^\nu)\right], \tag{29}$$

$$\mathrm{Re}(\delta)>0, \mathrm{Re}(\nu)>0.$$

Now, if $\delta = 1$ in (28), then for $f(t) = 1$

$$N(t) = \frac{N_0}{c^\mu} t^{-\mu} \left[\frac{1}{\Gamma(1-\mu)} - (ct)^{\nu-\mu} E_{\nu-\mu,\nu-2\mu+1}(-c^{\nu-\mu} t^{\nu-\mu}) \right], \tag{30}$$

$$0 \leq \mathrm{Re}(\mu) < 1, \mathrm{Re}(\nu-\mu) > 0,$$

this for $\mu = 0$ produces

$$N(t) = N_0 \ \left[1 - (ct)^\nu E_{\nu,\nu+1}(-c^\nu t^\nu) \right], \quad \mathrm{Re}(\nu) > 0. \tag{31}$$

(II) If $p = q = 0$, then for $f(t) = t^{\delta-1} \ {}_0F_0(-;-;-at) = \ t^{\delta-1} e^{-at}$:

$$\begin{aligned} N(t) = \frac{N_0}{c^\mu} \Gamma(\delta) t^{\delta-\mu-1} \Bigg[& \frac{1}{\Gamma(\delta-\mu)} \ {}_1F_1\left(\delta; \delta-\mu; -at\right) - \\ & (ct)^{\nu-\mu} \sum_{k=0}^{\infty} \frac{(-1)^k (ct)^{(\nu-\mu)k}}{\Gamma((\nu-\mu)k+\nu+\delta-2\mu)} \quad \times \\ & {}_1F_1(\delta; (\nu-\mu)k+\nu+\delta-2\mu; -at) \Bigg], \end{aligned} \tag{32}$$

$$\mathrm{Re}(\mu) \geq 0, \mathrm{Re}(\delta-\mu) > 0, \mathrm{Re}(\nu-\mu) > 0, \mathrm{Re}(\delta) > 0.$$

Making $\mu = 0$,

$$\begin{aligned} N(t) = N_0 \Gamma(\delta) t^{\delta-1} \Bigg[& \frac{1}{\Gamma(\delta)} e^{-at} - (ct)^\nu \quad \times \\ & \sum_{k=0}^{\infty} \frac{(-1)^k (ct)^{\nu k}}{\Gamma(\nu k+\nu+\delta)} \ {}_1F_1(\delta; \nu k+\nu+\delta; -at) \Bigg], \end{aligned} \tag{33}$$

$$\mathrm{Re}(\delta) > 0, \mathrm{Re}(\nu) > 0.$$

(III) If $p = 1, q = 0$, then for $f(t) = t^{\delta-1} \ {}_1F_0(\rho; -; -at) \ = t^{\delta-1}(1+at)^{-\rho}$:

$$\begin{aligned} N(t) = \frac{N_0}{c^\mu} \Gamma(\delta) t^{\delta-\mu-1} \Bigg[& \frac{1}{\Gamma(\delta-\mu)} \ {}_2F_1\left(\rho, \delta; \delta-\mu; -at\right) - \\ & (ct)^{\nu-\mu} \sum_{k=0}^{\infty} \frac{(-1)^k (ct)^{(\nu-\mu)k}}{\Gamma((\nu-\mu)k+\nu+\delta-2\mu)} \quad \times \\ & {}_2F_1(\rho, \delta; (\nu-\mu)k+\nu+\delta-2\mu; -at) \Bigg], \end{aligned} \tag{34}$$

$$\operatorname{Re}(\mu) \geq 0, \operatorname{Re}(\delta-\mu) > 0, \operatorname{Re}(\nu-\mu) > 0, \operatorname{Re}(\delta) > 0.$$

If we set $\mu = 0$ in the above result, then we get

$$N(t) = N_0 \Gamma(\delta) t^{\delta-1} \left[\frac{1}{\Gamma(\delta)} (1+at)^{-\rho} - (ct)^{\nu} \quad \times \right.$$

$$\left. \sum_{k=0}^{\infty} \frac{(-1)^k (ct)^{\nu k}}{\Gamma(\nu k + \nu + \delta)} \ {}_2F_1(\rho, \delta; \nu k + \nu + \delta; -at) \right], \tag{35}$$

$$\operatorname{Re}(\delta) > 0, \operatorname{Re}(\nu) > 0.$$

(IV) If $p = q = 1,\ A_1 = 1, B_1 = \delta = w + 1$, then for

$$f(t) = \frac{t^w}{\Gamma(w+1)} \ {}_1F_1(1; w+1; at) \ = E_t(w, a):$$

$$N(t) = \frac{N_0}{c^{\mu}} t^{w-\mu} \left[\frac{1}{\Gamma(w-\mu+1)} \ {}_1F_1\left(1; w-\mu+1; at\right) - \right.$$

$$(ct)^{\nu-\mu} \sum_{k=0}^{\infty} \frac{(-1)^k (ct)^{(\nu-\mu)k}}{\Gamma((\nu-\mu)k + \nu + w - 2\mu + 1)} \quad \times$$

$$\left. {}_1F_1(1; (\nu-\mu)k + \nu + w - 2\mu + 1; at) \right], \tag{36}$$

that is,

$$N(t) = \frac{N_0}{c^{\mu}} \left[E_t(w-\mu, a) - c^{\nu-\mu} \quad \times \right.$$

$$\sum_{k=0}^{\infty} (-1)^k c^{(\nu-\mu)k} E_t((\nu-\mu)k + \nu + w - 2\mu, a), \tag{37}$$

$$\operatorname{Re}(\mu) \geq 0, \operatorname{Re}(w-\mu+1) > 0, \operatorname{Re}(\nu-\mu) > 0, \operatorname{Re}(w+1) > 0.$$

If $\mu = 0$ in (37),

$$N(t) = N_0 \left[E_t(w, a) - c^{\nu} \sum_{k=0}^{\infty} (-1)^k c^{\nu k} E_t(\nu k + \nu + w, a) \right], \tag{38}$$

$$\operatorname{Re}(\nu) > 0, \operatorname{Re}(w+1) > 0.$$

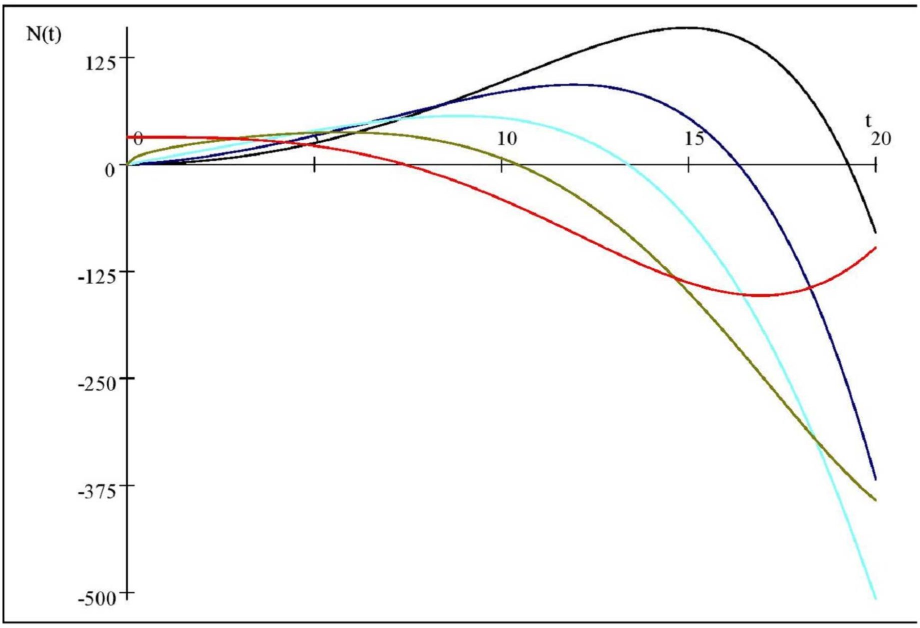

Figure 1: $N(t)$ for $f(t) = t^2$, $N_0 = 1, c = 0.25, \nu = 5$ and different values of μ; black $\mu = 0$ blue $\mu = \frac{1}{2}$ cyan $\mu = 1$ green $\mu = \frac{3}{2}$ red $\mu = 2$

9. Graphic Representation

In this section some figures are presented which show the behavior of the solution $N(t)$ for different $f(t)$ and selected values of the parameters.

10. An Alternative Method

Here we extended the fractional kinetic equation (19) to become

$$c^{\mu} D_t^{-\mu} N(t) - N_0 f(t) = -c^{\nu+\mu} D_t^{-\nu-\mu} N(t) \tag{39}$$

with $\mathrm{Re}(\mu) \geq 0, \mathrm{Re}(\nu) > 0, c > 0,$ $N_0 > 0$ is a constant.

Using the techniques from Al-Saqabi and Tuan [1] for solving differential equations, by applying the operator $(-c^{\nu})^m D_t^{-m\nu}$ to both sides of (39), we get

$$(-c^{\nu})^m D_t^{-m\nu} \left\{ c^{\mu} D_t^{-\mu} N(t) \right\} - (-c^{\nu})^m D_t^{-m\nu} \left\{ (-c^{\nu})\, c^{\mu} D_t^{-\nu-\mu} N(t) \right\}$$

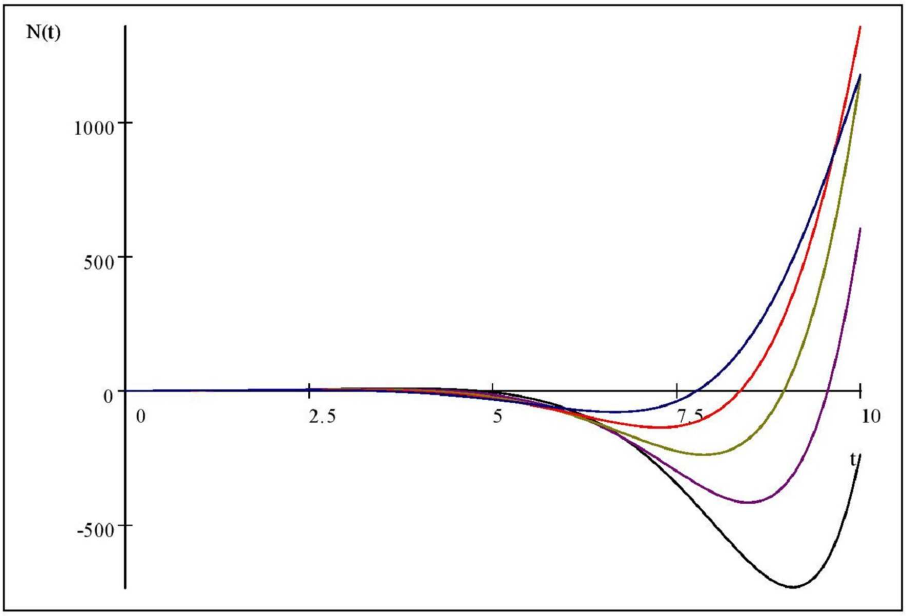

Figure 2: $N(t)$ for $f(t) = t^2$, $N_0 = 1 = c, \nu = 5$ and different values of μ black $\mu = 0$ purple $\mu = \frac{1}{4}$ green $\mu = \frac{1}{2}$ red $\mu = \frac{3}{4}$ blue $\mu = 1$

$$= N_0 \left(-c^{\nu}\right)^m D_t^{-m\nu} \{f(t)\}, \tag{40}$$

where $\nu > 0$ and $m = 0, 1, 2, \cdots$. Taking the sum over m yields

$$c^{\mu} \sum_{m=0}^{\infty} \left(-c^{\nu}\right)^m D_t^{-m\nu-\mu} \{N(t)\} - c^{\mu} \sum_{m=0}^{\infty} \left(-c^{\nu}\right)^{m+1} D_t^{-\nu(m+1)-\mu} \{N(t)\}$$

$$= N_0 \sum_{m=0}^{\infty} \left(-c^{\nu}\right)^m D_t^{-m\nu} \{f(t)\}.$$

Changing the index in the second series, we obtain

$$c^{\mu} \sum_{m=0}^{\infty} \left(-c^{\nu}\right)^m D_t^{-m\nu-\mu} \{N(t)\} - c^{\mu} \sum_{m=1}^{\infty} \left(-c^{\nu}\right)^m D_t^{-\nu m-\mu} \{N(t)\}$$

$$= N_0 \sum_{m=0}^{\infty} \left(-c^{\nu}\right)^m D_t^{-m\nu} \{f(t)\},$$

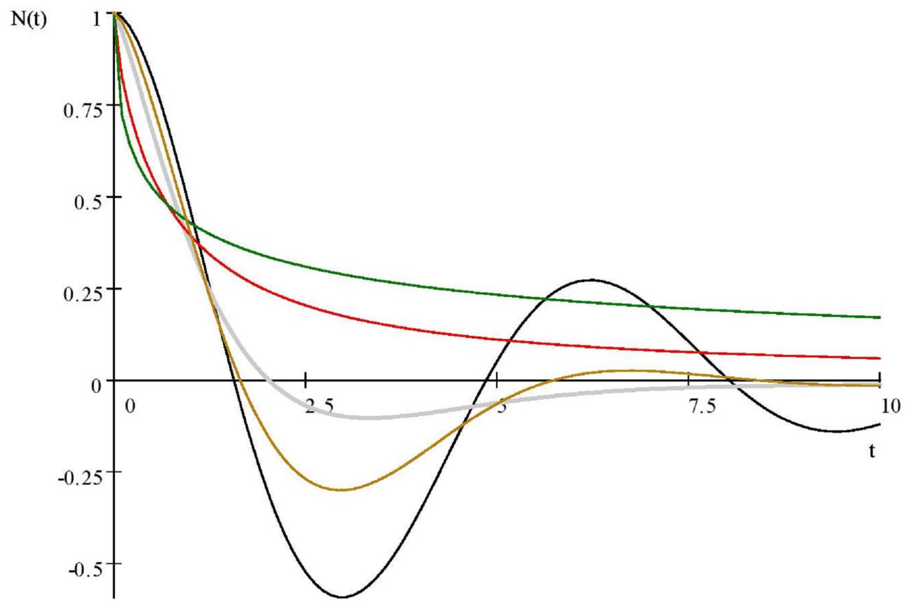

Figure 3: $N(t)$ for $f(t) = 1$, $N_0 = 1 = c, \mu = 0$ and different values of ν green $\nu = \frac{1}{2}$ red $\nu = \frac{3}{4}$ gray $\nu = \frac{5}{4}$ sienna $\nu = \frac{3}{2}$ black $\nu = \frac{7}{4}$

and by canceling out equal terms on the left of this equation it reduces to

$$c^{\mu} D_t^{-\mu} \{N(t)\} = N_0 \sum_{m=0}^{\infty} (-c^{\nu})^m D_t^{-m\nu} \{f(t)\} . \tag{41}$$

For $\mu = 0$, we obtain the correct form of the solution of Saxena and Kalla [14].

References

[1] B.N. Al-Saqabi, V.K. Tuan, Solutions of fractional differintegral equations, *Integral Transf. Spec. Funct.*, **4**, No 4 (1996), 321-326.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Table of Integral Transforms*, **II**, McGraw-Hill, New York (1954).

[3] W.G. Glöckle, T.F. Nonnenmacher, Fractional integral operators and Fox function in the theory of viscoelasticity, *Macromolecules*, **24** (1991), 6426-6434.

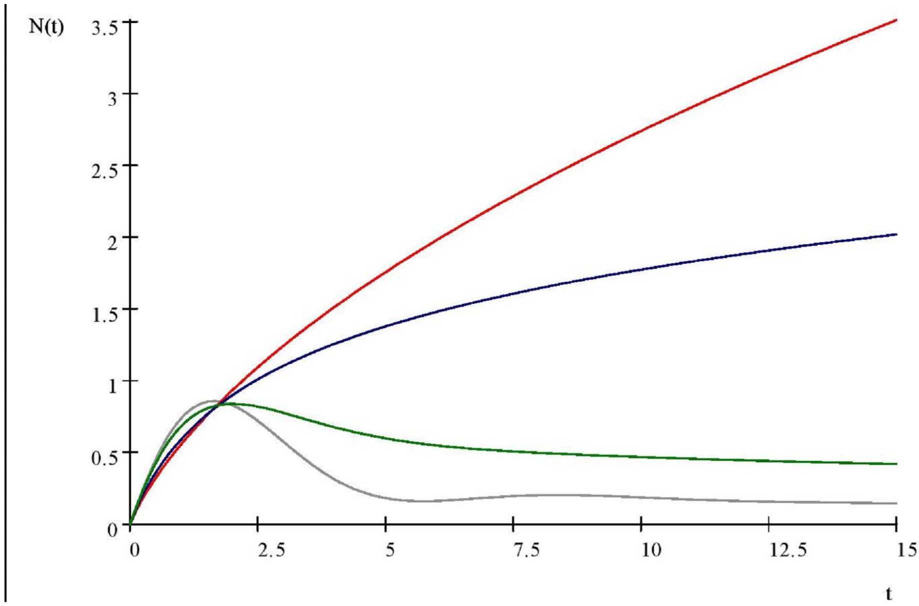

Figure 4: $N(t)$ for $f(t) = t$, $N_0 = 1 = c, \mu = 0$ and different values of ν red $\nu = \frac{1}{2}$ blue $\nu = \frac{3}{4}$ green $\nu = \frac{5}{4}$ gray $\nu = \frac{3}{2}$

[4] H.J. Haubold, A.M. Mathai, The fractional kinetic equation and thermonuclear functions, *Astrophys. Space Sci.*, **273** (2000), 53-63.

[5] R. Hilfer (Ed.), *Applications of Fractional Calculus in Physics*, World Scientific Publishing Co., New York (2000).

[6] E. Hille, J.D. Tamarkin, On the theory of linear integral equations, *Ann. Math.*, **31**, No 3 (1930), 479-528.

[7] V. Kiryakova, *Generalized Fractional Calculus and Applications*, John Wiley & Sons, New York (1994).

[8] A.M. Mathai, R.K. Saxena, *The H-function with Applications in Statistics and Other Disciplines*, John Wiley & Sons, New York (1978).

[9] K.S Miller, B. Ross, *An Introduction of Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York (1993).

[10] G.M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, *C.R. Acad. Sci. Paris*, **137** (1903), 554-558.

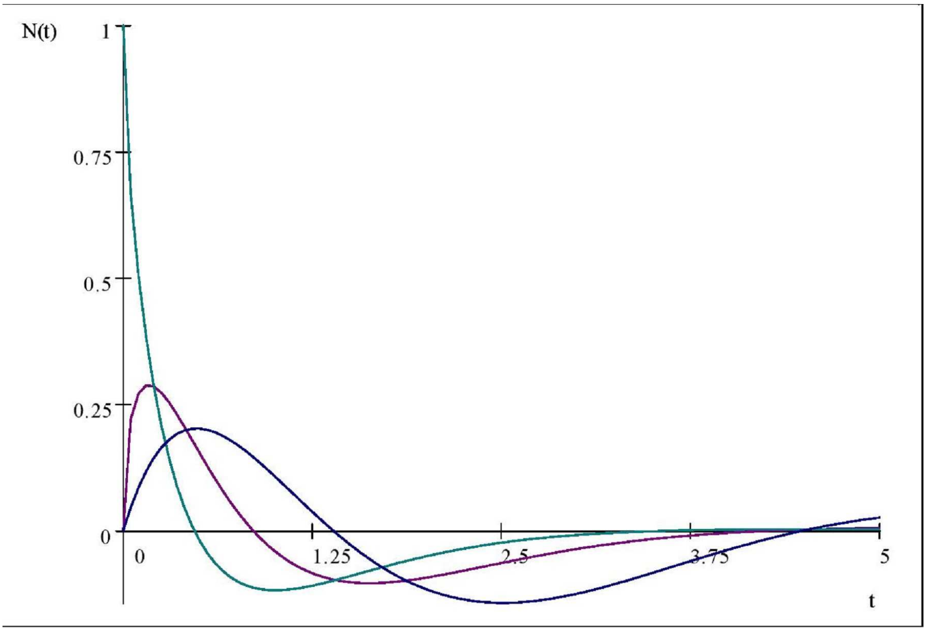

Figure 5: $N(t)$ for $f(t) = te^{-1.5t}$, $N_0 = 1 = c, \nu = 3/2$ and different values of μ blue $\mu = 0$ purple $\mu = \frac{1}{2}$ bluegreen $\mu = 1$

[11] K.B. Oldham, J. Spanier, *The Fractional Calculus*, Academic Press, New York (1974).

[12] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego (1999).

[13] A. Saichev, M. Zaslavsky, Fractional kinetic equations: solutions and applications, *Chaos*, **7**, No 4 (1997), 753-764.

[14] R.K. Saxena, S.L. Kalla, On the solutions of certain fractional kinetic equations, *Applied Mathematics and Computation*, **199** (2008), 504-511.

[15] R.K. Saxena, A.M. Mathai, H.J. Haubold, On fractional kinetic equations, *Astrophys. Space Sci.*, **282** (2002), 281-287.

[16] R.K. Saxena, A.M. Mathai, H.J. Haubold, On generalized fractional kinetic equations, *Physica A*, **344** (2004), 653-664.

[17] R.K. Saxena, A.M. Mathai, H.J. Haubold, Unified fractional kinetic equa-

tions and a fractional diffusion equations, *Astrophys. Space Sci.*, **290**, No 3-4 (2004), 299-310.

[18] R.K. Saxena, A.M. Mathai, H.J. Haubold, Fractional reaction-diffusion equations, *Astrophys. Space Sci.*, **305**, No 3 (2006), 289-296.

[19] R.K. Saxena, A.M. Mathai, H.J. Haubold, Reaction-diffusion systems and nonlinear waves, *Astrophys. Space Sci.,* **305**, No 3 (2006), 297-303.

[20] H.M. Srivastava, R.K. Saxena, Operators of fractional integration and their applications, *Appl. Math. Comput.,* **118** (2001), 1-52.

[21] H.M. Srivastava, Per W. Karlsson, *Multiple Gaussian Hypergeometric Series*, John Wiley & Sons, New York (1985).

[22] A. Wiman, Über den fundamentalsatz in der funktionen $E_\alpha(x)$, *Acta Math.,* **29** (1905), 191-201.

[23] M. Zaslavsky, Fractional kinetic equation for Hamiltonian chaos, *Physica D*, **78** , No 1-3 (1994), 110-122.