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CONVERGENCE AND STABILITY ANALYSIS OF EXTENDED EXPONENTIAL GENERAL LINEAR METHODS

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Abstract: This paper dwells on the stability analysis of extended exponential general linear methods. Like the paper of Butcher [1], we are able to show using the root locus method that the various methods constructed posses favorable stability properties as they are zero stable as their parasitic roots lie in a unit circle. It is also shown that for positive stepsizes, the error estimate holds with a positive constant, independent of n and the stepsizes (h). Experimental experience reveals that our scheme converges.

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1. Introduction

Even though the theory of numerical methods for time integration is well established for a general class of problems, recently due to improvements in the

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efficient computation of the exponential function, exponential integrators for problems

$$y'(t) = Ly(t)) + N(y(t)), \quad 0 \leq t \leq T, \text{ given } \; y(0)\,,$$

have emerged as a viable alternative. In the early fifties, the phenomenon of stiffness was first discovered by Curtis and Hirschfelder [4]. The stiffiness effectively yields explicit integrators useless, as the stability rather than the accuracy governs how the integrator performs. It could be said that more integrators have been developed to overcome the phenomenon of stiffiness, than any other property that a differential equation may have. Butcher and Wright [2] provided a novel approach to solving stiff problems. Integrators were constructed, which solve exactly the linear part of the problem and then used a change of variables to cast the problem in a form, which a traditional explicit method can be used to solve the transformed equation, the approximate solution is then back transformed. These methods are commonly known as Integrating Factor (IF) methods. Calvo and Palencia[3] constructed a class of methods known as the RKMK methods. Enright and Muir [5] introduced the Generalized Integrating Factor (GIF) methods which were shown to exhibit large improvements in accuracy over the IF, ETD and CF methods. It was concluded that the ETD methods constructed though stable but could be improved on.

2. Mathematical Formulation

Here, we provide the convergence behaviour of the method

$$y_{n+1} = e^{hL}y_n + h\sum_{i=1}^{s}(hL)N(Y_{ni}) + h\sum_{k=1}^{q-1}V_k(hL)N(y_{n-k}), \tag{1}$$

$$Y_{ni} = e^{c_ihL}y_n + h\sum_{j=1}^{i-1}A_{ij}^{(1)}(hL)N(Y_{nj}) + \; h\sum_{k=1}^{q-1}U_{ik}(hL)^{(1)}N(y_{n-k})$$

$$+h^2\sum_{j=1}^{i-1}A_{ij}^{(2)}(hL)N'(Y_{nj}) + h^2 sum_{k=1}^{q-1}U_{ik}(hL)^{(2)}N'(y_{n-k}), \tag{2}$$

as an extension of General Linear Methods of Butcher and Wright [2] for the problem

$$y'(t) = Ly(t)) + N(y(t)), \quad 0 \leq t \leq T, \text{ given } \; y(0), \tag{3}$$

which have generated a lot of interest in recent years. We assume that the nonlinear term is locally Lipchitz-continuous, i.e.

$$||N(V) - N(W)|| \leq K(\epsilon)||V - W||, ||V|| + ||W|| \leq \epsilon, \quad (4)$$

also we employ the assumption that the starting values $y_0, y_1, y_2, \cdots, y_{q-1}$ have been computed using some starting procedure, we suppose that the method of coefficients is sufficiently regular and satisfies

$$||A_{ij}^{(1)}(hL)|| + ||B_i(hL)|| + ||U_{ik}^{(1)}(hL)|| + ||V_k(hL)|| + ||A_{ij}^{(2)}(hL)||$$

$$+||U_{ik}^{(2)}(hL)|| \leq Kh^{\lambda}, \quad 1 \leq \lambda \leq 2. \quad (5)$$

Theorem 1. *The extended Exponential General linear method (1), when applied to the initial value (3), satisfies the order conditions also, suppose that $f^{(Q)}(t)$ and suppose $f^{(P)}(t)$ exists, then for stepsize h>0 sufficiently small the estimate*

$$||y(t_n) - y_n|| \leq K \sum_{\ell=1}^{q-1} ||y(t_\ell) - y_\ell|| + Kh^{Q+1} \sup_{0 \leq t \leq t_n} ||f^{(Q)}(t)||$$

$$+ Kh^{(P)} \sup_{0 \leq t \leq t_n} ||f^{(P)}(t)||, \quad t_q \leq t_n \leq T$$

holds with a constant k>0 independent of n and h.

Proof: We estimate e_n as follows:

$$||e_n|| \leq ||e^{(t_n - t_{q-1})L}||||e_{q-1}|| + ||\sum_{\ell=q}^{n} e^{(t_n - t_\ell)L} d\ell$$

$$+ h|| \sum_{\ell=q-1}^{n-1} \sum_{i=1}^{s} ||e^{(t_n - t_{\ell+1})L} B_i(hL)||||\triangle_{\ell i}||$$

$$+ h|| \sum_{\ell=q}^{n-1} \sum_{k=1}^{q-1} ||e^{(t_n - t_{\ell+1})L} V_k(hL)||||\triangle_{\ell-k}||, \quad (6)$$

consequently using the Lipschitz property

$$||e_n|| \leq K||e_{q-1}|| + \sum_{\ell=q}^{n} e^{(t_n - t_\ell)L} d\ell||$$

$$+Kh\sum_{t=q-1}^{n-1} e^{(t_n-t_\ell)L}(\sum_{i=1}^{s}||E_{\ell i}||+\sum_{k=1}^{q-1}||e_{\ell-k}||)$$

for the error of the internal stages (2), we have

$$||E_{\ell i}|| \le ||e^{c_i hL}||||e_\ell|| + h\sum_{j=1}^{i-1}||A_{ij}^{(1)}(hL)||||\triangle N_{\ell j}|| + h\sum_{k=1}^{q-1}||U_{ik}^{(1)}(hL)||||\triangle N_{\ell-k}||$$

$$+h^2\sum_{j=1}^{i-1}||A_{ij}^{(2)}(hL)||||\triangle N'_{\ell j} + h^2\sum_{k=1}^{q-1}||U_{ik}^{(2)}(hL)||||\triangle'_{\ell-k}|| + ||D_{\ell i}||,$$

$$||e_{\ell i}|| \le K||e_\ell|| + Kh\sum_{k=1}^{q-1}||e_{\ell-k}|| + K\sum_{j=1}^{i}||D_{\ell j}|| + ||\sum_{\ell=q}^{n} e^{(t_n-t_\ell)}d\ell||,$$

$$||E_{\ell i}|| \le K||e_\ell|| + Kh\sum_{k=1}^{q-1}||e_{\ell j}|| + Kh\sum_{k=1}^{q-1}||e_{\ell-k}|| + ||D_{\ell i}||,$$

and therefore the estimate (6) follows:

$$||e_n|| \le K||e_{q-1}|| + Kh\sum_{\ell=0}^{n-1}(t_n-t_\ell)||||e_\ell|| + Kh\sum_{\ell=q}^{n-1}\sum_{i=1}^{s}(t_n-t_\ell)||D_{\ell i}||$$

$$+||\sum_{\ell=q}^{n} e^{(t_n-t_\ell)L}d\ell||. \tag{7}$$

The constant $K>0$ in particular depends on T, but is independent of h.

3. Evaluating the ψ_ℓ Function

The exponential integrators use the exponential and related functions; the most common related functions used in exponential integrators are the so called ψ_ℓ function defined below. For integers $\ell \ge 0$ we define ψ_ℓ as

$$\psi_\ell(z) = \int_0^1 e^{(1-x)z}\frac{\tau^{\ell-1}}{(\ell-1)!}d\tau,\ \ell \ge 1, \tag{8}$$

$$\psi_0(z) = e^z \quad (z = hL).$$

Consequently, the recurrence relation

$$\psi_\ell(z) = \frac{1}{\ell!} + z\psi_{\ell+1}(z), \quad z \in c, \ \ell \geq 0 \tag{9}$$

is valid. We are going to use the Pade approximant to evaluate the ψ_ℓ function. The general form of the (d,d)-Pade approximant of ψ_ℓ is

$$\psi_\ell(z) = \frac{N_d^\ell(z)}{D_d^\ell(z)} + 0(z^{2d+1}), \tag{10}$$

where the unique polynomials N_d^ℓ andD_d^ℓ are defined as

$$N_d^\ell(z) = \frac{d!}{(2d+\ell)!}\sum_{i=0}^{d}[\sum_{j=0}^{i}\frac{(2d+\ell-j)!(-1)^j}{j!(d-j)!(\ell+i-j)!}]z^i$$

and

$$D_d^\ell(z) = \frac{d!}{(2d+\ell)!}[\sum_{i=0}^{d}\frac{(2d+\ell-i)!}{i!(d-i)!}](-z)^i. \tag{11}$$

When $\ell = 0$, these reduce to the diagonal Pade approximations of the exponential function.

4. Stability Analysis

4.1. Stabilities of order two step two and stage order one (221) Scheme

We recall order two step two and stage order one (221) Scheme

$$y_{n+1} = e^{hL}y_n + hB_1(hL)N(Y_{n1}) + hB_2(hL)N(Y_{n2}) \tag{12}$$

with $Y_{n1} = y_n$

$$Y_{n2} = e^{c_2hL}y_n + hA_{21}^{(2)}(hL)N(y_n) + h^2A_{21}^{(2)}N'(y_n). \tag{13}$$

The first characteristic polynomial is given by

$$y_{n+1} = e^z y_n = 0 \quad (hL = z).$$

Dividing through by y_n, we get

$$\frac{y_{n+1}}{y_n} - \frac{e^z y_n}{y_n} = 0,$$

$$r - e^z = 0. \tag{14}$$

The stability graph is shown below.

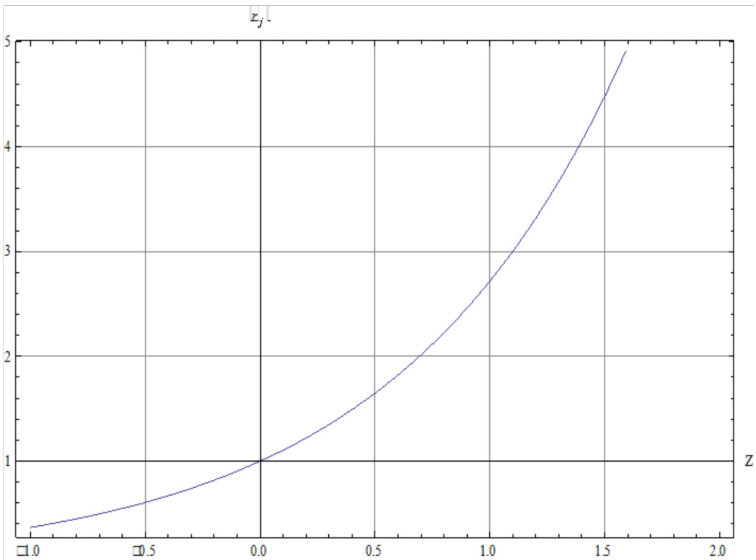

Figure 1: (EEGLM 221) The stability graph of order two step two and stage order one (221) Scheme

The graph shoes that the method is zero stable since the first characteristic polynomial lies in a unit circle.

4.2. Interval of stability of order two step two and stage order one (221) Scheme

To determine the interval of stability of order two step two and stage order one (221) Scheme, we write

$$y_{n+1} = e^{hL} y_n + hB_1(hL)f(Y_{n1}) + hB_2(hL)f(Y_{n2}),$$

$$Y_{n1} = y_n.$$

The internal stage two is defined through

$$Y_{n2} = e^{c_2 hL} y_n + hA_{21}^{(1)}(hL) f(y_n) + h^2 A_{21}^{(2)}(hL) f'(y_n)$$

using the root locus method

$$y' = qy(x)$$

$$y_{n+1} = e^{hL} y_n + z(\psi_1 - \psi_2) y_n + z\psi_2 (Y_{n2})$$

$$Y_{n2} = e^{hL} y_n + zA_{21}^{(1)} y_n + z^2 A_{21}^{(2)} y_n.$$

The above equation can be simplify as:

$$Y_{n2} = (e^{hL} + zA_{21}^{(1)} + z^2 A_{21(2)}) y_n.$$

Inserting the above values of Y_{n2} into the numerical solution above we have

$$y_{n+1} = e^{hL} y_n + z(\psi_1 - \psi_2) y_n + z\psi_2 (e^{hL} + zA_{21}^{(1)} + z^2 A_{21}^{(2)}) y_n.$$

Simplifying the above equation yields

$$y_{n+1} = (e^{hL} + z(\psi_1 - \psi_2) + z\psi_2 (e^{hL} + zA_{21}^{(1)} + z^2 A_{21}^{(2)})) y_n, \tag{15}$$

$$M(z) = e^z + zB_1 + zB_2 e^z + z^2 A_{21}^{(1)} B_2 + z^3 A_{21}^{(2)} B_2. \tag{16}$$

Equation (16) is called the stability matrix:

$$\begin{aligned} Det(\lambda I - M(z)) &= \left[Det \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 0 & \\ 0 & z^3 b_2 a_{21}^{(2)} & 0 \end{pmatrix} \right. \\ &\quad - \begin{pmatrix} 0 & 0 & \\ 0 & z^3 b_2 a_{21}^{(1)} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \\ 0 & z^2 b_2 a_{21}^{(1)} & 0 \end{pmatrix} \\ &\quad \left. - zb_1 - \tfrac{99}{100} zb_2 - \tfrac{99}{100} \right] \end{aligned}$$

$$= \frac{-99\lambda}{50} + \lambda^2 - 2\lambda z b_1 - \frac{-99\lambda}{50} z b_2 - \frac{99\lambda}{100} z^2 b_2 a_{21}^{(1)} - \frac{99\lambda}{100} z^3 b_2 a_{21}^{(2)} - z^3 b_2 b_1$$

$$-z^4 b_2 a_{21}^{(2)} b_1 - \frac{99\lambda}{100} z^3 b_2^2 a_{21}^{(1)} - \frac{-99\lambda}{100} z^4 b_2^2 a_{21}^{(2)}. \tag{17}$$

Using the Pade approximant giving in (11) and inserting into (17) where appropriate, we have

$$\frac{4925279z^3}{100000000} - \frac{4930304480z^4}{100000000} - \frac{99\lambda}{50} - \frac{9910z\lambda}{50000} + \lambda^2.$$

Solving for λ gives

$$\lambda_1 = \frac{1}{1000000}(990000 + 991010z - \sqrt{10}\sqrt{(98010000000} + 196219980000z + 9821008200z^2 + 49252797000 + z^3 + 49303044800z^4)),$$

$$\lambda_2 = \frac{1}{1000000}(990000 + 991010z + \sqrt{10}\sqrt{(98010000000} + 196219980000z + 9821008200z^2 + 49252797000 + z^3 + 49303044800z^4)).$$

The stability graph of order two step two and stage order one (221) Scheme using *MATHEMATICA* is shown in Figure 2.

4.3. Stabilities of order three step two and stage order two (322) Scheme

We recall order three step two and stage order two (322) Scheme given by

$$y_{n+1} = e^{hL}y_n + hB_1(hL)N(Y_{n1}) + hB_2(hL)N(Y_{n2}) + hV_1N(y_{n-1}) \tag{18}$$

with $Y_{n1} = y_n$

$$Y_{n2} = e^{c_2hL}y_n + hA_{21}^{(2)}(hL)N(y_n)$$

$$+ hU_{21}^{(1)}(hL)N(y_{n-1}) + h^2A_{21}^{(2)}N'(y_n) + h^{(2)}U_{21}^{(2)}N'(y_{n-1}). \tag{19}$$

Again the first characteristic polynomial is given by

$$y_{n+1} = e^z y_n = 0 \quad (hL = z).$$

Dividing through by y_n

$$\frac{y_{n+1}}{y_n} - \frac{e^z y_n}{y_n} = 0,$$

$$r - e^z = 0. \tag{20}$$

The stability graph of order three step two and stage order two (322) Scheme is shown in Figure 3.

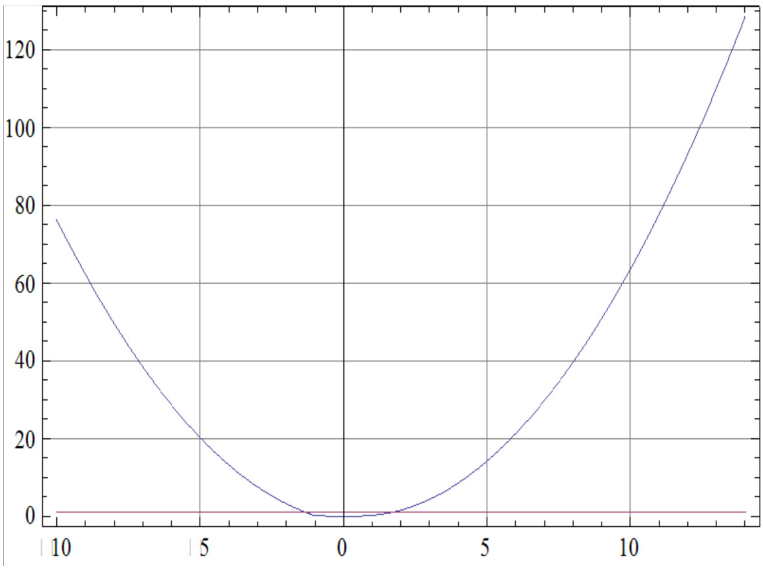

Figure 2: From the graph the interval of stability lies between -1.5 and 2.0.

4.4. Interval of stability of order three step two and stage order two methods

To determine the interval of stability of order three step two and stage order two (322) Scheme, we write

$$y_{n+1}=e^{hL}y_n+hB_1(hL)N(Y_{n1})+hB_2(hL)N(Y_{n2})+hV_1N(y_{n-1}) \tag{21}$$

with $Y_{n1} = y_n$

$$Y_{n2} = e^{c_2hL}y_n + hA_{21}^{(2)}(hL)N(y_n)$$

$$+ hU_{21}^{(1)}(hL)N(y_{n-1}) + h^2A_{21}^{(2)}N'(y_n) + h^{(2)}U_{21}^{(2)}N'(y_{n-1}). \tag{22}$$

The stability matrix is

$$M(z) = e^z + zB_1 + zB_2e^z + z^2B_2A_{21}^{(1)} + z^3B_2A_{21}^{(2)} + z^3B_2U_{21}^{(2)} + z^{(2)}B_2U_{21}^{(1)} + zV_1, \tag{23}$$

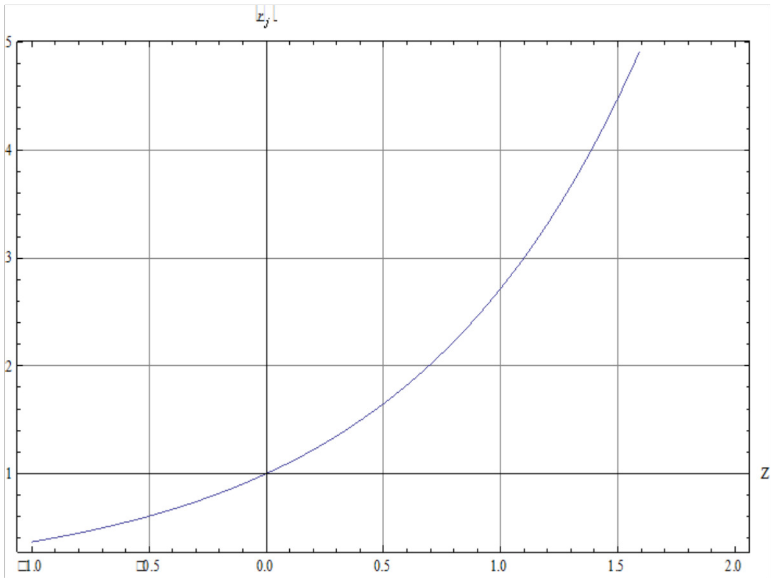

Figure 3: (SEEGLM 322) The graph shows by *MATHEMATICA* that order three step two and stage order two (322) Scheme is zero stable as the first characteristics polynomial lies in a unit circle.

$$
\begin{aligned}
Det(\lambda I - M(z)) \quad = \quad & \left[Det \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 0 & \\ 0 & z^3 b_2 a_{21}^{(2)} & 0 \end{pmatrix} \right. \\
& - \begin{pmatrix} 0 & 0 & \\ 0 & z^3 b_2 U_{21}^{(2)} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \\ 0 & z^2 b_2 a_{21}^{(1)} & 0 \end{pmatrix} \\
& - \begin{pmatrix} 0 & 0 & \\ 0 & z^2 b_2 U_{21}^{(1)} & 0 \end{pmatrix} - \\
& \left. - \begin{pmatrix} 0 & 0 & \\ 0 & z^2 b_2 U_{21}^{(1)} & 0 \end{pmatrix} - z b_1 - Z V_1 - \tfrac{99}{100} z b_2 - \tfrac{99}{100} \right]
\end{aligned}
$$

$$= \frac{-99\lambda}{50} + \lambda^2 - 2\lambda z b_1 - \frac{99\lambda}{50} z b_2 - \frac{99}{100} z^2 b_2 a_{21}^{(1)} - \frac{99}{100} z^3 b_2 a_{21}^{(1)} - z^3 b_2 a_{21}^{(1)} b_1$$
$$- z^4 b_2 a_{21}^{(2)} b_1 - \frac{99}{100} z^2 b_2^2 a_{21}^{(1)} - \frac{99}{100} z^4 b_2^2 a_{21}^{(2)} - \frac{99}{100} z^2 b_2^2 u_{21}^{(1)} - \frac{99}{100} z^3 b_2 u_{21}^{(2)} - z^4 b_2 u_{21}^{(2)} b_1$$
$$- \frac{99}{100} z^4 b_2^2 u_{21}^{(2)} - 2z\lambda v_1 - z^3 a_{21}^{(1)} b_2 v_1 - z^4 a_{21}^{(2)} b_2 v_1 - z^3 u_{21}^{(1)} b_2 v_1$$
$$- z^4 u_{21}^{(2)} b_2 v_1. \quad (24)$$

Inserting the values of the coefficients we have

$$\frac{-4109586921z^2}{2500000000} - \frac{51990721655067z^3}{25000000000} - \frac{14928815844279z^4}{500000000000}$$
$$- \frac{99\lambda}{50} - \frac{163473z\lambda}{500000} + \lambda^2. \quad (25)$$

Solving equation (25) we have

$$\lambda_1 = \frac{3}{10000000}(3300000 + 54410z$$
$$-\sqrt{2}\sqrt{(544500000 + 179820300z + 9280878834z^2}$$
$$+1155349370 + 1658757316z^4)),$$
$$\lambda_2 = \frac{3}{10000000}(3300000 + 54410z + \sqrt{2}\sqrt{(544500000 + 179820300z}$$
$$+ 9280878834z^2 + 1155349370 + 1658757316z^4)).$$

The graph is shown in Figure 4.

5. Numerical Experiments

We consider a negative exponential

$$y' = -y, \quad y(0) = 1$$

with theoretical solution given as

$$y - e^{-t}.$$

Our scheme is convergent. If we were to continue to reduce the steplength, then the point t_n at which the numerical solution differs from the theoretical by a given amount (fraction) would keep moving to the right. In the limit as $h \to 0$, as $n \to \infty$ the numerical solution would converge to the theoretical solution but for any fixed positive h, no matter how small, the absolute value of the numerical solution tends to infinity as n tends to infinity.

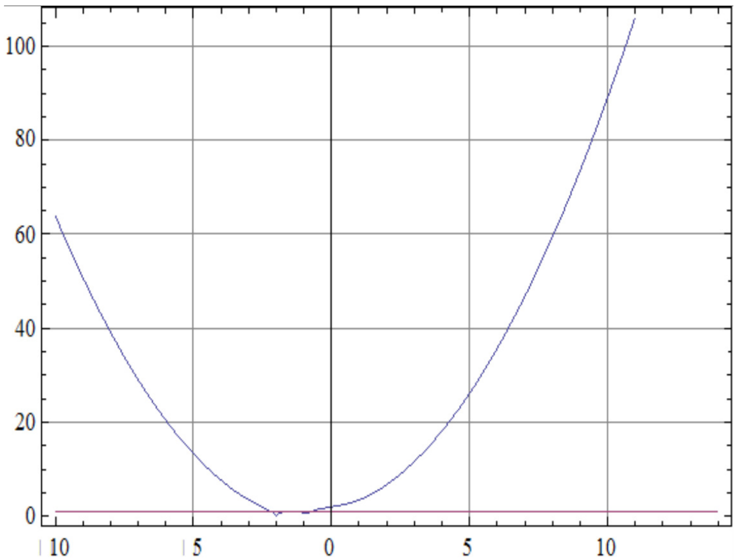

Figure 4: (SEEGLM) The stability graph for order three step two and stage order two (322) Scheme using *MATHEMATICA* indicates that the stability lies in (0.5 2.3).

6. Conclusion

This paper has investigated the stability analysis of Extended Exponential General Linear Methods. Our investigations reveal that our methods are zero stable with parasitic roots lie on a unit disc. Our scheme is convergent. If we were to continue to reduce the steplength, then the point t_n at which the numerical solution differs from the theoretical by a given amount (fraction) would keep moving to the right. In the limit as $h \to 0$, as $n \to \infty$ the numerical solution would converge to the theoretical solution but for any fixed positive h, no matter how small, the absolute value of the numerical solution tends to infinity as n tends to infinity.

Number of iterations	h=0.01 (error)	h=0.02 (error)	h=0.03 (error)
1	2.27503 x10-08	8.357503 x10-07	7.451126 x10-06
2	4.50463 x10-08	1.6548666 x10-06	5.627241 x10-05
3	6.71561 x10-08	2.4585499 x10-06	3.421479 x10-05
4	8.91526 x10-08	3.2451526 x10-06	4.624177 x10-05
5	1.105016 x10-07	4.0155016 x10-06	4.723760 x10-05
6	1.305845 x10-07	4.7705845 x10-06	2.463071 x10-05
7	1.50907 x10-07	5.510907 x10-06	2.257107 x10-05
8	1.703874 x10-07	6.234387 x10-06	3.724172 x10-05
9	1.902722 x10-07	7.085241 x10-06	3.864216 x10-05

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