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IMPLICIT COLLOCATION METHOD FOR A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract: The aim of this paper is to present a numerical method to solve of a class of singular second order nonlinear differential equations. The singularity of the nonlinear differential equation is modified by L'Hospital rule and boundary condition $y'(0) = 0$. The collocation method yields an nonlinear system and Newton's iterative method solves the nonlinear system. Numerical examples are presented to illustrate the efficiency of the presented method with respect to the other methods.

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1. Introduction

Many numerical methods for singular second order nonlinear differential equations have been proposed in recent years. For example, Pandey and Singh [6] employed a finite difference method for a class of singular second order nonlinear differential equations. Kadalbajoo and Aggarwal [4] solved the differential equation on two subintervals, which is called spline Chebyshev economized method. Kanth and Bhattacharya [7] reduced the nonlinear term to linear term

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and applied B-spline functions to solve the differential equation by modifying the resulting set around the singular point. Abukhaled et al. [1] modified the singularity by L'Hospital's rule then two methods are presented. The first method is spline Chebyshev economized method introduced in Abukhaled et al. [1] and the second method is adaptive splines method introduced in Khuri and Sayfy [5]. For more information, see Chawla and Katti [2] and El-Gebeily and Abbu-Zaid [3]. This paper focuses on the following singular second order nonlinear differential equation

$$\begin{cases} y''(x) + \dfrac{q(x)}{x} y'(x) = f(x, y), \qquad 0 \leq x \leq 1, \\ y'(0) = 0, \\ \alpha y(1) + \beta y'(1) = \gamma. \end{cases} \tag{1}$$

We apply L'Hospital's rule idea introduced in Abukhaled et al. [1] and Khuri and Sayfy [5] to overcome singularity. Implicit collocation method based on Legendre polynomials is employed for (1) and this method yields a nonlinear system. One can solve the nonlinear system by Newton's iterative method, but the main problem to apply Newton's iterative method is to choose initial vector. The linearization strategy help us to find the initial vector for Newton's iterative method. We will explain this method in the next section completely.

2. Implicit Collocation Method

This section presents a numerical method to solve singular second-order differential equation (1) by Implicit Collocation method. If the collocation method is applied for (1) with the collocation points $x_0 = 0$ and $x_i = i + h, h = 1/N$ then a problem will be appeared by substituting $x_0 = 0$ as collocation point. This problem is appeared in the second term of (1) which is $\dfrac{q(x)y'(x)}{x}$. To modify this problem we apply L'Hospital's rule on the second term of (1) as x approaches zero. Note that boundary condition $y'(0) = 0$ helps us to apply L'Hospital's rule on the second term of (1). This idea has been presented in Abukhaled et al. [1] and Khuri and Sayfy [5]. Problem (1) can be written as follows at $x_0 = 0$ by L'Hospital's rule,

$$(1 + q(x_0))y''(x_0) + q'(x_0)y'(x_0) = f(x_0, y). \tag{2}$$

Now, we can introduce Implicit collocation method for singular second order differential equation (1). Let N be an arbitrary natural number. Consider the following approximate solution,

$$y_N(x) = \sum_{i=0}^{N+2} c_i L_i(x), \tag{3}$$

with unknown $c_i, i = 0, \ldots, N+2$ and $L_i(x)$ is shifted Legendre polynomial on $[0,1]$, that is,

$$\begin{cases} L_i(x) = P_i(2x-1), \qquad 0 \le x \le 1, \qquad i \ge 0, \\ P_0(x) = 1, \qquad P_1(x) = x, \qquad -1 \le x \le 1, \\ P_{k+1}(x) = \left(\frac{2k+1}{k+1}\right) P_k(x) - \left(\frac{x}{k+1}\right) P_{k-1}(x), \qquad k \ge 1. \end{cases}$$

We will obtain the above unknown coefficients by the following nonlinear system,

$$\begin{cases} g_1(C) := (1+q(x_0))y''_N(x_0) + q'(x_0)y'_N(x_0) - f(x_0, y_N(x_0)) = 0, \\ \\ g_{j+1}(C) := y''_N(x_j) + \dfrac{q(x_j)}{x_j} y'_N(x_j) - f(x_j, y_N(x_j)) = 0, \ 1 \le j \le N, \end{cases} \tag{4}$$

where $x_j = x_0 + jh$, $x_0 = 0$, $h = 1/N$, $C = [c_0, \ldots, c_{N+3}]^t$ and y_N introduced in (3) which consists c_i's unknowns. The first equation of (4) has been obtained from (2). The system (4) consists of $N+1$ nonlinear equations with $N+3$ unknowns. Substitution of approximate solution y_N into boundary condition of (1) yields:

$$\begin{cases} g_{N+2}(C) := \displaystyle\sum_{i=0}^{N+2} c_i L'_i(x_0) = 0, \\ \\ g_{N+3}(C) := \alpha \displaystyle\sum_{i=0}^{N+2} c_i L_i(x_N) + \beta \sum_{i=0}^{N+2} c_i L'_i(x_N) - \gamma = 0. \end{cases} \tag{5}$$

The system of nonlinear equations (4)-(5) consist $N+3$ equations with $N+3$ unknowns. We can drive the nonlinear system (4)-(5) as follows:

$$G(C) := (g_1(C), \ldots, g_{N+3}(C)) = 0. \tag{6}$$

To solve the nonlinear system (6) by Newton's iterative method we need initial vector $C^{(0)}$ for Newton's iterations

$$C^{(k+1)} = C^{(k)} - J^{-1}_{G(C^{(k)})}(G(C^{(k)})), \tag{7}$$

J_G is the Jacobian matrix for vector value function G, and $\|C^{(k+1)} - C^{(k)}\|_2 < \varepsilon$ is the stoping criterion of (7) for given $\varepsilon > 0$. The main problem in the Newton's iterative method is to determine the initial vector $C^{(0)}$. To overcome this problem, we convert the nonlinear system (4) to a linear system by Taylor expansion of the nonlinear term $f(x, y)$. System (5) is linear, then we will obtain a linear system of $N + 3$ equations with $N + 3$ unknowns. By solving the linear system we can obtain an initial vector $C^{(0)}$ for iterative method (7). Note that one can solve problem (1) by the mentioned linear system but the approximate solution converges to exact solution slowly, because the linear system is approximate system. We prefer to solve nonlinear system (4)-(5).

3. Numerical Examples

In this section, we solve two examples. L'Hospital's rule modifies the singularity, then the collocation method yields a nonlinear system. To solve nonlinear system, we convert the nonlinear system to a linear system for getting initial vector of the Newton's iterative method. To stop iterative method (7) we employ the stopping criterion $\|C^{(k+1)} - C^{(k)}\|_2 < 10^{-40}$ for both examples. All computations have been run by Maple 15 software.

Example 3.1. Consider the following singular second order boundary value problem

$$\begin{cases} y''(x) + \dfrac{x+b}{x} y'(x) = \dfrac{5x^3(5x^5 \exp(y) - x - b - 4)}{4 + x^5}, \\ y'(0) = 0, \\ y(1) + 5y'(1) = -(5 + \ln(5)). \end{cases}$$

The exact solution is $-\ln(x^5 + 4)$. This problem is related to oxygen diffusion problem. L'Hospital's rule and the linearization of the collocation system yield the initial vector $C^{(0)}$. Tables 1–3 show the error of implicit collocation method (ICM) for $b = 0.25, 1, 8$, the number of iteration (NI) of Newton's iterations, the error of spline method (SM) in Abukhaled et al. [1] and also, the error of finite difference method (FDM) in Pandey and Singh [6].

Example 3.2. Consider the following singular second order boundary value problem

$$\begin{cases} y''(x) + \dfrac{2}{x}y'(x) + (y(x))^5 = 0, \\ y'(0) = 0, \\ y(1) = \frac{\sqrt{3}}{2}. \end{cases}$$

The exact solution is $(1+\frac{x^2}{3})^{\frac{-1}{2}}$. This problem is related to astronomy; the equilibrium of isothermal gas spheres problem. The linearization and L'Hospital's rule imply that the initial vector $C^{(0)}$. Table 4 shows the error of implicit collocation method (ICM), the number of iteration (NI) of Newton's iterations, and the error of adaptive spline method (ASM) in Khuri and Sayfy [5].

Table 1: Example 3.1: b=0.25

N	ICM	NI	SM	FDM
16	$1.1e-07$	2	$7.7e-04$	$1.1e-03$
32	$9.5e-15$	3	$1.9e-04$	$3.0e-04$
64	$1.3e-28$	4	$4.9e-05$	$7.6e-05$

Table 2: Example 3.1: $b=1$

N	ICM	NI	SM	FDM
16	$1.1e-07$	2	$7.7e-04$	$1.4e-03$
32	$1.9e-15$	3	$1.9e-04$	$3.6e-04$
64	$1.3e-28$	4	$4.9e-05$	$9.2e-05$

4. Conclusion

The presented method solves a class of singular second order differential equation numerically. L'Hospital's rule and boundary condition $y'(0) = 0$ help us to modify the singularity of differential equation, then the collocation method yields a nonlinear system. The main problem to solve a nonlinear system by Newton's iterative method is to choose an initial vector for Newton's iterative

Table 3: Example 3.1: $b = 8$

N	ICM	NI	SM	FDM
16	$1.4e-07$	2	$2.5e-03$	$4.1e-03$
32	$1.1e-14$	3	$6.3e-04$	$9.7e-04$
64	$1.6e-28$	4	$1.5e-04$	$2.3e-04$

Table 4: Example 3.2.

N	ICM	NI	ASM
16	$1.8e-14$	4	$1.4e-07$
32	$1.2e-25$	4	$9.0e-09$
64	$2.3e-48$	5	$5.6e-10$

formula. To overcome of this problem, we apply the linearization strategy by Taylor expansion, which yields an approximate linear system. The solution of the linear system is proposed as initial vector for Newton's iterative method. The presented method has been employed for two applied examples and the comparison of the methods illustrates the efficiency of the presented method with respect to the other methods.

References

[1] M. Abukhaled, S.A. Khuri and A. Sayfy, A numerical approach for solving a class of singular boundary value problems arising in physiology, *Int. J. Numer. Anal. Model.* **8** (2011), 353-363.

[2] M.M. Chawla and C.P. Katti, Finite difference methods and their convergence for a class of singular two point boundary value problem, *Numer. Math.* **39** (1982), 341-350.

[3] M.A. El-Gebeily and I.T. Abbu-Zaid, On a finite difference method for singular two-point boundary value problem, *IMA J. Numer. Anal.* **18** (1998), 179-190.

[4] M.K. Kadalbajoo and V.K. Aggarwal, Numerical solution of singular

boundary value problems via Chebyshev polynomial and B-spline, *Appl. Math Comput.* **160** (2005), 851-863.

[5] S.A. Khuri and A. Sayfy, A Twofold spline Chebychev linearization approach for a class of singular second-order nonlinear differential equation, *Results. Math.* **63** (2013), 817-835.

[6] R.K. Pandey and A. K. Singh, On the convergence of a finite difference method for a class of singular boundary value problem arising in physiology, *J. Comput. Appl. Math.* **166** (2004), 553-564.

[7] A.S.V. Ravi Kanth and V. Bhattacharya, Cubic spline polynomial for a class of non-linear singular two-point boundary value problem arising in physiology. *Appl. Math Comput.* **174** (2006), 768-774.