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CAUCHY PROBLEMS WITH MODIFIED CONDITIONS FOR THE EULER-POISSON-DARBOUX EQUATIONS IN THE SPHERICAL SPACE

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Abstract: In this note we give the solutions of the Cauchy problems for the Euler-Poisson-Darboux equations (EPD) with modified conditions in the spherical space with application to the wave equation.

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1. Introduction

In El-Hafedh and Ould Moustapha [1] and El-Hafedh et al. [2], there are obtained the explicit solutions of Cauchy problems with modified conditions for the Euler-Poisson-Darboux equations in Euclidean and hyperbolic spaces. Here we give the explicit solutions of Cauchy problems with modified conditions for the Euler-Poisson-Darboux equation in spherical space. The classical Cauchy problem for the Euler-Poisson-Darboux equation in spherical space is considered in Fusaro [3] and in Kipriyanov and Ivanov [4] and [5]:

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$$\left(E_\mu^n\right)'' \begin{cases} (a)\ L_n U(t,\theta) = A_t^\mu U(t,\theta)\ 0<t<\pi,\ \theta \in \mathbb{S}^n \\ (b)''\ U(0,\theta) = f(\theta),\ \partial_t U(0,\theta) = 0; f \in C^\infty(\mathbb{S}^n). \end{cases}$$

More specifically, we are interested in the family of problems:

$$\left(E_\mu^n\right) \begin{cases} (a)\ L_n U(t,\theta) = A_t^\mu U(t,\theta)\ 0<t<\pi,\ \theta \in \mathbb{S}^n \\ (b)\ U(0,\theta) = f(\theta),\ \lim_{t\to 0} t^{1-2\mu}\partial_t U(t,\theta) = g(\theta);\ f,g \in C^\infty(\mathbb{S}^n) \end{cases}$$

$$\left(E_\mu^\nu\right)' \begin{cases} (a)' A_\theta^\nu U(t,\theta) = A_t^\mu U(t,\theta) 0<t<\pi, 0<\theta<\pi \\ (b)' U(0,\theta) = f(\theta), \lim_{t\to 0} t^{1-2\mu}\partial_t U(t,\theta) = g(\theta); f,g \in C^\infty([0,\pi]) \end{cases}$$

with L_n the Laplace-Beltrami operator associated with the spherical Riemannian space $\mathbb{S}^n$ given by polar geodetic coordinates:

$$L_n = \frac{\partial^2}{\partial r^2} + (n-1)\cot r \frac{\partial}{\partial r} - (\frac{n-1}{2})^2 + \Lambda(r), \tag{1.1}$$

and $\Lambda(r)$ being a differential operator of second order on the sphere $\mathbb{S}^{n-1}(r)$ of radius r.

The operator A_θ^ν is given by:

$$A_\theta^\nu = \frac{\partial^2}{\partial \theta^2} + (1-2\nu)\cot\theta \frac{\partial}{\partial\theta} - \left(\frac{1-2\nu}{2}\right)^2. \tag{1.2}$$

Note that in $\left(E_\mu^n\right)''$, the second data is zero ($g=0$) as a solution of equation (a) can not be regular for $t=0$ if its first derivative with respect to t are not zero. The modified conditions (b) and $(b)'$ allow to take the second data as any function g, void or while covering the classical Cauchy conditions $(b)''$. Thus the Cauchy problems $\left(E_{\frac{1}{2}}^n\right)$ and $\left(E_{\frac{1}{2}}^\nu\right)'$ correspond to the classical (see Bunke and Olbrich[7]) and radial (Theorem 2) wave equations in $\mathbb{S}^n$.

The main results of this note - Theorems 1, 2 and 3 - are given below, and their applications are in Section 6.

2. Theorems

Theorem 1. (Classical EPD with modified initial conditions) *Let* $\mu \in (0, \frac{1}{2})$. *The Cauchy problem* $\left(E_\mu^n\right)$ *with modified conditions for the classical Euler-Poisson-Darboux equation on the spherical space has the unique solution given* *by:*

$$U(t,\theta)=\alpha_{n,-\mu}(\sin t)^{2\mu}\left(\frac{\partial}{\sin t\partial t}\right)^{\frac{n-1}{2}}\int_{r<t} f(\theta')\left(\sin^2\frac{t}{2}-\sin^2\frac{r}{2}\right)^{-\mu-\frac{1}{2}}d\mu(\theta')$$

$$+\frac{1}{2\mu}\alpha_{n,\mu}\left(\frac{\partial}{\sin t\partial t}\right)^{\frac{n-1}{2}}\int_{r<t} g(\theta')\left(\sin^2\frac{t}{2}-\sin^2\frac{r}{2}\right)^{\mu-\frac{1}{2}}d\mu(\theta'),$$

when n is odd;

$$U(t,\theta)=\beta_{n,-\mu}(\sin t)^{2\mu}\left(\frac{\partial}{\sin t\partial t}\right)^{\frac{n}{2}}\int_{r<t} f(\theta')\left(\sin^2\frac{t}{2}-\sin^2\frac{r}{2}\right)^{-\mu}d\mu(\theta')$$

$$+\frac{1}{2\mu}\beta_{n,\mu}\left(\frac{\partial}{\sin t\partial t}\right)^{\frac{n}{2}}\int_{r<t} g(\theta')\left(\sin^2\frac{t}{2}-\sin^2\frac{r}{2}\right)^{\mu}d\mu(\theta'),$$

when n is even; where $\alpha_{n,\mu}=\dfrac{1}{2}\dfrac{\Gamma(1+2\mu)}{(2\pi)^{\frac{n-1}{2}}\Gamma^2\left(\frac{1}{2}+\mu\right)}$, $\beta_{n,\mu}=\dfrac{4^\mu}{(2\pi)^{\frac{n}{2}}}$ *and* $r=d(\theta,\theta')$ *is the geodesic distance between θ and θ' in $\mathbb{S}^n$.*

Theorem 2. (Radial wave equation) *Let $1-2\nu=2q$ be a positive integer. The Cauchy problem $\left(E^\nu_{\frac{1}{2}}\right)'$ for the radial wave equation on the spherical space has the unique solution given by:*

$$U(t,\theta)=\int_0^\pi f(\theta')\frac{\partial}{\partial t}W(t,\theta,\theta')(\sin\theta')^{1-2\nu}d\theta'+\int_0^\pi g(\theta')W(t,\theta,\theta')(\sin\theta')^{1-2\nu}d\theta',$$

where

$$W(t,\theta,\theta')=4^{\nu-1}i\left(\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^{\nu}\left(\cos\frac{\theta'}{2}\right)^{2\nu}$$

$$\times\int_0^{+\infty}\int_{ze^{-i\frac{t}{2}}}^{ze^{i\frac{t}{2}}} J_{-\nu}(z\sin\frac{\theta}{2})J_{-\nu}(z'\sin\frac{\theta'}{2}) \tag{2.1}$$

and J_ν is the Bessel function (see [6], p. 65).

Theorem 3. (Radial EPD with modified initial conditions) *Let $1-2\nu=2q$ be a positive integer and $\mu\in(0,\frac{1}{2})$. The Cauchy problem $\left(E^\nu_\mu\right)'$ with modified conditions for the radial Euler-Poisson-Darboux equation on the spherical space has the unique solution given by:*

$$U(t,\theta)=(\sin t)^{2\mu}\int_0^\pi f(\theta')W_{-\mu}(t,\theta,\theta')(\sin\theta')^{1-2\nu}d\theta'$$

$$+\frac{1}{2\mu}\int_0^{\pi} g(\theta')W_\mu(t,\theta,\theta')(\sin\theta')^{1-2\nu}d\theta',$$

where

$$W_\mu(t,\theta,\theta') = 4^{\nu+\mu-1}i\frac{\Gamma(1+\mu)}{\sqrt{\pi}\Gamma(\frac{1}{2}+\mu)}\left(\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^{\nu}\left(\cos\frac{\theta'}{2}\right)^{2\nu}$$

$$\times\int_0^t\left(\sin^2\frac{t}{2}-\sin^2\frac{y}{2}\right)^{\mu-\frac{1}{2}}\frac{\partial}{\partial y}\int_0^{+\infty}\int_{ze^{-i\frac{y}{2}}}^{ze^{i\frac{y}{2}}} J_{-\nu}(z\sin\frac{\theta}{2})J_{-\nu}(z'\sin\frac{\theta'}{2})$$

$$\times J_0\left(\sqrt{z^2+z'^2-2zz'\cos\frac{y}{2}}\right)z^{-\nu}z'^{\nu}dz'dzdy. \tag{2.2}$$

3. Preliminaries

In this section we recall the continuous Jacobi transform, see Walter and Zayed [8], and we give some lemmas.

If $1-2\nu = 2q$ is a positive integer and $f(x)(1+x)^{\nu} \in L^1\left\{(-1,1),(1-x^2)^{-\nu}\right\}$, the continuous Jacobi transform $\widehat{f}(\lambda)$ of $f(x)$ is defined by

$$\widehat{f}(\lambda) = \frac{1}{4^q}\int_{-1}^{1} f(x)P_\lambda^{\nu}(x)\left(1-x^2\right)^{-\nu}dx, \quad \lambda > \nu-\frac{1}{2}, \tag{3.1}$$

where $P_\lambda^{\nu}(x)$ is the Jacobi function of the first kind, namely:

$$P_\lambda^{\nu}(x) = \frac{\Gamma(\lambda+1-\nu)}{\Gamma(\lambda+1)\Gamma(1-\nu)}\ F\left(-\lambda,\lambda+1-2\nu,1-\nu,\frac{1-x}{2}\right), \tag{3.2}$$

and $F(a,b,c,z)$ is the Gauss hypergeometric function, see [6].

Lemma 1. *We have* $\widehat{A_\theta^{\nu}f}(\lambda) = -\left(\lambda+q\right)^2\widehat{f}(\lambda)$, *where* $\lambda \in \mathbb{R}^+,\ q = \frac{1}{2}-\nu$.

Proof. It suffices to write

$$A_\theta^{\nu} = \frac{1}{(\sin\theta)^{1-2\nu}}\frac{\partial}{\partial\theta}\left(\sin\theta\right)^{1-2\nu}\frac{\partial}{\partial\theta}+\left(\frac{1-2\nu}{2}\right)^2,$$

to do two integrations by parts and the change of variables $x = -\cos\theta$ in (3.1).

Lemma 2. *An inverse transform of the continuous Jacobi transform* (3.1) *is given by:*

$$f(x) = 4\pi \int_0^{+\infty} \left[\frac{\Gamma(\lambda+q)}{\Gamma(\lambda+\frac{1}{2})}\right]^2 \widehat{f}(\lambda-q) P^{\nu}_{\lambda-q}(-x) \frac{\lambda\cot[(q-\lambda)\pi]}{\Gamma(q-\lambda)\Gamma(q+\lambda)} d\lambda. \quad (3.3)$$

Proof. By using the properties of the Gamma function (see Magnus et al. [6], p. 2), we unify several formulas for the inverse transform in Walter and Zayed [8] ((5.1),(5.4),(5.9) and (5.11)) when $2q$ is a positive integer.

Lemma 3. *For* $0 < t,\ \theta,\ \theta' < \pi$,

$$J(t,\theta,\theta') = \int_0^{+\infty} \int_{ze^{-i\frac{t}{2}}}^{ze^{i\frac{t}{2}}} J_{-\nu}(z\sin\frac{\theta}{2}) J_{-\nu}(z'\sin\frac{\theta'}{2})$$

$$\times J_0\left(\sqrt{z^2 + z'^2 - 2zz'\cos\frac{t}{2}}\right) z^{-\nu} z'^{\nu} dz dz',$$

then if t *is sufficiently small, we have the following asymptotic formula:*

$$J(t,\theta,\theta') \approx \frac{i\sin\frac{t}{2}}{2\pi\sin\frac{\theta}{2}\sin\frac{\theta'}{2}} \int_{-1}^{1} \frac{1}{\sqrt{Z}} F\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu, \frac{1}{2}, Z\right) dp,$$

where $Z = \dfrac{a^2-(b-c)^2}{4bc}$, $b-a < c < b+a$, $a = \sqrt{1-p^2}\sin\dfrac{t}{2}$, $b = \sin\frac{\theta}{2}$, $c = \sin\frac{\theta'}{2}$.

Lemma 4. *If* W^{ν}_{μ} *is a solution of* $(a)'$, *then we have:*

(i) $A^{\mu}_t\left[(\sin t)^{2\mu} W^{\nu}_{-\mu}(t,\theta)\right] = (\sin t)^{2\mu} A^{-\mu}_t W^{\nu}_{-\mu}(t,\theta)$;

(ii) $(\sin t)^{2\mu} W^{\nu}_{-\mu}(t,\theta)$ *satisfies equation* $(a)'$ *in* $\left(E^{\nu}_{\mu}\right)'$;

(iii) $W^{1-\frac{n}{2}}_{\mu}(t,r)$ *and* $(\sin t)^{2\mu} W^{1-\frac{n}{2}}_{-\mu}(t,r)$ *satisfies equation* (a)

with $r = d(\theta,\theta')$.

Lemma 5. *For $0 < t < \pi$ and $\theta,\ \theta' \in \mathbb{S}^n$ let*

$$W_{n,\mu}(t,\theta,\theta') = C_{n,\mu}\left(\sin^2\frac{t}{2} - \sin^2\frac{r}{2}\right)^{\mu-\frac{n}{2}}$$

with $C_{n,\mu} = \dfrac{4^\mu\Gamma(1+\mu)}{2^n\pi^{\frac{n}{2}}\Gamma(1+\mu-\frac{n}{2})}$ *and $r = d(\theta,\theta')$, then we have:*

$$(i)\, W_{n,\mu}(t,\theta,\theta') = \begin{cases} \alpha_{n,\mu}(\frac{\partial}{\sin t\partial t})^{\frac{n-1}{2}}\left(\sin^2\frac{t}{2} - \sin^2\frac{r}{2}\right)^{\mu-\frac{1}{2}} & \text{when } n \text{ is odd} \\ \beta_{n,\mu}(\frac{\partial}{\sin t\partial t})^{\frac{n}{2}}\left(\sin^2\frac{t}{2} - \sin^2\frac{r}{2}\right)^{\mu} & \text{when } n \text{ is even,} \end{cases}$$

(ii) $W_{n,\mu}(t,x,x')$ *satisfies the equation* (a).

Lemma 6. *Let J_ν be the Bessel function, then we have:*

(i) $A_\theta^\nu\left[\sin^\nu\frac{\theta}{2}J_{-\nu}(z\sin\frac{\theta}{2})\right] = -\frac{1}{4}\sin^\nu\frac{\theta}{2}B_z^\nu\left[J_{-\nu}(z\sin\frac{\theta}{2})\right]$,
(ii) $A_\theta^\nu\left[\sin^\nu\frac{\theta}{2}\cos^{2\nu}\frac{\theta}{2}J_{-\nu}(z\sin\frac{\theta}{2})\right] = -\frac{1}{4}\sin^\nu\frac{\theta}{2}\cos^{2\nu}\frac{\theta}{2}B_z^{-\nu}\left[J_{-\nu}(z\sin\frac{\theta}{2})\right]$
where $B_z^\nu = z^2\frac{\partial^2}{\partial z^2} + (3-2\nu)z\frac{\partial}{\partial z} + (1-\nu)^2 + z^2$,
(iii) $\int(B_z^\nu\phi)\,\psi dz = \int\phi\,(C_z^\nu\psi)\,dz$, *for* $\phi \in L^1_{loc}(\mathbb{R}^+)$ *and* $\psi \in D(\mathbb{R}^+)$
with $C_z^\nu = z^2\frac{\partial^2}{\partial z^2} + (1+2\nu)z\frac{\partial}{\partial z} + \nu^2 + z^2$,
(iv) The function $\psi(t,z,z') = z^{-\nu}J_0\left(\sqrt{z^2+z'^2-2zz'\cos\frac{t}{2}}\right)$ *satisfies the equation* $-\frac{1}{4}C_z^\nu\psi(t,z,z') = \frac{\partial^2}{\partial t^2}\psi(t,z,z')$.

The proofs of Lemmas 3, 4, 5 and 6 are analogous of the corresponding lemmas in El-hafed et al. [2] and are left to the reader.

4. The Classical Euler-Poisson-Darboux Equation

Proof of Theorem 1. – To prove that $U(t,\theta)$ satisfies equation (a), we use Lemmas 4 and 5.

– To see the initial conditions, we introduce the polar coordinates centralized in θ: $\theta' = \theta + \tan\frac{r}{2}\omega,\ \omega \in \mathbb{S}^{n-1}$, and the change of variable $\sin\frac{r}{2} = (\sin\frac{t}{2})s,\ 0 < s < 1$, we obtain:

$$U(t,\theta) = 2^{n+2\mu}C_{n,-\mu}\cos^{2\mu}\frac{t}{2}\int_0^1 f_\theta^\#(\varphi\sin\frac{t}{2})(1-s^2)^{-\mu-\frac{n}{2}}$$

$$\times\left(1+s^2\sin^2\frac{t}{2}\right)^{\frac{n-2}{2}}s^{n-1}ds + \frac{C_{n,\mu}}{2\mu}2^n\sin^{2\mu}\frac{t}{2}$$

$$\times \int_0^1 g_\theta^\#(\varphi \sin\frac{t}{2})(1-s^2)^{\mu-\frac{n}{2}} \left(1+s^2\sin^2\frac{t}{2}\right)^{\frac{n-2}{2}} s^{n-1}ds,$$

where $f_\theta^\#(r) = \int_{S^{n-1}} f(\theta + r\omega)d\sigma(\omega)$ and $\varphi = \dfrac{s}{\sqrt{1+s^2\sin^2\frac{t}{2}}}$,

since $\int_{S^{n-1}} d\sigma(\omega) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ and $\int_0^1 (1-s^2)^{-\mu-\frac{n}{2}} s^{n-1}ds = \frac{\Gamma(1-\mu-\frac{n}{2})\Gamma(\frac{n}{2})}{2\Gamma(1-\mu)}$.

5. The Radial Wave Equation

Proof of Theorem 2. – To prove that the kernel $W(t,\theta,\theta')$ given in (2.1) satisfies the equation $(a)'$, $(\mu = \frac{1}{2})$, we use Lemma 6.

– To see the initial conditions, we use Lemma 3 and the change of variables:

$$\theta' = 2\arcsin\left(\frac{\sin\frac{\theta}{2} + q\sqrt{1-p^2}\sin\frac{t}{2}}{\cos\frac{t}{2} + ip\sin\frac{t}{2}}\right), \quad -1 < p, q < 1.$$

Remark 1. By applying the Jacobi transform (3.1) to this problem, we have from Lemma 1:

$$\widehat{U}(t,\lambda - q) = \cos(\lambda t)\widehat{f}(\lambda - q) + \frac{\sin(\lambda t)}{\lambda}\widehat{g}(\lambda - q). \tag{5.1}$$

By using the inversion formula (3.3) and interchange the order of integration we have from Lemma 2:

$$U(t,\theta) = \int_0^\pi f(\theta')\frac{\partial}{\partial t}W(t,\theta,\theta')(\sin\theta')^{1-2\nu}d\theta' + \int_0^\pi g(\theta')W(t,\theta,\theta')(\sin\theta')^{1-2\nu}d\theta',$$

$$W(t,\theta,\theta') = \frac{\pi}{4^{q-1}}\int_0^{+\infty}\left[\frac{\Gamma(\lambda+q)}{\Gamma(\lambda+\frac{1}{2})}\right]^2 P_{\lambda-q}^\nu(\cos\theta)P_{\lambda-q}^\nu(\cos\theta') \times \frac{\sin(\lambda t)\cot[(q-\lambda)\pi]}{\Gamma(q-\lambda)\Gamma(q+\lambda)}d\lambda.$$

6. The Radial Euler-Poisson-Darboux Equation

Proof of Theorem 3. – To prove that $U(t,\theta)$ satisfies equation $(a)'$, we use Lemmas 4, 5 and 6.

– To see the initial conditions, we use Lemma 3 and the change of variables:

$$\sin\frac{y}{2} = s\sin\frac{t}{2} \text{ and } \theta' = 2\arcsin\left(\sin\frac{\theta}{2} + q\sqrt{1-p^2}s\sin\frac{t}{2}\right), \quad -1<q<1.$$

Remark 2. By applying the Jacobi transform to this problem, we have from Lemma 1:

$$\widehat{U}(t,\lambda-q) = (\cos\frac{t}{2})^{2\mu}\ {}_2F_1(\frac{1}{2}-\lambda,\frac{1}{2}+\lambda,1-\mu,\sin^2\frac{t}{2})\widehat{f}(\lambda-q)$$

$$+\frac{1}{2\mu}(2\sin\frac{t}{2})^{2\mu}\ {}_2F_1(\frac{1}{2}-\lambda,\frac{1}{2}+\lambda,1+\mu,\sin^2\frac{t}{2})\widehat{g}(\lambda-q). \tag{6.1}$$

By the inversion formula and interchanging the order of integration, we have from Lemma 2:

$$U(t,\theta) = (\sin t)^{2\mu}\int_0^\pi f(\theta')W_{-\mu}(t,\theta,\theta')(\sin\theta')^{1-2\nu}d\theta'$$

$$+\frac{1}{2\mu}\int_0^\pi g(\theta')W_\mu(t,\theta,\theta')(\sin\theta')^{1-2\nu}d\theta',$$

$$W_\mu(t,\theta,\theta') = \frac{\pi}{4^{q-1}}\left(2\sin\frac{t}{2}\right)^{2\mu}\int_0^{+\infty}\left[\frac{\Gamma(\lambda+q)}{\Gamma(\lambda+\frac{1}{2})}\right]^2 P^\nu_{\lambda-q}(\cos\theta)P^\nu_{\lambda-q}(\cos\theta')$$

$$\times\,{}_2F_1(\frac{1}{2}-\lambda,\frac{1}{2}+\lambda,1+\mu,\sin^2\frac{t}{2})\frac{\lambda\cot[(q-\lambda)\pi]}{\Gamma(q-\lambda)\Gamma(q+\lambda)}d\lambda.$$

7. Applications

Corollary 1. (Bunke and Olbrich [7], Proposition 2.2) The classical wave equation in the spherical space of dimension n). *We let $\mu\to\frac{1}{2}$ in Theorem 1, we obtain the solution of the Cauchy problem for the classical wave equation in $\mathbb{S}^n$ ($f=0$):*

$$U(t,\theta) = \frac{1}{2(2\pi)^m}\left(\frac{\partial}{\sin t\partial t}\right)^{m-1}\frac{1}{\sin t}\int_{S_t(\theta)} g(\theta')d\mu(\theta'),$$

when n is odd ($n = 2m+1$), where $S_t(\theta)$ is the sphere of radius t around θ;

$$U(t,\theta) = \frac{1}{\sqrt{2}(2\pi)^m}\left(\frac{\partial}{\sin t\partial t}\right)^{m-1}\int_{\mathbb{S}^n} g(\theta')\Re\frac{1}{\sqrt{\cos(t)-\cos(d(\theta,\theta'))}}d\mu(\theta'),$$

when n is even ($n = 2m$), where $d(\theta,\theta')$ is the spherical distance between θ and θ'.

Corollary 2. (The radial wave equation in the spherical space one-dimensional) *We let $\mu \to \frac{1}{2}$ in Theorem 3. We obtain the solution of the Cauchy problem for the radial wave equation (see Theorem 2).*

8. Numerical Trials

Example. When $\nu = -\frac{1}{2}$ the radial wave problem

$$(P)\begin{cases} (\frac{\partial^2}{\partial x^2} + 2\cot x\frac{\partial}{\partial x} - 1)U(t,x) = \frac{\partial^2}{\partial t^2}U(t,x), \\ \\ U(0,x) = 0, \quad U_t(0,x) = \sin x \end{cases}$$

has a unique solution given by

$$U(t,x) = \frac{2t - \sin(2t)\cos(2x)}{4\sin x}.$$

We compare the exact solution with the approximate solution obtained by discretization of an interval $[A,B]$, $A > 0$ with a step Δx and a discretization of time with a step Δt (see Figure 1).

Let $x_j = A + j\Delta x$, $1 \le j \le n_x$, $L = B - A$, $\Delta x = L/(n_x+1)$ and $t_n = n_t\Delta t$.

Numerically solving the problem (P) means finding a discrete function U defined in points (x_j, t_n), we note U_j^n the values of U at these points. The function U is obtained as the solution of a discrete problem

$$[1-\theta-(1-\eta)R_j]U_{j-1}^{n+1} - [2(1-\theta)+r_1]U_j^{n+1} + [1-\theta+(1-\eta)R_j]U_{j+1}^{n+1}$$
$$= (\eta R_j - \theta)U_{j-1}^n + (2\theta - 2r_1 - r_2)U_j^n - (\theta + \eta R_j)U_{j+1}^n + r_1 U_j^{n-1}$$
$$U_j^0 = 0, \quad U_j^{-1} = -(\Delta t)g(x_j),$$

where $r_1 = \dfrac{(\Delta x)^2}{(\Delta t)^2}$, $r_2 = -(\Delta x)^2$ and $R_j = \dfrac{\Delta x}{\tan(x_j)}$. (We take $A = 0.5$, $B = 3$, $n_x = 10$, $n_t = 30$, $\Delta t = 0.01$ and $\theta = \eta = 0.5$).

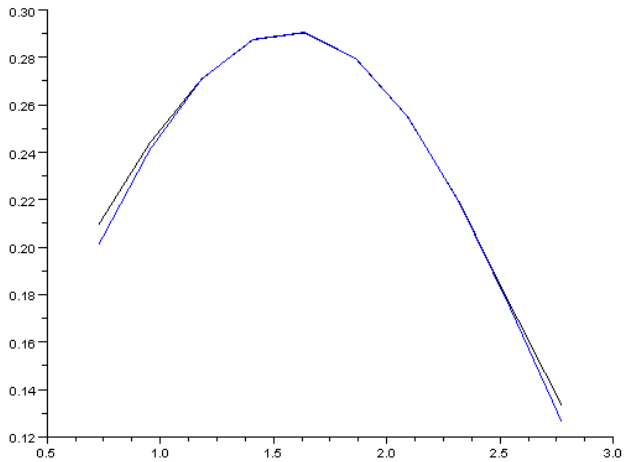

Figure 1: Representation of the two solutions to the radial wave problem

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