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NEW A PRIORI ESTIMATIONS OF THE SOLUTION OF QUASI-INVERSE PROBLEM

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Abstract: In R. Almomani and H. Almefleh [1], the authors formulated the control problem of heat conduction problem with inverse direction of time and integral boundary conditions and they show the non-wellposedness of this problem. In H. Almefleh [2], the author reduced the solution of the control problem of the inhomogeneous heat equation to the homogeneous case. In H. Almefleh, R. Almomani [3] the authors established a priori estimate for the solution of quasi-inverse problem. In this paper we establish a new priori estimate for the same problem and the same order but with another weight function. The solution of our problem plays an important role in optimal control in heat conduction theory and in plasma physics, that is, in those problems where we have an integral restriction on a function.

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We establish a priori estimate of the solution of

$$\frac{\partial U_\epsilon}{\partial t} - \frac{\partial^2 U_\epsilon}{\partial x^2} - \epsilon \frac{\partial^3 U_\epsilon}{\partial x^2 dt} = 0, \quad (x,t) \in Q,\ t < T$$

$$U_\epsilon(x,T) = X(x), \quad 0 \le x \le 1$$

$$U_\epsilon(0,t) = 0, \ \int_0^1 U_\epsilon(x,t)dx = 0,\ 0 \le t \le T \tag{1}$$

of the same order as in

$$\int\limits_Q (1-x)\left(\frac{\partial U_\epsilon}{\partial t}\right) dxdt + \frac{\epsilon}{2}\int\limits_Q (1-x)\left(\frac{\partial^2 U_\epsilon}{\partial x \partial t}\right) dxdt$$

$$\le \frac{T}{2} e^{\frac{4T^2}{\epsilon^2}} \int\limits_0^1 (1-x)\left(X'(x)\right)^2 dx, \tag{2}$$

H. Almefleh, R. Almomani [3] but with another weight function.

By scalar multiplication of (1) in $L_2(Q_\tau)$ by the function

$$(1-x)^2 \frac{\partial U_\epsilon}{\partial t} + 2(1-x) J \frac{\partial U_\epsilon}{\partial t}, \tag{3}$$

and doing the same procedure as of proving (2), we get

$$\int\limits_0^\tau \int\limits_0^1 \left(\frac{\partial U_\epsilon}{\partial t} - \frac{\partial^2 U_\epsilon}{\partial x^2} - \epsilon \frac{\partial^3 U_\epsilon}{\partial x^3 \partial t}\right)$$

$$\times \left[(1-x)^2 \frac{\partial U_\epsilon}{\partial t} + 2(1-x) J \frac{\partial U_\epsilon}{\partial t}\right] dxdt = 0. \tag{4}$$

Using $\frac{\partial U_\epsilon}{\partial t} = \frac{\partial}{\partial x} J \frac{\partial U_\epsilon}{\partial t}$, integrating by parts, and taking into account the boundary conditions for the function $U_t(x,t)$, we set the following equalities

$$2\int\limits_0^\tau \int\limits_0^1 (1-x)\frac{\partial U_\epsilon}{\partial t} J \frac{\partial U_\epsilon}{\partial t} dxdt = \int\limits_0^\tau \int\limits_0^1 \left(J \frac{\partial U_\epsilon}{\partial x}\right)^2 dxdt, \tag{5}$$

$$\int_0^\tau\int_0^1 \frac{\partial^2 U_\epsilon}{\partial x^2}\left[(1-x)^2\frac{\partial U_\epsilon}{\partial t}+2(1-x)J\frac{\partial U_\epsilon}{\partial t}\right]dxdt$$
$$= -\int_0^\tau\int_0^1 (1-x)^2\frac{\partial U_\epsilon}{\partial x}\frac{\partial^2 U_\epsilon}{\partial x\partial t}dxdt - 2\int_0^\tau\int_0^1 U_\epsilon\frac{\partial U_\epsilon}{\partial t}dxdt, \tag{6}$$

$$-\epsilon\int_0^\tau\int_0^1 \frac{\partial^3 U_\epsilon}{\partial x^2\partial t}\left[(1-x)^2\frac{\partial U_\epsilon}{\partial t}+2(1-x)J\frac{\partial U_\epsilon}{\partial t}\right]dxdt$$
$$= \epsilon\int_0^\tau\int_0^1 (1-x)^2\left(\frac{\partial^2 U_\epsilon}{\partial x\partial t}\right)^2 dxdt + 2\epsilon\int_0^\tau\int_0^1\left(\frac{\partial U_\epsilon}{\partial x}\right)^2 dxdt. \tag{7}$$

From (4), (5), (6) and (7) we imply the following equality

$$\int_0^\tau\int_0^1 (1-x)^2\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt + \int_0^\tau\int_0^1\left(J\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt$$
$$+\epsilon\int_0^\tau\int_0^1 (1-x)^2\left(\frac{\partial^2 U_\epsilon}{\partial x\partial t}\right)^2 dxdt + 2\epsilon\int_0^\tau\int_0^1\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt$$
$$= \int_0^\tau\int_0^1 (1-x)^2\frac{\partial U_\epsilon}{\partial x}\frac{\partial^2 U_\epsilon}{\partial x\partial t}dxdt + 2\int_0^\tau\int_0^1 U_\epsilon\frac{\partial U_\epsilon}{\partial t}dxdt. \tag{8}$$

We estimate the integrals in the right hand side of (8) by means of the integrals in the left hand side in the following way:

$$\int_0^\tau\int_0^1 (1-x)^2\frac{\partial U_\epsilon}{\partial x}\frac{\partial^2 U_\epsilon}{\partial x\partial t}dxdt$$
$$\leq \frac{\epsilon}{2}\int_0^\tau\int_0^1 (1-x)^2\left(\frac{\partial^2 U_\epsilon}{\partial x\partial t}\right)^2 dxdt + \frac{1}{2\epsilon}\int_0^\tau\int_0^1\left(\frac{\partial U_\epsilon}{\partial x}\right)^2 dxdt,$$

$$2\int_0^\tau\int_0^1 U_\epsilon\frac{\partial U_\epsilon}{\partial t}dxdt \leq \epsilon\int_0^\tau\int_0^1\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt + \frac{1}{\epsilon}\int_0^\tau\int_0^1 U_\epsilon^2 dxdt.$$

As a result, we get the following inequality

$$\int_0^\tau\int_0^1 (1-x)^2\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt+\int_0^\tau\int_0^1\left(J\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt$$
$$+\frac{\epsilon}{2}\int_0^\tau\int_0^1 (1-x)^2\left(\frac{\partial^2 U_\epsilon}{\partial x\partial t}\right)^2 dxdt+\epsilon\int_0^\tau\int_0^1\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt$$
$$\leq \frac{1}{2\epsilon}\int_0^\tau\int_0^1 (1-x)^2\left(\frac{\partial U_\epsilon}{\partial x}\right)^2 dxdt+\frac{1}{\epsilon}\int_0^\tau\int_0^1 U_\epsilon^2 dxdt. \quad (9)$$

The second term in the right hand side of (9) can be estimated by the first term using the inequality

$$\int_0^1 U^2 dx\leq 4\int_0^1 (1-x)^2\left(\frac{\partial U_\epsilon}{\partial x}\right)^2 dx. \quad (10)$$

Which is correct for any continuous function $U(x)$ and this function turns to zero at $x=0$, that is $U(0)=0$.

Thus the right hand side of (9) is bounded above by

$$\frac{1}{2\epsilon}\int_0^\tau\int_0^1 (1-x)^2\left(\frac{\partial U_\epsilon}{\partial x}\right)^2 dxdt. \quad (11)$$

In addition, since (8) holds by analogy of (10) but with factor $(1-x)^2$, the right hand side of (9) can be estimated above by

$$\frac{18T}{\epsilon}\int_0^\tau dt\int_0^t\int_0^1 (1-x)^2\left(\frac{\partial^2 U_\epsilon}{\partial x\partial y}\right)^2 dxdy+\frac{9T}{\epsilon}\int_0^1 (1-x)^2\left(X'(x)\right)^2 dx. \quad (12)$$

And from (9) we get

$$\int_0^\tau\int_0^1 (1-x)^2\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt + \int_0^\tau\int_0^1 \left(J\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt$$
$$+\frac{\epsilon}{2}\int_0^\tau\int_0^1 (1-x)^2\left(\frac{\partial^2 U_\epsilon}{\partial x\partial t}\right)^2 dxdt + \epsilon\int_0^\tau\int_0^1 \left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt$$
$$\leq \frac{18T}{\epsilon}\int_0^\tau dt\int_0^t\int_0^1 (1-x)^2\left(\frac{\partial^2 U_\epsilon}{\partial x\partial y}\right)^2 dxdy$$
$$+\frac{9T}{\epsilon}\int_0^1 (1-x)^2\left(X'(x)\right)^2 dx. \tag{13}$$

From (13) and using Gronwall's inequality, we get the following a prior estimation for the solution of (1):

$$\int_0^\tau\int_0^1 (1-x)^2\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt + \int_0^\tau\int_0^1 \left(J\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt$$
$$+\frac{\epsilon}{2}\int_0^\tau\int_0^1 (1-x)^2\left(\frac{\partial^2 U_\epsilon}{\partial x\partial t}\right)^2 dxdt + \epsilon\int_0^\tau\int_0^1 \left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt$$
$$\leq \frac{9T}{\epsilon}e^{36T^2/\epsilon^2}\int_0^1 (1-x)^2\left(X'(x)\right)^2 dx, \tag{14}$$

or in the following more compact form:

$$\int_Q (1-x)^2\left(\frac{\partial U_\epsilon}{\partial t}\right)^2 dxdt + \frac{\epsilon}{2}\int_Q (1-x)^2\left(\frac{\partial^2 U_\epsilon}{\partial x\partial t}\right)^2 dxdt$$
$$\leq \frac{9T}{\epsilon}e^{36T^2/\epsilon^2}\int_0^1 (1-x)^2\left(X'(x)\right)^2 dx.$$

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