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A FRACTIONAL NEWTON SERIES FOR THE RIEMANN ZETA FUNCTION

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Abstract: In recent years much attention has been given to Newton series representations of a regularized Zeta function. Such representations are limited as they do not lead to a series expansion for the Zeta function that converges in the critical strip. In this paper, we define a fractional Newton series which serves as the meromorphic continuation of a classical Newton Series. We show that the Riemann Zeta function can be represented by a fractional Newton series in the critical strip. Under this representation the coefficients are given in terms of differences of the zeta function evaluated at the positive half integers instead of the usual situation of evaluating at the integers. Using the method of stationary phase, we derive an asymptotic formula for these zeta differences.

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1. Introduction

A Newton series on the positive integers is an expression of the form

$$c_1 + c_2(s-1) + c_3(s-1)(s-2) + c_4(s-1)(s-2)(s-3) + \ldots\ldots = \sum_{k=1}^{\infty} c_k \frac{\Gamma(s)}{\Gamma(s-k+1)}. \tag{1}$$

If such a series converges, then it will always converge on a half plane $Re(s) > \lambda$,

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where λ is referred to as the abscissa of convergence. This paper is based on the recent interest in the representation of a regularized Riemann Zeta function by such a series. In this paper we use the standard notation for the Zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

One of the main papers in this area is [2], where the regularized Zeta function

$$\zeta(s) - \frac{1}{1-s} \tag{2}$$

is represented by a Newton series whose nodes are the non-negative integers, and estimates are made for the asymptotics of the coefficients. Whereas in [5], a Newton series representation for the function

$$\zeta(s)(1-s) \tag{3}$$

was studied. Here we state the main result from [2].

Theorem A *For $s \in \mathbf{C}$ we have*

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=0}^{\infty} (-1)^n b_n \binom{s}{n}, \tag{4}$$

where $\binom{s}{n}$ is a binomial coefficient. Furthermore, the coefficients b_n satisfy

$$b_n = \left(\frac{2n}{\pi}\right)^{1/4} e^{-2\sqrt{\pi n}} \cos\left(2\sqrt{\pi n} - \frac{5\pi}{8}\right) + O(e^{-2\sqrt{\pi n}} n^{-1/4}). \tag{5}$$

This theorem gives an asymptotic expansion for the coefficients of the Newton series in (4). The series is convergent in the whole plane. In this paper we look at the idea of expanding a non-regularized Zeta function in a Newton series. Since the Zeta function has a pole at $s = 1$, we would use the set $\{2, 3, 4, ...\}$ as the nodes for the expansion and look for a representation of the form

$$\zeta(s) = c_0 + c_1(s-2) + c_2(s-2)(s-3) + ... \tag{6}$$

In light of the work done in [2], the most efficient way to do this would be to first form the expansion

$$\zeta(s) - \frac{1}{s-1} = d_0 + d_1(s-2) + d_2(s-2)(s-3) + ... \tag{7}$$

which would be valid in the whole plane, then find a similar expansion for the function $1/(s-1)$ and add it to both sides. Since the expansion for the function $1/(s-1)$ would only be valid in the half plane $Re(s) > 1$ the resulting expansion for the Zeta function would only be valid in the same half plane.

In this paper we introduce a series expansion related to a Newton series that we will refer to as a fractional Newton series. We will then show that we can expand the Zeta function in terms of the fractional Newton series and obtain an expansion that is valid for $Re(s) > 0$. In other words we will find that the fractional Newton series provides a meromorphic continuation of the classical Newton series for the Riemann Zeta function into the critical strip. At this point we introduce the main definition and state the two main results. For cosmetic purposes all of the Newton series we consider in the remainder of this paper have nodes existing in a left half plane.

Definition 1. Let μ be a complex number. We shall refer to any series of the form

$$\sum_{k=1}^{\infty} c_k \frac{\Gamma\left(\mu+\frac{k}{2}\right)}{\Gamma(\mu+1)} \tag{8}$$

as a fractional Newton series.

Note that the terms indexed by an even k are polynomials that appear in a standard Newton series with the set of nodes $\{-1,-2,-3,...\}$. The terms indexed by an odd k are the additional feature of a fractional Newton series.

Definition 2. Throughout this paper we will refer to the sets Ω and $\widetilde{\Omega}$, defined to be

$$\Omega = \{\mu | Re(\mu) < -1/2\} \setminus \{-1, -3/2, -2, \cdots\},$$
$$\widetilde{\Omega} = \{\mu | Re(\mu) < 1/2\} \setminus \{-1/2, -1, -3/2, \cdots\}.$$

Our two main results are as follows:

Theorem 3. *Let $\mu \in \widetilde{\Omega}$. Then*

$$\frac{1}{2\mu+1} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)} \frac{\Gamma\left(\mu+\frac{k}{2}\right)}{\Gamma(\mu+1)}. \tag{9}$$

The above theorem gives the fractional Newton series expansion for a simple pole.

Theorem 4. *Let $\mu \in \widetilde{\Omega}$. Then*

$$-\frac{1}{2}\zeta\left(-\mu+\frac{1}{2}\right)=\sum_{k=1}^{\infty}(-1)^{k+1}\gamma_k\frac{\Gamma\left(\mu+\frac{k}{2}\right)}{\Gamma(\mu+1)}, \tag{10}$$

where

$$\gamma_k=\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)} \qquad k \text{ odd}$$

and

$$\gamma_k=\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}+\frac{1}{k!}\left[\left(\sqrt{\pi k}\right)e^{-2\sqrt{\pi k}}\cos\left(2\sqrt{\pi k}\right)+O\left(e^{-2\sqrt{\pi k}}\right)\right] \tag{11}$$
$$k\to\infty \qquad k \text{ even.}$$

Remark 1. The critical strip for the Riemann Zeta function appearing on the left hand side of (10) is contained in $\widetilde{\Omega}$.

Remark 2. In [3] and [4], for $b \in (0,1)$ the expression

$$\sum_{k=1}^{\infty}(-b)^{k+1}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)}\frac{\Gamma\left(\mu+\frac{k}{2}\right)}{\Gamma(\mu+1)} \tag{12}$$

was studied and was shown to be equal to a certain integral over the set $[-1,-b]\cup[b,1]$. Whilst it is possible to prove Theorem **3** by taking a limit as $b\to 1^-$ it is of interest to have a direct proof that does not rely on the introduction of the parameter b.

In the next section we give proofs of Theorems **3** and **4**.

2. Proof of Main Results

We begin this section with three lemmas that will be necessary for the proofs. The first lemma is a generalized version of Dirichlet's convergence test. We present the proof for convenience since in the literature it is mainly the standard Dirichlet test that is presented.

Lemma 5. *Suppose that* $\{a_n\}$ *is a sequence such that* $|\sum_{j=1}^{n} a_j| < M$ *for all* n. *If* $\{c_n\}$ *is of bounded variation and* $\lim_{n\to\infty} c_n = 0$, *then* $\sum a_n c_n$ *converges.*

Proof. Let $A_k = \sum_{j=1}^{n} a_k$ and let $A_0 = 0$. Then

$$\begin{aligned}\sum_{j=1}^{n} a_j c_j &= \sum_{j=1}^{n} (A_j - A_{j-1}) c_j \\ &= \sum_{j=1}^{n} A_j c_j - \sum_{j=1}^{n} A_{j-1} c_j \\ &= \sum_{j=1}^{n} A_j c_j - \sum_{j=0}^{n-1} A_j c_{j-1} \\ &= a_n c_n + \sum_{j=0}^{n-1} A_j (c_j - c_{j-1}).\end{aligned}$$

Therefore, if $n > m$, then

$$\left|\sum_{j=m+1}^{n} a_j c_j\right| \leq |a_n c_n - a_m c_m| + M \sum_{j=m}^{n-1} |c_j - c_{j+1}|. \tag{13}$$

Thus, it is easy to see that the partial sums of $\sum a_j c_j$ form a Cauchy sequence. □

Lemma 6. *Let* $\Theta = \{\mu | Re(\mu) < -1\} \setminus \mathbf{Z}$. *If* $\mu \in \Theta$, *then*

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+\mu)}{\Gamma(n)} = 0.$$

Proof. Let K be a compact subset of Θ and $\mu \in K$. Since $\lim_{n\to\infty} \dfrac{\Gamma(n+\mu)}{\Gamma(n) n^{\mu}} = 1$ it follows that for some $M > 0$ and $n = 1, 2, \cdots$

$$\left|\frac{\Gamma(n+\mu)}{\Gamma(n)}\right| < M |n^{\mu}|.$$

Thus from Weistrass M-test it follows that $\displaystyle\sum_{n=1}^{m} \frac{\Gamma(n+\mu)}{\Gamma(n)}$ converges uniformly on K as m tends to ∞. For each $n = 0, 1, 2, ...$, the function $\Gamma(n+\mu)$ is analytic on Θ. Therefore, if $f(\mu) = \sum_{n=1}^{\infty} \frac{\Gamma(n+\mu)}{\Gamma(n)}$, then f is analytic on Θ. If $\mu \in \Theta$, then

$$\begin{aligned} f(\mu) &= \sum_{n=1}^{\infty} \frac{\Gamma(n-1+\mu+1)}{\Gamma(n)} \\ &= \sum_{n=1}^{\infty} (n-1+\mu) \frac{\Gamma(n-1+\mu)}{\Gamma(n)} \\ &= \sum_{n=1}^{\infty} (n-1) \frac{\Gamma(n-1+\mu)}{\Gamma(n)} + \sum_{n=1}^{\infty} \mu \frac{\Gamma(n-1+\mu)}{\Gamma(n)} \\ &= \sum_{n=2}^{\infty} \frac{\Gamma(n-1+\mu)}{\Gamma(n-1)} + \mu \sum_{n=1}^{\infty} \frac{\Gamma(n+(\mu-1))}{\Gamma(n)}. \end{aligned}$$

Thus for all $\mu \in \Theta$,

$$f(\mu) = f(\mu) + \mu f(\mu - 1).$$

Therefore $f(\mu-1) = 0$. But f is analytic on Θ hence $f(\mu) = 0$ for $\mu \in \Theta$. □

Lemma 7. *Let $\mu \in \widetilde{\Omega}$. If*

$$b_k = \frac{\Gamma(\mu + k/2)}{\Gamma(k/2 + 1/2)}$$

for $k = 1, 2, \cdots$, then $\sum_{k=1}^{\infty} |b_k - b_{k+1}|$ converges.

Proof. If α, β are complex constants and if $z+\alpha, z+\beta$ are not negative integers, then

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta} \left(1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{2z} + O(|z|^{-2}) \right) \tag{14}$$

as $z \to \infty$ (see [8, pg. 133]). Let $z = k/2, \alpha = \mu$ and $\beta = 1/2$. Then

$$b_k = \left(\frac{k}{2}\right)^{\mu-1/2} \left(1 + \frac{(\mu-1/2)(\mu-1/2)}{k} + O(k^{-2}) \right).$$

Now let $z = k/2, \alpha = \mu + 1/2$ and $\beta = 1$ in (14). Then

$$b_{k+1} = \left(\frac{k}{2}\right)^{\mu-1/2} \left(1 + \frac{(\mu-1/2)(\mu+1/2)}{k} + O(k^{-2}) \right).$$

Therefore

$$b_k - b_{k+1} = 2^{1/2-\mu}\left(\frac{1/2-\mu}{k^{3/2-\mu}} + \frac{1}{k^{3/2-\mu}}O(k^{-2})\right) \tag{15}$$

for k large enough. Since $Re(\mu) < 1/2$, it follows that $Re(3/2-\mu) > 1$. Hence $\sum |k^{\mu-3/2}|$ converges. Thus the convergence of $\sum |b_k - b_{k+1}|$ follows. Since the absolute values of terms on the right side of (15) depends only on k and the real part of μ, it is easy to see that the series converges uniformly on compact subsets of $\widetilde{\Omega}$. □

We now give the proof of Theorem **3**.

Proof. Let

$$f(\mu) = \frac{1}{2\mu+1}.$$

From [6], pages 307-308, the function f has the Newton series representation

$$\frac{1}{2\mu+1} = -\sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)} \frac{\Gamma(\mu+k)}{\Gamma(\mu+1)}, \qquad Re(\mu) < -\frac{1}{2}. \tag{16}$$

The Newton series above consists of the even terms from (9). We now use the fact that for $\mu \in \Omega$ the odd terms from (9) sum to zero,

$$\sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{2k-1}{2}+\frac{1}{2}\right)} \frac{\Gamma\left(\mu+\frac{2k-1}{2}\right)}{\Gamma(\mu+1)} = 0. \tag{17}$$

This follows from Lemma **6**. Both sums (16) and (17) are divergent for $Re(\mu) > -\frac{1}{2}$ however if we interlace the terms we will see that the resulting series converges on the set $\widetilde{\Omega}$. Now, for $\mu \in \Omega$

$$\begin{aligned} \frac{1}{2\mu+1} &= -\sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)} \frac{\Gamma(\mu+k)}{\Gamma(\mu+1)} \\ &= -\sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)} \frac{\Gamma(\mu+k)}{\Gamma(\mu+1)} + \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{2k-1}{2}+\frac{1}{2}\right)} \frac{\Gamma\left(\mu+\frac{2k-1}{2}\right)}{\Gamma(\mu+1)} \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)} \frac{\Gamma\left(\mu+\frac{k}{2}\right)}{\Gamma(\mu+1)}. \end{aligned} \tag{18}$$

Using Lemma **5** together with Lemma **7** we see that the series appearing in the third line is uniformly convergent on compact subsets of $\widetilde{\Omega}$. The result follows by meromorphic continuation. □

We now give a proof of Theorem **4**. The method is based on the method of stationary phase applied to Norlund-Rice type integrals. We follow very closely the technique used in [2], pages 62 – 66.

Proof. We begin with the entire function

$$h(\mu) = -\frac{1}{2}\zeta\left(-\mu + \frac{1}{2}\right) - \frac{1}{2\mu + 1}$$

on the negative integers. By the second theorem on page 311 of [6], the function h can be represented by a Newton series on the negative integers.

$$h(\mu) = \sum_{k=1}^{\infty} \beta_k \frac{\Gamma(\mu + k)}{\Gamma(\mu + 1)} \qquad \text{where} \qquad \beta_k = \sum_{j=1}^{k} \frac{(-1)^{j-1} h(-j)\binom{k}{j}}{k!}.$$

The β_k can be written in terms of a Norlund integral representation, see [7] and [2], and a closed form expression from (16). We have,

$$\beta_k = \frac{(-1)^k}{4\pi i} \int_C \zeta\left(s + \frac{3}{2}\right) \frac{1}{s(s-1)\dots(s-[k-1])}\, ds + \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(k + \frac{1}{2}\right)},$$

where C encircles the integers $\{0, 1, 2, \dots, k-1\}$ in a positive direction and is contained in $Re(s) \geq -\frac{1}{2}$. The contour can be chosen to be made up of a line from $-1/4 - iT$ to $-1/4 + iT$ followed by a clockwise arc of radius T, that lies to the right of the line. As $T \to \infty$ the contribution from the arc vanishes. Hence

$$\beta_k = \frac{(-1)^{k-1}}{4\pi i} \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \zeta\left(s + \frac{3}{2}\right) \frac{1}{s(s-1)\dots(s-[k-1])}\, ds + \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(k + \frac{1}{2}\right)}.$$

We then move the contour to the left and account for the residue at $s = -\frac{1}{2}$ to obtain

$$\beta_k = \frac{(-1)^{k-1}}{4\pi i} \int_{-2-i\infty}^{-2+i\infty} \zeta\left(s + \frac{3}{2}\right) \frac{1}{s(s-1)\dots(s-[k-1])}\, ds. \tag{19}$$

We now use the reflection formula for the zeta function

$$\zeta(s) = 2\Gamma(1-s)(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \zeta(1-s) \tag{20}$$

to obtain

$$\beta_k = \frac{(-1)^{k-1}}{2\pi i} \int_{-2-i\infty}^{-2+i\infty} \Gamma\left(-s - \frac{1}{2}\right) \sin\left(\frac{\pi\left(s + \frac{3}{2}\right)}{2}\right) f(s)\, ds, \tag{21}$$

where

$$f(s) = \frac{(2\pi)^{s+\frac{1}{2}} \zeta\left(-s - \frac{1}{2}\right)}{s(s-1)\ldots(s - [k-1])}.$$

We then make the substitution $t = -s - 3/2$ to obtain

$$\beta_k = -\frac{1}{k!}\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \Gamma(t+1) \sin\left(\frac{\pi t}{2}\right) g(t)\, dt, \tag{22}$$

where

$$g(t) = \frac{k!(2\pi)^{-t-1}\zeta(t+1)}{(t+3/2)(t+5/2)\ldots(t+k+1/2)}.$$

The integral appearing in the previous expression is, up to a constant, identical to the integral appearing in [2], equation (19), except that the zeroes in the denominator have half integer instead of integer values. We now make approximations to the integrand as $|t| \to \infty$. In particular we make use of the approximations

$$\zeta(t) = 1 + O\left(2^{-Re(t)}\right), \qquad Re(t) \geq 3/2$$

and also in the upper half plane

$$\sin\left(\frac{\pi t}{2}\right) = \frac{-1}{2i} e^{\frac{-i\pi t}{2}} + \frac{1}{2i} O\left(e^{-\frac{\pi}{2} Im(t)}\right).$$

Therefore we think of (22) as

$$\beta_k = -\frac{1}{k!}\frac{1}{4\pi} \int_{1/2-i\infty}^{1/2+i\infty} e^{\omega(t)}\, dt, \tag{23}$$

where a model for $\omega(t)$ in the upper half plane is

$$\begin{aligned} \omega_{upper}(t) \simeq (-t-1)\log(2\pi) - i\frac{\pi t}{2} \\ + \log\left[\frac{\Gamma(k+1)\Gamma(t+1)\Gamma(t+3/2)}{\Gamma(t+k+3/2)}\right]. \end{aligned} \tag{24}$$

Furthermore we make use of Stirling's approximation

$$\Gamma(t+1) = \sqrt{2\pi}\, t^{t+1/2} e^{-t} \left(1 + O\left(\frac{1}{|t|}\right)\right), \qquad Re(t) \geq 0.$$

Letting $t = x\sqrt{k}$ to take into account the scaling of the saddle points, we obtain

$$\begin{aligned}
\omega_{upper}(x\sqrt{k}) &= x\sqrt{k}\left(-\log(2\pi) - \frac{i\pi}{2} - 2 + 2\log x\right) + \frac{1}{4}\log k \\
&\qquad + \frac{3}{2}\log x - \frac{x^2}{2} + O(k^{-1/2}), \\
\omega'_{upper}(x\sqrt{k}) &= \left(-\log(2\pi) - \frac{i\pi}{2} + 2\log x\right) \\
&\qquad + \frac{3}{2x\sqrt{k}} - \frac{x}{\sqrt{k}} + O\left(\frac{1}{k}\right), \\
\omega''_{upper}(x\sqrt{k}) &= \frac{2}{x\sqrt{k}} + O\left(\frac{1}{k}\right).
\end{aligned} \tag{25}$$

From these equations we see that an approximation for the location of the saddle point is $t = e^{i\pi/4}\sqrt{2\pi}$ and the value of δ (the angle of approach for the contour) is $5\pi/8$. These quantities are the same as in [2]. Consequently the saddle point analysis proceeds in the same way. The contour and the second order scaling are chosen as in [2]. Therefore, we substitute the approximations (25) into (23) and obtain the asymptotic approximation for the β_k (taking into account the factor of 2 for the contribution to the contour from the lower half plane). We obtain

$$\beta_k = -\frac{1}{k!}\left[\left(\sqrt{\pi k}\right) e^{-2\sqrt{\pi k}} \cos\left(2\sqrt{\pi k}\right) + O\left(e^{-2\sqrt{\pi k}}\right)\right]. \tag{26}$$

So far we have shown that

$$h(\mu) = \sum_{k=1}^{\infty} \beta_k \frac{\Gamma(\mu+k)}{\Gamma(\mu+1)}, \qquad \mu \in \mathbf{C},$$

where β_n satisfies (26). Now we use the result from theorem **3** and add the fractional Newton series for the pole. In doing so we obtain

$$-\frac{1}{2}\zeta\left(-\mu + \frac{1}{2}\right) = \sum_{k=1}^{\infty} (-1)^{k+1} \gamma_k \frac{\Gamma\left(\mu + \frac{k}{2}\right)}{\Gamma(\mu+1)}, \qquad \mu \in \widetilde{\Omega},$$

where

$$\gamma_k = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}, \qquad \text{k odd,}$$

and

$$\gamma_k = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)} + \frac{1}{k!}\left[\left(\sqrt{\pi k}\right) e^{-2\sqrt{\pi k}} \cos\left(2\sqrt{\pi k}\right) + O\left(e^{-2\sqrt{\pi k}}\right)\right], \tag{27}$$
$$k \to \infty, \qquad \text{k even.}$$

$\square$

3. Discussion

We have found that a Newton series for the Riemann Zeta function will converge in a half plane but that half plane will not contain the critical strip. For convergence in the critical strip one must consider a fractional Newton series. By doing so one naturally ends up using the half integers as nodes and the coefficients of the even terms in the fractional Newton series are divided differences of the Zeta function evaluated at these nodes.

In [3] and [4] an integral over the set $E_2 = [-1, -b] \cup [b, 1]$ was expanded in a fractional Newton series which is the expression given in (12). The integrand for this integral is very closely related to a set of orthogonal polynomials on the set E_2. A natural question to consider in the future is what kind of integral and possibly what kind of orthogonal polynomials are associated with the fractional Newton series

$$\sum_{k=1}^{\infty} (-b)^{k+1} \gamma_k \frac{\Gamma\left(\mu+\frac{k}{2}\right)}{\Gamma(\mu+1)}, \qquad b \in (0,1) \ \ ? \tag{28}$$

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