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A HILBERT SPACE ON LEFT-DEFINITE STURM-LIOUVILLE DIFFERENCE EQUATIONS

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Abstract: We investigate the discrete Sturm-Liouville problems

$$-\Delta(p\Delta u)(n-1) + q(n)u(n) = lw(n)u(n),$$

where p is strictly positive, q is nonnegative and w may change sign. If w is positive, the ℓ^2-space weighted by w is a Hilbert space and it is customary to use that space for setting the problem. In the present situation the right-hand-side of the equation does not give rise to a positive-definite quadratic form and we use instead the left-hand-side to definite such a form. We prove in this paper that this form determines a Hilbert space (such problems are called left-definite).

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1. Introduction

Let $\mathbf{N}$ be the set of natural number. Define $S(\mathbf{N})$ to be the set of all the sequences over $\mathbf{N}$ which are complex valued. If $u \in S(\mathbf{N})$ then define $\triangle : S(\mathbf{N}) \longrightarrow S(\mathbf{N})$ to be the first forward difference operator given by

$$(\triangle u)(n) = u(n+1) - u(n).$$

Using this definition,

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$$(\Delta(fg))(n) = g(n+1)(\Delta f)(n) + f(n)(\Delta g)(n). \tag{1.1}$$

Also, using the fact $\sum_{i=j}^{k}(\Delta u)(i) = u(k+1) - u(j)$, we get the summation by parts formula:

$$\sum_{n=j}^{N} g(n+1)(\Delta f)(n) = (fg)(N+1) - (fg)(j) - \sum_{n=j}^{N} f(n)(\Delta g)(n). \tag{1.2}$$

This equation implies

$$\sum_{n=1}^{N}(p\Delta u)(n)\overline{\Delta v(n)}$$

$$= (p\Delta u)(N)\overline{v(N+1)} - (p\Delta u)(0)\overline{v(1)} - \sum_{n=1}^{N}\Delta(p\Delta u)(n-1)\overline{v(n)}. \tag{1.3}$$

We associate the term *left-definite problem* with an inner product associated with the left hand side of the equation $Lu = wf$.

The left-definite spectral problem was first raised by Weyl in his seminal paper [10] and treated by him in [9]. There is now a large body of literature on the problem of determining spectral properties for such systems. We mention here for instance Niessen and Schneider [7], Krall [3, 4], Marletta and Zettl [6], Littlejohn and Wellman [5].

In this paper, we are interested in studying an inner product determined by the left-hand-side of the difference equation

$$-(\Delta(p\Delta u))(n-1) + q(n)u(n) = \lambda w(n)u(n); \quad n \geq 2, \tag{1.4}$$

Some spectral properties were discussed in [1] related to left-hand-side of the equation

$$-(\Delta^2 u)(n-1) + q(n)u(n) = \lambda w(n)u(n); \quad n \geq 2. \tag{1.5}$$

Unlike the continuous case, the equation (1.4) can not be transformed to (1.5).

Now, for the solutions ϕ and θ of the equation (1.4), we define the *Wronskian*, $W_{\phi,\theta}$, to be

$$W_{\phi,\theta}(n) = p(n)(\phi(n)(\Delta\theta)(n) - (\Delta\phi)(n)\theta(n)).$$

Proposition 1.1. *$W_{\phi,\theta}(n)$ is constant for all $n \in \mathbf{N}$.*

Proof. Using the product rule (1.1)

$$\begin{aligned}(\Delta W_{\phi,\theta})(n) &= \phi(n+1)(\Delta(p\Delta\theta))(n) + (\Delta\phi)(n)(p\Delta\theta)(n)\\ &\quad - (\theta(n+1)(\Delta(p\Delta\phi))(n) + (p\Delta\phi)(n)(\Delta\theta)(n))\end{aligned}$$

Using the fact that ϕ, θ are solutions for (1.4), then

$$\begin{aligned}&(\Delta W_{\phi,\theta})(n)\\ &= \phi(n+1)((q-\lambda w)\theta)(n+1) - \theta(n+1)((q-\lambda w)\phi)(n+1) = 0.\end{aligned}$$

Hence, the Wronskian is constant. □

Our main interest is studying the equation (1.4) where λ is a complex parameter and where q and w are sequences with q is defined on $\mathbf{N}_0$ and assumes non-negative real values but is not identically equal to zero, w is defined on $\mathbf{N}$ and real-valued, and p is defined on $\mathbf{N}_0$ and assumes strictly positive real values

Consider the operator on the left-hand side of (1.4) by L, i.e.,

$$(Lu)(n) = -(\Delta(pu\Delta))(n-1) + (qu)(n), \quad n \in \mathbf{N}.$$

Note that L operates from $\mathbf{C}^{\mathbf{N}_0}$ to $\mathbf{C}^{\mathbf{N}}$.

2. Main Result

Due to the fact that the sign of w is indefinite it is not convenient to phrase the spectral and scattering theory in the usual setting of a weighted ℓ^2-space, since it is not a Hilbert space. Instead the requirement that q is non-negative but not identically equal to zero allows us to define an inner product associated with the left hand side of the equation $Lu = wf$ giving rise to the term *left-definite problem.* To do so define the set

$$\mathcal{H}_1 = \{u \in \mathbf{C}^{\mathbf{N}_0} : \sum_{n=0}^{\infty}(p(n)|(\Delta u)(n)|^2 + q(n)|u(n)|^2) < \infty\}$$

and introduce the scalar product

$$< u, v >= \sum_{n=0}^{\infty}(p(n)(\Delta u)(n)\overline{(\Delta v)(n)} + q(n)u(n)\overline{v(n)}).$$

The associated norm is denoted by $\|\cdot\|$. We will also use the norm in $\ell^2(\mathbf{N}_0)$ which we denote by $\|\cdot\|_2$. We claim $\mathcal{H}_1$ with this norm is a complete space. Such a result plays a role in studying the spectral properties of (1.4).

We start with the following sequence of lemmas:

Lemma 2.1. *If $m \geq n$, then for $u \in S(\boldsymbol{N})$*

$$|u(m)| \leq |u(n)| + (\sum_{l=1}^{\infty} p(l)|(\Delta u)(l)|^2)^{1/2}(\sum_{l=n}^{m-1} \frac{1}{p(l)})^{1/2}. \tag{2.1}$$

Proof.

$$|u(m)| - |u(n)| \leq |u(m) - u(n)|,$$

and

$$|u(m) - u(n)| = |\sum_{l=n}^{m-1} (\Delta u)(l)| \leq \sum_{l=n}^{m-1} |\Delta u(l)|.$$

Now, the inequality of Cauchy-Schwarz gives that

$$\begin{aligned}\sum_{l=n}^{m-1} |\Delta u(l)| &= (\sum_{l=n}^{m-1} \sqrt{p}(l)|\Delta u|(l)(\frac{1}{\sqrt{p}(l)}) \\ &\leq (\sum_{l=n}^{m-1} p(l)|(\Delta u)(l)|^2)^{1/2}(\sum_{l=n}^{m-1} \frac{1}{p(l)})^{1/2}.\end{aligned}$$

By combining the previous inequalities, we get:

$$|u(m)| - |u(n)| \leq (\sum_{l=n}^{m-1} p(l)|(\Delta u)(l)|^2)^{1/2}(\sum_{l=n}^{m-1} \frac{1}{p(l)})^{1/2},$$

this inequality implies the required result since

$$(\sum_{l=n}^{m-1} p(l)|\Delta u)(l)|^2)^{1/2} \leq (\sum_{l=1}^{\infty} p(l)|\Delta u(l)|^2)^{1/2}.$$

□

Lemma 2.2. *If r satisfies $\sum_{n=1}^{r} q(n) > 0$, then for $1 \leq n \leq m \leq r < \infty$ and $u \in S(\boldsymbol{N})$*

$$\begin{aligned}|u(m)| \sum_{n=1}^{r} q(n) &\leq (\sum_{n=1}^{r} q(n))^{1/2}(\sum_{n=1}^{r} q(n)|u(n)|^2)^{1/2} \\ &+ C_r(\sum_{l=1}^{\infty} p(l)|\Delta u(l)|^2)^{1/2} \sum_{n=1}^{r} q(n),\end{aligned}$$

where $C_r = (\sum_{l=1}^{r} \frac{1}{p(l)})^{1/2}$.

Proof. The equation (2.1) gives that

$$|u(m)| \leq |u(n)| + C_r(\sum_{l=1}^{\infty} p(l)|(\Delta u)(l)|^2)^{1/2}.$$

Multiplying by $q(n)$ and taking the sum from 1 to r with respect to n give

$$|u(m)|\sum_{n=1}^{r} q(n) \leq \sum_{n=1}^{r} q(n)|u(n)| + C_r(\sum_{l=1}^{\infty} p(l)|\Delta u(l)|^2)^{1/2}\sum_{n=1}^{r} q(n), \tag{2.2}$$

Now, the inequality of Cauchy-Schwarz gives

$$\sum_{n=1}^{r} q(n)|u(n)| \leq (\sum_{n=1}^{r} q(n))^{1/2}(\sum_{n=1}^{r} q(n)|u(n)|^2)^{1/2}.$$

Then (2.2) becomes

$$|u(m)|\sum_{n=1}^{r} q(n) \leq (\sum_{n=1}^{r} q(n))^{1/2}(\sum_{n=1}^{r} q(n)|u(n)|^2)^{1/2}$$
$$+C_r(\sum_{l=1}^{\infty} |p(l)\Delta u(l)|^2)^{1/2}\sum_{n=1}^{r} q(n).$$

□

We are ready to prove the following lemma:

Lemma 2.3. *For any $N \in \mathbf{N}$, there exists C_N such that*

$$|u(m)| \leq C_N\|u\|_{\mathcal{H}_1},$$

for any m such that $1 \leq m \leq N$ and any $u \in \mathcal{H}_1$.

Proof. For any $N \in \mathbf{N}$ there exists $r \geq N$ such that $\sum_{n=1}^{r} q(n) > 0$. Now, Lemma 2.2 implies

$$|u(m)|\sum_{n=1}^{r} q(n) \leq \|u\|_{\mathcal{H}_1}(\sum_{n=1}^{r} q(n)^{1/2} + C_r\sum_{n=1}^{r} q(n)),$$

or

$$|u(m)| \leq C_N\|u\|_{\mathcal{H}_1},$$

where

$$C_N = C_r + (\sum_{n=1}^{r} q(n))^{-1/2}.$$

□

The following lemma gives some properties for the Cauchy sequences in $\mathcal{H}_1$.

Lemma 2.4. *Let $n \mapsto u_n(\cdot)$ be a Cauchy sequence in $\mathcal{H}_1$, then*

1. *There exists $v(\cdot)$ such that $(\sqrt{p}\Delta u_n)(\cdot) \longrightarrow v(\cdot)$ in $l^2(\mathbf{N})$ as $n \longrightarrow \infty$.*
2. *$\sqrt{q(\cdot)}u_n(\cdot) \longrightarrow \sqrt{q(\cdot)}u(\cdot)$ in $l^2(\mathbf{N})$, where $u(k) = \lim_{n\longrightarrow\infty} u_n(k)$ in $\boldsymbol{C}$ for all $k \in \mathbf{N}$.*

Proof. 1. If $n \mapsto u_n(\cdot)$ is a Cauchy sequence in $\mathcal{H}_1$, then for $\varepsilon > 0$, there exists n_0 such that for all $m, n \geq n_0$

$$\|u_m(\cdot) - u_n(\cdot)\|_{\mathcal{H}_1} < \varepsilon \tag{2.3}$$

consequently,

$$\|(\sqrt{p}\Delta u_m)(\cdot) - (\sqrt{p}\Delta u_n)(\cdot)\|_{l^2(\mathbf{N})} < \varepsilon,$$

this means by the completeness of $l^2(\mathbf{N})$ that there exists $v(\cdot)$ such that, as $n \longrightarrow \infty$,

$$(\sqrt{p}\Delta u_n)(\cdot) \longrightarrow v(\cdot) \text{ in } l^2(\mathbf{N}). \tag{2.4}$$

Therefore,

$$(\sqrt{p}\Delta u_n)(k) \longrightarrow v(k) \text{ in } \mathbf{C}. \tag{2.5}$$

2. Lemma 2.3 gives K_r such that if $k \leq r$

$$|u_m(k) - u_n(k)| \leq K_r\|u_m(\cdot) - u_n(\cdot)\|_{\mathcal{H}_1} < K_r\varepsilon,$$

this means that $n \mapsto u_n(k)$ is a Cauchy sequence in $\mathbf{C}$. The completeness of the complex numbers $\mathbf{C}$ gives the existence of $u \in S(\mathbf{N})$ such that, as $n \longrightarrow \infty$,

$$u_n(k) \longrightarrow u(k) \text{ in } \mathbf{C} \tag{2.6}$$

and hence

$$\sqrt{q(k)}u_n(k) \longrightarrow \sqrt{q(k)}u(k) \text{ in} \mathbf{C}. \tag{2.7}$$

Moreover, equation (2.3) gives

$$\|\sqrt{q(\cdot)}u_m(\cdot) - \sqrt{q(\cdot)}u_n(\cdot)\|_{l^2(\mathbf{N})} < \varepsilon.$$

Again by the completeness of $l^2(\mathbf{N})$ then there exists $\nu(\cdot)$ such that as $n \longrightarrow \infty$, $\sqrt{q(\cdot)}u_n(\cdot) \longrightarrow \nu(\cdot)$ in $l^2(\mathbf{N})$ this means $\sum_{k=1}^{\infty} |\sqrt{q(k)}u_n(k) - \nu(k)|^2 \longrightarrow 0$. Hence, for any k,

$$\sqrt{q(k)}u_n(k) \longrightarrow \nu(k) \text{ in } \mathbf{C},$$

which implies by (2.7) $\nu(k) = \sqrt{q(k)}u(k)$.

□

Proposition 2.5. *The space $\mathcal{H}_1$ is complete.*

Proof. First, assume $n \mapsto u_n(\cdot)$ is a Cauchy sequence. Then using Lemma 2.4 there exist $u \in S(\mathbf{N})$ such that u_n converges to u pointwise and $v(\cdot) \in l^2(\mathbf{N})$ such that $(\Delta u_n)(\cdot) \longrightarrow v(\cdot)$. This proves that $u(k) = u(1) + \sum_{j=1}^{k-1} v(j) \in \mathcal{H}_1$. Also, this lemma implies $(\sqrt{p}\Delta u_n)(\cdot) \longrightarrow (\sqrt{p}\Delta u)(\cdot)$ and $(\sqrt{q}u_n)(\cdot) \longrightarrow (\sqrt{q}u)(\cdot)$ in $l^2(\mathbf{N})$.

Moreover, one can prove that $u_n(\cdot) \longrightarrow u(\cdot)$ in $\mathcal{H}_1$ as follows. Since

$$\|u_n(\cdot) - u(\cdot)\|_{\mathcal{H}_1} = \sum_{k=1}^{\infty} p(k)|(\Delta u_n)(k) - (\Delta u)(k)|^2 + \sum_{k=1}^{\infty} q(k)|u_n(k) - u(k)|^2,$$

then

$$\|u_n(\cdot) - u(\cdot)\|_{\mathcal{H}_1} = \sum_{k=1}^{\infty} |(\sqrt{p(k)}(\Delta u_n)(k) - v(k))|^2 + \sum_{k=1}^{\infty} |(\sqrt{q(k)}(u_n(k) - u(k)))^2|.$$

Using Lemma2.4 and the last equation, we get $\|u_n(\cdot) - u(\cdot)\|_{\mathcal{H}_1} \longrightarrow 0$, which means $u_n(\cdot) \longrightarrow u(\cdot)$ in $\mathcal{H}_1$, and hence the Cauchy sequence in $\mathcal{H}_1$ is convergent, this means $\mathcal{H}_1$ is complete. □

References

[1] R. AlAhmad and R. Weikard, On inverse problems for left-definite discrete Sturm-Liouville equations, *Operators and Matrices*, **7**, No 1 (2013), 35-70.

[2] G.Š. Guseĭnov, Determination of an infinite nonselfadjoint Jacobi matrix from its generalized spectral function, *Mat. Zametki*, **23**, No 2 (1978), 237-248.

[3] A.M. Krall, Regular left definite boundary value problems of even order. *Quaestiones Math.*, **15**, No 1 (1992), 105-118.

[4] A.M. Krall, Singular left-definite boundary value problems. *Indian J. Pure Appl. Math.*, **29**, No 1 (1998), 29-36.

[5] L.L. Littlejohn and R. Wellman, A general left-definite theory for certain self-adjoint operators with applications to differential equations, *J. Differential Equations*, **181**, No 2 (2002), 280-339.

[6] M. Marletta and A. Zettl, Counting and computing eigenvalues of left-definite Sturm- Liouville problems, *J. Comput. Appl. Math.*, **148**, No 1 (2002), 65-75 (Special Issue on the Occasion of the 65th Birthday of Professor Michael Eastham).

[7] H.D. Niessen and A. Schneider, Spectral theory for left-definite singular systems of differential equations, In: *Spectral Theory and Asymptotics of Differential Equations (Proc. Conf. Scheveningen, 1973)*, North-Holland Math. Studies, # **13**. North-Holland, Amsterdam (1974), 29-43.

[8] R. Weikard, A local Borg-Marchenko theorem for difference equations with complex coefficients, In: *Partial Differential Equations and Inverse Problems*, # **362** of Contemp. Math. Ser., Amer. Math. Soc., Providence – RI (2004), 403-410.

[9] H. Weyl, Uber gewöhnliche lineare Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen (2. note), *Gött. Nachr.* (1910), 442-467.

[10] H. Weyl, Uber gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen, *Math. Ann.*, **68** (1910), 220-269.