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HEAT, RESOLVENT AND WAVE KERNELS WITH BI-INVERSE SQUARE POTENTIAL ON THE EUCLIDIAN PLANE

Mohamed Vall Ould Moustapha

Université des Sciences, de Technologie et de la Médecine (USTM)
Faculté des Sciences et Techniques
Département de Mathématiques et Informatique
Unité de Recherche: Analyse, EDP et Modélisation (AEDPM)
B.P: 5026, Nouakchott – MAURITANIE

Abstract: We give explicit formulas for the heat, resolvent and wave kernels associated to the Schrödinger operator with bi-inverse square potential on the Euclidian plane $I\!R^2$.

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1. Introduction

In this paper we give explicit formulas for the Schwartz integral kernels of the heat, resolvent and wave operators $e^{t\Delta_{\nu,\mu}}$, $\left(\Delta_{\nu,\mu}+\lambda^2\right)^{-1}$ and $\cos t\sqrt{-\Delta_{\nu,\mu}}$ attached to the Schrödinger operator with bi-inverse square potential on the Euclidian plane $I\!R^2$:

$$\Delta_{\nu,\mu}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1/4-\nu^2}{x^2}+\frac{1/4-\mu^2}{y^2}, \tag{1.1}$$

where ν,μ are real parameters.

The inverse square potential is an interesting potential which arises in several contexts, one of them being the Schrödinger equation in non relativistic

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quantum mechanics. For example the Hamiltonian for a spin zero particle in Coulomb field gives rise to a Schrödinger operator involving the inverse square potential see Case [4]. Note that the Schrödinger operator with bi-inverse square potential (1.1) is considered in Boyer [1], in the case of the time dependent Schrödinger equation.

First of all we recall the following formulas for the modified Bessel function of the first kind I_ν and the Hankel function of the first kind $H_\nu^{(1)}$:

$$I_\nu(z) = \frac{(2z)^\nu e^z}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^1 e^{-2zt} \left[t(1-t)\right]^{\nu-1/2} dt \tag{1.2}$$

(see Temme [11], p. 237);

$$I_\nu(x) \sim \frac{(x/2)^\nu}{\Gamma(\nu+1)} \qquad x \longrightarrow 0 \quad \nu \neq -1, -2, ... \tag{1.3}$$

(see Temme [11], p. 234);

$$I_\nu(x) \sim e^x (2\pi x)^{-1/2} \qquad x \longrightarrow \infty \tag{1.4}$$

(see Temme [11] p. 240);

$$H_\nu^{(1)}(z) = \frac{2}{i\sqrt{\pi}\Gamma(1/2-\nu)} (z/2)^{-\nu} \int_1^\infty e^{izt} \left(t^2-1\right)^{-\nu-1/2} dt \tag{1.5}$$

(see Erdélyi et al. [7], p. 83);

$$H_\nu^{(1)}(z\sqrt{\alpha^2}) = \frac{-i}{\pi} e^{-i\nu\pi/2} (\alpha^2)^{\nu/2} \int_0^\infty e^{i\frac{z}{2}\left(t+\frac{\alpha^2}{t}\right)} t^{-\nu-1} dt \tag{1.6}$$

$\mathcal{I}\, z > 0$ and $\mathcal{I}\, \alpha^2 z > 0$ (see Magnus et al [9], p. 84). Recall also the two variables double series, called F_2 Appell hypergeometric function (see Erdélyi et al [6], p. 224):

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', z, z') = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m! n!} z^m z'^n$$

$$(|z| + |z'| < 1), \tag{1.7}$$

and its integral representation (see Erdélyi et al [6], p.230) for $\mathcal{R}\, \beta > 0$, $\mathcal{R}\, \beta' > 0$, $\mathcal{R}\,(\gamma - \beta) > 0$ and $\mathcal{R}\,(\gamma' - \beta') > 0$:

$$F_2\left(\alpha, \beta, \beta', \gamma, \gamma', z, z'\right) = c \int_0^1 \int_0^1 (1-u)^{\gamma-\beta-1}$$

$$\times (1-v)^{\gamma'-\beta'-1} u^{\beta-1} v^{\beta'-1} \left(1-uz-vz'\right)^{-\alpha} dudv, \tag{1.8}$$

where

$$c = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}. \tag{1.9}$$

We recall the following formulas for the heat kernel associated to the Schrödinger operator with inverse square potential $L_\nu = \frac{\partial^2}{\partial x^2} + \frac{1/4-\nu^2}{x^2}$ (see Calin et al. [3], p. 68):

$$e^{tL_\nu} = \frac{(r\rho)^{1/2}}{2t} e^{\frac{-(r^2+\rho^2)}{4t}} I_\nu\left(\frac{r\rho}{2t}\right), \tag{1.10}$$

where I_ν is the modified Bessel function of the first kind.

Proposition 1.1. *The Schwartz integral kernel of the heat operator with bi-inverse square potential $e^{t\Delta_{\nu,\mu}}$ can be written for $p=(x,y)$, $p'=(x',y') \in \mathbb{R}_+^2$ and $t \in \mathbb{R}_+$ as*

$$H_{\nu,\mu}(t,p,p') = \frac{(xx'yy')^{1/2}}{4\pi t^2} e^{-(|p|^2+|p'|^2)/4t} I_\nu((xx')/2t) I_\mu((yy')/2t) \tag{1.11}$$

where I_ν is the modified Bessel function of the first kind.

Proof. The formula (1.11) is a direct consequence of the formula (1.10) and the properties of the operator (1.1).

2. Resolvent Kernel with Bi-Inverse Square Potential on the Euclidian Plane

Theorem 2.1. *The Schwartz integral kernel for the resolvent operator $\left(\Delta_{\nu,\mu}+\lambda^2\right)^{-1}$ is given by the formula*

$$G_{\nu,\mu}(\lambda,p,p') = \frac{i^{-(\nu+\mu)} e^{\frac{(\nu+\mu+1)}{2} i\pi}}{4\pi} \frac{(xx')^{\nu+1/2}(yy')^{\mu+1/2}}{\Gamma(\nu+1/2)\Gamma(\mu+1/2)}$$

$$\times \int_0^1 \int_0^1 \left(\frac{\sqrt{|p-p'|^2 + 4xx'u + 4yy'v}}{2\lambda} \right)^{-(\nu+\mu+1)}$$

$$\times H^{(1)}_{\nu+\mu+1}(\lambda\sqrt{|p-p'|^2 + 4xx'u + 4yy'v})$$

$$\times [u(1-u)]^{\nu-1/2} [v(1-v)]^{\mu-1/2} dudv, \tag{2.1}$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind.

Proof. We use the well known formula connecting the resolvent and the heat kernels:

$$G_{\nu,\mu}(\lambda,p,p')=\int_0^\infty e^{\lambda^2 t}H_{\nu,\mu}(t,p;p')dt, \qquad \mathcal{R}\,\lambda^2<0, \tag{2.2}$$

we combine the formulas (2.2) and (1.11) then use the formulas (1.3), (1.4) to apply the Fubini Theorem and in view of the formula (1.6), we get the formula (2.1) and the proof of Theorem 2.1 is finished.

Theorem 2.2. *The Schwartz integral kernel of the resolvent operator $\left(\Delta_{\nu,\mu}+\lambda^2\right)^{-1}$ can be written as*

$$G_{\nu,\mu}(\lambda,p,p')=c_2(2xx')^{\nu+1/2}(2yy')^{\mu+1/2}\int_{|p-p'|}^\infty e^{i\lambda s}\left(s^2-|p-p'|^2\right)^{-3/2-\nu-\mu}$$

$$\times F_2\left(a,b_1,b_2,2b_1,2b_2,\frac{4xx'}{s^2-|p-p'|^2},\frac{-4yy'}{s^2-|p-p'|^2}\right)ds, \tag{2.3}$$

with $a=3/2+\nu+\mu$, $b_1=1/2+\nu$ and $b_2=1/2+\mu$,

$$c_2=(-1)^{\nu+\mu+1}\frac{\Gamma(1/2+\nu)\Gamma(1/2+\mu)}{4\pi^{3/2}\Gamma(2\nu+1)\Gamma(2\mu+1)\Gamma(-1/2-\nu-\mu)}, \tag{2.4}$$

where F_2 is the two variables Appell hypergeometric function (1.7).

Proof. We use the formula (2.1) and (1.5) as well as the Fubini theorem to arrive at the announced formula (2.3).

3. Wave Kernel with Bi-Inverse Square Potential on the Euclidian Plane

In physics, the nature tells us that energy and information can only be transmitted with finite speed, smaller or equal to the speed of light. The mathematical framework, which allows an analysis and proof of this phenomenon, is the theory of hyperbolic differential equations and in particular of the wave equation. The result, which may be obtained, runs under the name finite propagation speed (see Cheeger et al [5]). The following result illustrates this principle for the case of the the Schrödinger operator with bi-inverse square potential.

Theorem 3.1. (*Finite propagation speed*) *Let $w_a(t,x,x')$ be the Schwartz integral kernel of the wave operator $\frac{\sin t\sqrt{-\Delta_{\nu,\mu}}}{\sqrt{-\Delta_{\nu,\mu}}}$, then we have*

$$w_a(t,x,x') = 0, \quad \text{whenever} \quad |x-x'| > t. \tag{3.1}$$

Proof. The proof of this result use an argument of analytic continuation from the identity

$$\frac{\sin t\sqrt{-\Delta_{\nu,\mu}}}{\sqrt{-\Delta_{\nu,\mu}}} = \frac{1}{2i}\left(\frac{e^{it\sqrt{-\Delta_{\nu,\mu}}}}{\sqrt{-\Delta_{\nu,\mu}}} - \frac{e^{-it\sqrt{-\Delta_{\nu,\mu}}}}{\sqrt{-\Delta_{\nu,\mu}}}\right). \tag{3.2}$$

We recall the formula [12], p.50,

$$\frac{e^{-t\lambda}}{t} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-ut^2}u^{-1/2}e^{-\lambda^2/4u}du. \tag{3.3}$$

By setting $t = \sqrt{-\Delta_{\nu,\mu}}$ and $\lambda = s$ in (3.3) we can write

$$e^{-s\sqrt{-\Delta_{\nu,\mu}}} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-s^2/4u}u^{-1/2}e^{u\Delta_{\nu,\mu}}du, \tag{3.4}$$

and let $P(x,x',s)$ be the integral kernel of $e^{-s\sqrt{-\Delta_{\nu,\mu}}}$ then we can write

$$P(x,x',s) = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-s^2/4z}z^{-1/2}H_{\nu,\mu}(z,x,x')dz, \tag{3.5}$$

where $H_{\nu,\mu}(t,x,x')$ is the heat kernel with the bi-inverse square potential given by (1.11).

Consider the integral

$$J(\tau) = P(x,x',\tau) = \frac{1}{\sqrt{\pi}}\int_0^\infty |e^{-\tau^2/4z}z^{-1/2}H_{\nu,\mu}(z,x,x')|dz, \tag{3.6}$$

using (1.11) we have

$$J(\tau) = \frac{(xx'yy')^{1/2}}{4\pi^{3/2}}\int_0^\infty |e^{-\tau^2/4z}z^{-5/2}e^{-(|p|^2+|p'|^2)/4z}I_\nu((xx')/2z)I_\mu((yy')/2z)|dz. \tag{3.7}$$

From (3.2) we have

$$w_a(t,x,x') = \frac{1}{2i}\left(P(it,x,x') - P(-it,x,x')\right) = \frac{1}{2i}\left(J(it) - J(-it)\right). \tag{3.8}$$

Now set

$$J(\tau) = J_1(\tau) + J_2(\tau), \tag{3.9}$$

where

$$J_1(\tau) = \frac{(xx'yy')^{1/2}}{4\pi^{3/2}} \int_0^1 |e^{-\tau^2/4z} z^{-5/2} e^{-(|p|^2+|p'|^2)/4z} \times I_\nu((xx')/2z) I_\mu((yy')/2z)| dz, \tag{3.10}$$

$$J_2(\tau) = \frac{(xx'yy')^{1/2}}{4\pi^{3/2}} \int_1^\infty |e^{-\tau^2/4z} z^{-5/2} e^{-(|p|^2+|p'|^2)/4z} \times I_\nu((xx')/2z) I_\mu((yy')/2z)| dz. \tag{3.11}$$

Using the formula (1.3) we see that the last integral $J_2(\tau)$ converge absolutely and is analytic in τ for $\nu + \mu + 3/2 > 0$.

For the first integral $J_1(\tau)$ we obtain

$$J_1(\tau) = \frac{(xx'yy')^{1/2}}{4\pi^{3/2}} \int_1^\infty e^{-\tau^2 z/4} z^{1/2} e^{-(|p|^2+|p'|^2)z/4} \times I_\nu((xx')z/2) I_\mu((yy')z/2)| dz, \tag{3.12}$$

and from the formula (1.4) we see that

$$J_1(\tau) \sim \frac{1}{4\pi^{5/2}} \int_1^\infty z^{-1/2} e^{-\left(\tau^2+|p-p'|^2\right)\frac{z}{4}} dz \tag{3.13}$$

is analytic in τ and converges if $\mathcal{R}e\left[\tau^2 + |p-p'|^2\right] > 0$, hence the integral $J(\pm it)$ is absolutely convergent if $(\pm it)^2 + |p-p'|^2 > 0$ (ie) $|p-p'| > t$ and in view of (3.8) we have $w_{\nu,\mu}(t,x,x') = 0$ for $|p-p'| > t$ and the proof of Theorem 3.1 is finished.

Theorem 3.2. *The Schwartz integral kernel for the wave operator* $\cos t\sqrt{-\Delta_{\nu,\mu}}$ *with bi-inverse square potential on the Euclidian plane can be written on the two following forms*

$$w_{\nu,\mu}(t,p,p') = \frac{(xx'yy')^{1/2}}{i\sqrt{(2\pi)^3}t^4} \int_{-\infty}^{0+} \exp\left[-\frac{u}{2t^2}\left(|p|^2+|p'|^2-t^2\right)\right] \times I_\nu\left(\frac{xx'}{t^2}u\right) I_\mu\left(\frac{yy'}{t^2}u\right) u^{3/2} du, \tag{3.14}$$

and

$$w_{\nu,\mu}(t,p,p') = 2\frac{(xx'yy')^{1/2}}{\sqrt{(2\pi)^3}t^4}\int_0^\infty \exp\left[\frac{r}{2t^2}\left(|p|^2+|p'|^2-t^2\right)\right]$$
$$\times I_\nu\left(\frac{-xx'}{t^2}r\right) I_\mu\left(-\frac{yy'}{t^2}r\right) r^{3/2}dr, \quad (3.15)$$

where I_ν is the first kind modified Bessel function of order ν.

Proof. We start by recalling the formula (see Magnus et al. [9], p.73),

$$\cos z = \sqrt{\pi z/2}\, J_{-1/2}(z), \quad (3.16)$$

where $J_\nu(.)$ is the Bessel function of first kind and of order ν defined by (see Magnus et al [9], p.83),

$$J_\nu(\alpha z) = \frac{z^\nu}{2i\pi}\int_{-\infty}^{0+} e^{(\alpha/2)(t-z^2/t)}t^{-\nu-1}dt, \quad (3.17)$$

provided that $\mathcal{R}\alpha > 0$ and $|argz| \leq \pi$. Here we should note that the integral in (3.17) can be extended over a contour starting at ∞, going clockwise around 0, and returning back to ∞ without cutting the real negative semi-axis.

For $\nu = -1/2$ the eq. (3.17) can be combined with eq. (3.16) to derive the following formula:

$$\cos(\alpha z) = \frac{1}{2i\sqrt{2\pi}}\int_{-\infty}^{0+} e^{(\alpha/2)(u-z^2/u)}u^{-1/2}du. \quad (3.18)$$

Putting $\alpha = 1$ and replacing the variable z by the symbol $t\sqrt{-\Delta_{\nu,\mu}}$ in (3.18), we obtain

$$\cos t\sqrt{-\Delta_{\nu,\mu}} = \frac{1}{2i\sqrt{2\pi}}\int_{-\infty}^{0+} e^{\left(u/2+(t^2/2u)\Delta_{\nu,\mu}\right)}u^{-1/2}du. \quad (3.19)$$

Finally making use of (1.11) in (3.19), we get, after an appropriate change of variable, the formula (3.14).

To see the formula (3.15), set

$$J = \int_{-\infty}^{0+} \exp\left[-\frac{u}{2t^2}\left(|p|^2+|p'|^2-t^2\right)\right] I_\nu\left(\frac{xx'}{t^2}u\right) I_\mu\left(\frac{yy'}{t^2}u\right) u^{3/2}du \quad (3.20)$$

and

$$I = \int_0^\infty \exp\left[\frac{u}{2t^2}\left(|p|^2+|p'|^2-t^2\right)\right] I_\nu\left(-\frac{xx'}{t^2}u\right) I_\mu\left(-\frac{yy'}{t^2}u\right) u^{3/2}du, \quad (3.21)$$

we have

$$J = J_1 + J_2 + J_3, \tag{3.22}$$

$$J_1 = \int_{\gamma_1} \exp\left[-\frac{u}{2t^2}\left(|p|^2 + |p'|^2 - t^2\right)\right] I_\nu\left(\frac{xx'}{t^2}u\right) I_\mu\left(\frac{yy'}{t^2}u\right) u^{3/2} du, \tag{3.23}$$

$$J_2 = \int_{\gamma_2} \exp\left[-\frac{u}{2t^2}\left(|p|^2 + |p'|^2 - t^2\right)\right] I_\nu\left(\frac{xx'}{t^2}u\right) I_\mu\left(\frac{yy'}{t^2}u\right) u^{3/2} du, \tag{3.24}$$

and

$$J_3 = \int_{\gamma_3} \exp\left[-\frac{u}{2t^2}\left(|p|^2 + |p'|^2 - t^2\right)\right] I_\nu\left(\frac{xx'}{t^2}u\right) I_\mu\left(\frac{yy'}{t^2}u\right) u^{3/2} du, \tag{3.25}$$

where the paths γ_1, γ_2 and γ_3 are given by

$$\gamma_1 : z = re^{i\pi}; \epsilon \leq r < \infty (\text{above the cut}),$$

$$\gamma_2 : z = re^{-i\pi}; \infty > r \geq \epsilon (\text{below the cut}),$$

$$\gamma_3 : z = \epsilon e^{i\theta}; -\pi < \theta < \pi (\text{rund the small circle}),$$

and as $\epsilon \longrightarrow 0$, we have $J_1 \longrightarrow e^{5i\pi/2} I$, $J_2 \longrightarrow -e^{-5i\pi/2} I$ and $J_3 \longrightarrow 0$.

Summing the integrals, we establish the required result $J = 2i \sin(5\pi/2)\, I$.

Theorem 3.3. *The integral kernel for the wave operator* $\cos t\sqrt{-\Delta_{\nu,\mu}}$ *with bi-inverse square potential on the Euclidian plane can be written as*

$$w_{\nu,\mu}(t,p,p') = C_2(-4xx')^{\nu+1/2}(-4yy')^{\mu+1/2}\left(t^2 - |p-p'|^2\right)_+^{-5/2-\nu-\mu}$$

$$\times F_2\left(\alpha, \beta, \beta', 2\beta, 2\beta', \frac{4xx'}{t^2 - |p-p'|^2}, \frac{4yy'}{t^2 - |p-p'|^2}\right), \tag{3.26}$$

where $F_2(\alpha, \beta, \beta', \gamma, \gamma'; z, z')$ *is the two variables Appell hypergeometric function* F_2 *given in* (1.7), $\alpha = 5/2 + \nu + \mu$, $\beta = \nu + 1/2$, $\beta' = \mu + 1/2$ *and the constant* C_2 *is given by*

$$C_2 = (-1)^{\nu+\mu}\left(\frac{2}{\pi}\right)^{5/2} \frac{\Gamma(1/2+\nu)\Gamma(1/2+\mu)}{\Gamma(2\nu+1)\Gamma(2\mu+1)}. \tag{3.27}$$

Proof. We use essentially the formula (3.15) of Theorem 3.2, the formulas (1.2), the Fubini theorem and the formula (1.8).

4. Applications and Further Studies

We give an application of Theorem 3.3.

Corollary 4.1. *The integral kernel of the heat operator* $e^{t\Delta_{\nu,\mu}}$ *can be written in the form*

$$H_{\nu,\mu}(t,p,p') = \frac{1}{\sqrt{\pi t}} C_2(-4xx')^{\nu+1/2}(-4yy')^{\mu+1/2}$$

$$\times \int_{|p-p'|}^{\infty} e^{-u^2/4t} u \left(u^2 - |p-p'|^2\right)^{-5/2-\nu-\mu}$$

$$\times F_2\left(a, b, b', 2b, 2b', \frac{4xx'}{u^2-|p-p'|^2}, \frac{4yy'}{t^2-|p-p'|^2}\right) du,$$

with $a = 5/2+\nu+\mu$ *,*$b = \nu + 1/2$*,* $b' = \mu + 1/2$*.*

Proof. We use the transmutation formula (see Greiner et al [8], p.362),

$$e^{t\Delta_{\nu,\mu}} = \frac{1}{\sqrt{\pi t}} \int_0^{\infty} e^{-u^2/4t} \cos u\sqrt{-\Delta_{\nu,\mu}} du.$$

We suggest here a certain number of open related problems connected to this paper. We are interested in the semi-linear wave and heat equations associated to the bi-inverse square potential and global solution and a possible blow up in finite times.

We can also to look for the dispersive and Strichartz estimates for the Schrödinger and the wave equations with bi-inverse square potential, for the case of inverse square potential (see Burg et al [2], Planchon et al. [2]).

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