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FUZZY GROUP RINGS

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Abstract: In this paper, first we define fuzzy group rings and then deduce some results about them.

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1. Introduction

The concept of Group Ring is relatively old. It appears implicitly in a paper of A. Cayley [1]. To get an idea about the importance of group rings in algebraic research, it is enough to observe that many of the great algebraists have worked in the area. Among them we can mention: D.S. Passman, S.K. Sehgal, S.A. Amitsur, H. Bass, E. Formanek, N.D. Gupta, I.N. Herstein, G. Higman, A.V. Jategaonkar, I. Kaplansky, W. May, K.W. Roggenkamp, W. Rudin, S.D. Berman, A.A. Bovdi, A.E. Zalesskii and A.V. Mikhalev, etc.

Let R be a unital ring and G an arbitrary group, RG be the set of all formal linear combinations of the form $\sum_{g\in G} a_g g$, where $a_g \in R$ and only finitely many of the a_g's are non-zero. This definition implies that two elements $\alpha = \sum_{g\in G} a_g g, \beta = \sum_{g\in G} b_g g \in RG$ are equal if and only if $a_g = b_g$ for all $g \in G$. The sum of elements of RG is defined by:

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g.$$

Also, given two elements $\alpha = \sum_{g\in G} a_g g, \beta = \sum_{h\in G} b_h h \in RG$, their product is

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defined by:

$$\alpha\beta = \sum_{g,h\in G} a_g b_h gh.$$

It is easily verified that RG with the above operations is a ring, which is called the group ring of G over R.

2. Main Results

First, we remind the definition of fuzzy polynomials (see [4]).

Definition 2.1. Let $[0,1]$ be a closed interval of the real line and R be a commutative ring with 1 or the field of reals. The fuzzy polynomial ring in the variable x with coefficients from R denoted by $R[x^{[0,1]}]$ consists of elements of the form

$$\left\{ a_0 + a_1x^{\gamma_1} \ldots + a_nx^{\gamma_n} \middle| \begin{array}{rr} a_0, a_1, \ldots, a_n \in R & \text{and} \\ \gamma_1, \gamma_2, \ldots, \gamma_n \in [0,1] & \text{with} \\ \gamma_1 < \gamma_2 < \ldots < \gamma_n & \end{array} \right\}.$$

If $p(x) = a_0 + a_1x^{\gamma_1} + \ldots + a_nx^{\gamma_n}$ and $q(x) = b_0 + b_1x^{s_1} + \ldots + b_mx^{s_m}$, then $p(x)+q(x) = c_0 + c_1x^{t_1} + \ldots + c_kx^{t_k}$ where $\gamma_1 < \gamma_2 < \ldots < \gamma_n$, $s_1 < s_2 < \ldots < s_m$ and $t_1 < t_2 < \ldots < t_k$ with $\gamma_i, s_j, t_p \in [0,1]$, $1 \le i \le n$, $1 \le j \le m$ and $1 \le p \le k$.

Now it is easily verified that $R[x^{[0,1]}]$ is an Abelian group under " $+$ "; with $0 = 0 + 0\ x^{\gamma_0} + \ldots + 0\ x^{\gamma_n}$ as the additive identity. The multiplication "o" of two fuzzy polynomials in $R[x^{[0,1]}]$ is defined for x^{s_1} and $x^{t_1} \in R[x^{[0,1]}]$ by the following rules

$$\begin{array}{rcll} x^{s_1}\, o\, x^{t_1} & = & x^{s_1+t_1} & \text{if } s_1 + t_1 \le 1, \\ & = & x^{s_1+t_1-1} & \text{if } s_1 + t_1 > 1. \end{array}$$

We extend this way of multiplication for any two polynomials $p(x), q(x) \in R[x^{[0,1]}]$. Clearly $R[x^{[0,1]}]$ is a semigroup under "o" and $(R[x^{[0,1]}], +, o)$, defined to be a fuzzy polynomial ring.

Now we define fuzzy group rings.

Definition 2.2. Let G be a group (semigroup) and R be a unital ring.

The fuzzy group ring or G over R is denoted by $RG^{[0,1]}$ and is defined as follows

$$\begin{aligned} RG^{[0,1]} &= \{\textstyle\sum_{i=1}^{n} a_{g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \ldots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}} g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \ldots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}} \mid \\ & g_{\alpha_{i,1}}, \ldots, g_{\alpha_{i,l_i}} \in G, \gamma_{\alpha_{i,1}}, \ldots, \gamma_{\alpha_{i,l_i}} \in [0,1], \quad n \in \mathbb{N}\}. \end{aligned}$$

In order to make a ring out of $RG^{[0,1]}$, we must be able to recognize when two elements in it are equal and then we must be able to add and multiply elements in $RG^{[0,1]}$ so that the axioms defining a ring to hold true for $RG^{[0,1]}$. Now, if

$$\sum_{i=1}^{n} a_{g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \ldots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}} g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \ldots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}$$

and

$$\sum_{i=1}^{m} b_{h_{\beta_{i,1}}^{\gamma_{\beta_{i,1}}} h_{\beta_{i,2}}^{\gamma_{\beta_{i,2}}} \ldots h_{\beta_{i,k_i}}^{\gamma_{\beta_{i,k_i}}}} h_{\beta_{i,1}}^{\gamma_{\beta_{i,1}}} h_{\beta_{i,2}}^{\gamma_{\beta_{i,2}}} \ldots h_{\beta_{i,k_i}}^{\gamma_{\beta_{i,k_i}}} \in RG^{[0,1]},$$

we have

$$\begin{aligned} &\sum_{i=1}^{n} a_{g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \ldots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}} g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \ldots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}} \\ &= \sum_{i=1}^{m} b_{g_{\beta_{i,1}}^{\gamma_{\beta_1}} g_{\beta_{i,2}}^{\gamma_{\beta_{i,2}}} \ldots h_{\beta_{i,k_i}}^{\gamma_{\beta_{i,k_i}}}} h_{\beta_{i,1}}^{\gamma_{\beta_{i,1}}} h_{\beta_{i,2}}^{\gamma_{\beta_{i,2}}} \ldots h_{\beta_{i,k_i}}^{\gamma_{\beta_{i,k_i}}}, \end{aligned}$$

if and only if $m = n$, and probably after reindexing, we have

$$a_{g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \ldots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}} = b_{g_{\beta_{i,1}}^{\gamma_{\beta_{i,1}}} g_{\beta_{i,2}}^{\gamma_{\beta_{i,2}}} \ldots h_{\beta_{i,k_i}}^{\gamma_{\beta_{i,k_i}}}}$$

and

$$g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \ldots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}} = h_{\beta_{i,1}}^{\gamma_{\beta_{i,1}}} h_{\beta_{i,2}}^{\gamma_{\beta_{i,2}}} \ldots h_{\beta_{i,k_i}}^{\gamma_{\beta_{i,k_i}}},$$

for each i. We define the summation " $+$ " like in group rings, if

$$\delta = \sum_{i=1}^{n} a_{g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \ldots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}} g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \ldots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}$$

and

$$\sigma = \sum_{i=1}^{n} b_{g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \ldots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}} g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \ldots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}},$$

then

$$\delta+\sigma=$$
$$\sum_{i=1}^{n} a_{g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \dots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}} g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \dots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}+$$
$$\sum_{i=1}^{n} b_{g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \dots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}} g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \dots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}=$$
$$\sum_{i=1}^{n} (a_{g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \dots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}}+b_{g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \dots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}}) g_{\alpha_{i,1}}^{\gamma_{\alpha_{i,1}}} g_{\alpha_{i,2}}^{\gamma_{\alpha_{i,2}}} \dots g_{\alpha_{i,l_i}}^{\gamma_{\alpha_{i,l_i}}}.$$

Now we define the multiplication "o" of two elements in $RG^{[0,1]}$ according to the following laws. For every $g, h \in G$ and every $\alpha, \beta \in [0,1]$ we have:

1. $$\begin{aligned} g^{\alpha} o g^{\beta} &= g^{\alpha+\beta} && \text{if} \quad \alpha+\beta \leq 1 \\ &= g^{\alpha+\beta}-1 && \text{if} \quad \alpha+\beta>1. \end{aligned}$$

2. $g^0=1$.

3. $g^{\alpha} o h^{\alpha}=(gh)^{\alpha}$.

We extend this way of multiplication for any two members of $RG^{[0,1]}$. Clearly, $RG^{[0,1]}$ is an Abelian group under " $+$ " with

$$0=\sum 0\ g_{\alpha_1}^{\gamma_{\alpha_1}} g_{\alpha_2}^{\gamma_{\alpha_2}} \dots g_{\alpha_i}^{\gamma_{\alpha_i}}$$

as the additive identity which for short will be denoted by 0 and under "o" it is a semigroup. Thus $(RG^{[0,1]}, +, o)$ is a ring.

Theorem 2.3. *Let $RG^{[0,1]}$ be the fuzzy group ring over the field of reals R. Then $RG^{[0,1]}$ is an infinite dimensional vector space over R.*

Proof. It follows from the fact that $RG^{[0,1]}$ is an additive Abelian group for any fuzzy group ring and we have for all $r \in R$ and $\alpha \in RG^{[0,1]}$, $r\alpha \in RG^{[0,1]}$. So we deduce that $RG^{[0,1]}$ is a vector space over R since the interval $[0,1]$ is of infinite cardinality we say $RG^{[0,1]}$ is a vector space with infinite basis. □

Theorem 2.4. *Let R be a field. Then $RG^{[0,1]}$ is a fuzzy group ring which is not a field or an integral domain.*

Proof. To show that $RG^{[0,1]}$ is not a field or an integral domain, we have to show that $RG^{[0,1]}$ contains two elements $0 \neq \alpha, 0 \neq \beta \in RG^{[0,1]}$ with $\alpha\, o\, \beta=0$. Take

$$\alpha=g+g^{0.5}\ , \quad \beta=1-g^{0.5}\ ,$$

where g is a non-identity element of G. We have

$$\begin{aligned} \alpha o \beta &= (g + g^{0.5})(1 - g^{0.5}) \\ &= g + g^{0.5} - g^{0.5} - g^{1} \\ &= g + g^{0.5} - g^{0.5} - g \\ &= 0, \end{aligned}$$

as we wanted. □

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