

A NOTE ON PAIRS OF REGULAR FOLIATIONS  
ON THE SOLID TORUS

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**Abstract:** We discuss the geometry of the pair of foliations on a solid torus given by the Reeb foliation together with discs transverse to the boundary of the torus.

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**Key Words:** foliations, solid torus, singularities

1. Introduction

An important feature of a pair of codimension one regular foliations in  $\mathbb{R}^3$  is its discriminant which is the locus of points where the foliations are tangent.

Assuming that the foliations are the leaves of germs of differential 1-forms  $\omega$  and  $\eta$ , then the discriminant  $D(\omega, \eta)$  of the pair  $(\omega, \eta)$  is the zero locus of  $\omega \wedge \eta$ , that is, the locus where the 1-form  $\omega$  is a multiple of  $\eta$ . This is generically a germ of a space curve. In [4] the authors show that the discriminant  $D(\omega, \eta)$  determines the local topological type of the pair  $(\omega, \eta)$  and obtain a complete list of discrete topological models. Because the discriminant plays a key role in

the topological classification, in [4] and [5] the authors analyze its singularities considering the discriminant, in local coordinates, given by the fibre of a map-germ  $F : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^2, 0)$ .

In this paper we show that the discriminant is also important to the study of the geometry of the pair of foliations on a solid torus  $S^1 \times D^2$  given by the Reeb foliation (i.e. has the boundary of the solid torus as the only compact leaf and all other leaves are homeomorphic to  $\mathbb{R}^2$  and accumulate only on the boundary [2], see Figure 1) together with discs  $D_\theta$  transverse to the boundary of the torus, i.e.  $D_\theta = \{\theta\} \times D^2$ , with  $\theta \in S^1$ . Its discriminant clearly yields good information about the geometric behavior of the leaves of the pair of foliations.

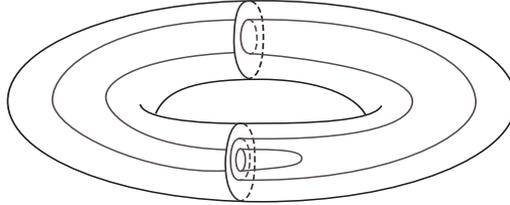


Figure 1: A pair of foliations on a solid torus given by discs and the Reeb foliation.

We shall assume that the discriminant has stable  $\mathcal{K}^*$ -singularities. Let  $\mathcal{E}_n$  be the local ring of germs of functions  $(\mathbb{R}^n, 0) \rightarrow \mathbb{R}$  and  $m_n$  its maximal ideal (which is the subset of germs that vanish at the origin). Denote by  $\mathcal{E}(n, p)$  the  $p$ -tuples of elements in  $\mathcal{E}_n$ . The contact group  $\mathcal{K}$  is the set of germs of diffeomorphisms  $\mathbb{R}^n \times (\mathbb{R}^p, 0) \rightarrow \mathbb{R}^n \times (\mathbb{R}^p, 0)$  which can be written in the form  $H(x, y) = (h(x), H_1(x, y))$ , with  $h \in \text{Diff}(\mathbb{R}^n, 0)$  and  $H_1(x, 0) = 0$  for  $x$  near 0. This means that  $\pi \circ H = h \circ \pi$  where  $\pi : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$  is the canonical projection. So  $H$  is a fibred mapping over the diffeomorphism  $h$  and preserves the 0-section  $\mathbb{R}^n \times \{0\}$ . The group  $\mathcal{K}$  acts on  $m_n \cdot \mathcal{E}(n, p)$  as follows:  $G = H.F$  if and only if  $(x, G(x)) = H(h^{-1}(x), F(h^{-1}(x)))$ ; see [6] for details.

The action of the group  $\mathcal{K}$  is a natural one to use when one seeks to understand the singularities of the zero fibres of germs in  $m_n \cdot \mathcal{E}(n, p)$ . Indeed, if two germs are  $\mathcal{K}$ -equivalent, then their zero fibres are diffeomorphic. The  $\mathcal{K}^*$  group is the subgroup of  $\mathcal{K}$  where the changes of coordinates in the source preserve the discs  $D_\theta$  transverse to the boundary of the torus. We observe that the  $\mathcal{K}^*$ -action preserves the contact of any germ in  $m_3 \cdot \mathcal{E}(3, 2)$  with  $D_\theta$ . Also if the discriminant, in local coordinates, is given by the fibre of a map-germ  $F : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^2, 0)$  and  $F$  has a finitely  $\mathcal{K}^*$ -determined singularity, then

the discriminant is transverse, away from the origin, to the foliation given by discs  $D_\theta$ ,  $\theta \in S^1$ . See [4] for various classifications of the singularities of the discriminant.

Since we are assuming that the discriminant has stable  $\mathcal{K}^*$ -singularities, so it is a smooth simple curve transverse to both foliations except maybe at some isolated points where it has ordinary tangency with the foliations. These points are labelled points of tangency. (When the discriminant is smooth, it can be parametrised by a germ of some map  $\phi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^3, p)$ . A leaf at  $p$  of one of the foliations is the zero set of a germ of some function  $g : (\mathbb{R}^3, p) \rightarrow (\mathbb{R}, 0)$ . The discriminant has an ordinary tangency at  $p$  with the leaf  $g^{-1}(0)$  if the function  $g \circ \phi$  satisfies  $g(\mathbf{0}) = g'(\mathbf{0}) = 0$  and  $g''(\mathbf{0}) \neq 0$ .) We show in this paper that the discriminant of the mentioned foliations is a finite disjoint union of curves homeomorphic to  $S^1$  and, if it has only one connected component then its class generates the fundamental group of the solid torus.

The motivation for this work came from [4], which is a generalization of [7] for pairs of germs or regular foliations in the plane. Pairs of foliations appear in, and have applications to, control theory, partial differential equations and differential geometry (see for example [3]).

## 2. Pairs of Foliations on the Solid Torus

We consider two regular codimension one foliations on the solid torus  $M = S^1 \times D^2$ . One, denoted by  $\mathcal{G}$ , is given by discs  $G_\theta = \{\theta\} \times D^2$ ,  $\theta \in S^1$ , transverse to the boundary and the other, denoted by  $\mathcal{F}$ , is the Reeb foliation. The discriminant of the pair is denoted by  $\Delta$ .

An embedding  $g : D^2 \rightarrow S^1 \times D^2$  is said to be in *general position* with respect to  $\mathcal{F}$  if for every distinguished map  $f$  of  $\mathcal{F}$ , the map  $f \circ g$  is locally of Morse type. The submanifold  $g(D^2)$  is said to be in *general position* with respect to  $\mathcal{F}$ . In this case,  $g$  induces a foliation  $\mathcal{F}^*$  in  $g(D^2)$  whose leaves are the connected components of the intersection of the leaves of  $\mathcal{F}$  with  $g(D^2)$  (the *traces* of  $\mathcal{F}$  in  $g(D^2)$ ). Furthermore, the singularities of  $\mathcal{F}^*$  are the points where  $g(D^2)$  is tangent to a leaf of  $\mathcal{F}$ . So  $g$  induces a  $C^r$ -foliation  $g^*(\mathcal{F})$  on  $D^2$  whose leaves are the connected components of the sets  $g^{-1}(F)$ , with  $F$  a leaf of  $\mathcal{F}$ . Therefore we have the foliation  $\mathcal{F}^*$  in  $g(D^2)$  such that the singularities of  $\mathcal{F}^*$  are isolated, hence finite in number, and are saddles or centers.

Since the discriminant has stable  $\mathcal{K}^*$ -singularities then the inclusion  $j : D^2 \rightarrow S^1 \times D^2$ , such that  $j(D^2) = D_\theta$ , is in general position with respect to  $\mathcal{F}$ . So, the traces of  $\mathcal{F}$  in  $G_\theta$ , i.e. the intersections of the leaves of  $\mathcal{F}$  with

$G_\theta$ , induce a singular foliation  $\mathcal{F}_\theta$  in  $G_\theta$ , for all  $\theta \in S^1$ . The singularities of  $\mathcal{F}_\theta$  are the points of intersection of the discriminant with  $G_\theta$ . Therefore, the singularities of  $\mathcal{F}_\theta$  are isolated and consequently, they consist of a finite number (with total distinct from zero) of centres, saddles or points of tangency. As the non-compact leaves of  $\mathcal{F}$  are homeomorphic to  $\mathbb{R}^2$  and accumulate only on the boundary, the regular leaves of  $\mathcal{F}_\theta$  are closed curves, homoclinic curves or saddles connections.

**Theorem 1.** *The discriminant is a finite disjoint union of curves homeomorphic to  $S^1$  and at least one of them is not homotopic to constant in  $M$ .*

*Proof.* For each  $\theta \in S^1$  we consider a subset  $U_\theta = [\theta - \epsilon_\theta, \theta + \epsilon_\theta] \times D^2$  of  $M$  with  $\epsilon_\theta$  chosen as follows. As  $\Delta$  is generic, it intersects transversely  $G_\theta$  at a finite number of points and is maybe tangent to  $G_\theta$  at also a finite number of points. Therefore, there exists  $\epsilon_\theta > 0$  such that  $U_\theta$  contains a finite number of compact connected components of  $U_\theta \cap \Delta$ . Since  $M$  is compact and  $M = \bigcup_{\theta \in S^1} ]\theta - \epsilon_\theta, \theta + \epsilon_\theta[ \times D^2$ , there exist  $\theta_1, \dots, \theta_k$  in  $S^1$  such that  $M = \bigcup_{i=1}^k ]\theta_i - \epsilon_{\theta_i}, \theta_i + \epsilon_{\theta_i}[ \times D^2$ . As  $\Delta = \bigcup_{i=1}^k U_{\theta_i} \cap \Delta$ , it follows that it is a compact set. Given that it is a simple curve, it is therefore a disjoint union of a finite number of simple closed curves all homeomorphic to  $S^1$ .

Let  $\theta \in S^1$  such that  $\Delta \cap G_\theta$  has  $c_\theta$  centres and  $s_\theta$  saddles and no points of tangency. Let  $X$  be a normal direction field to  $\mathcal{F}_\theta$  in  $G_\theta$ . As  $\mathcal{F}_\theta$  is orientable (see for example [1]),  $X$  can in fact be chosen to be a vector field. This vector field  $X$  is transverse to the boundary of  $G_\theta$  and has node singularities (resp. saddles) at the centres (resp. saddles) of  $\mathcal{F}_\theta$ . Given that the Euler characteristic of  $G_\theta$  is 1 and nodes and saddles are singularities of indices 1 and -1, respectively, we conclude that  $c_\theta = s_\theta + 1$ , where  $c_\theta$  (resp.  $s_\theta$ ) denotes the number of centres (resp. saddles) in  $G_\theta$ . This implies that the number of points in  $\Delta \cap G_\theta$  is odd. Suppose that all the connected components of  $\Delta$  are homotopic to a constant in  $M$ . Then  $\Delta$  intersects  $G_\theta$  in an even number of points, a contradiction. Therefore at least one of the connected component of  $\Delta$  is not homotopic to constant in  $M$ .  $\square$

Since the number of points in  $G_\theta \cap \Delta$  is always odd when  $G_\theta$  does not contain points of tangency, so  $\Delta$  cannot be as in Figure 2. So, we conclude the following result:

**Corollary 2.** *If the discriminant has only one connected component then*

the class  $[\Delta]$  generates the fundamental group of  $M$ .

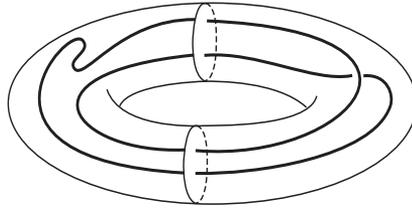


Figure 2: An impossible configuration of the discriminant.

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