International Journal of Applied Mathematics

Volume 27 No. 1 2014, 89-107
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v27i1.9

ASYMPTOTIC LINEAR ARBITRAGE AND UTILITY-BASED ASYMPTOTIC LINEAR ARBITRAGE IN MEAN-REVERTING FINANCIAL MARKETS

Mbele Bidima Martin Le Doux

University of Yaoundé I, Cameroon
African Institute for Mathematical Sciences
P.O. Box 15780 Yaoundé, CAMEROON

Abstract: Consider a general mean-reverting discrete-time model of financial markets in which the stock prices process is a time discretization of a stochastic differential equation. We introduce a new type of asymptotic arbitrage by proving existence of self-financing strategies that generate linear growing profits on investors' wealth with probability converging to 1 geometrically fast. We estimate the rate of this convergence using ergodic results on Markov chains and large deviations theory.

Next, we discuss asymptotic linear arbitrage in the expected utility sense and its link with the first type of asymptotic arbitrage.

AMS Subject Classification: 91G80, 60F10, 60J05
Key Words: asymptotic linear arbitrage, stochastic control in finance, mean-reverting process, Markov chain, discrete-time Markov processes, large deviations, large deviations principle (LDP)

1. Introduction

In Mathematical Finance, most models in discrete and continuous-time share the following feature: for any finite time horizon $T < \infty$, there is a possibility to exclude arbitrage opportunity from the market model, see for instance [3]. But, in long-term trading i.e., when $T \to \infty$, one may always generate riskless profit, which is known in the literature as asymptotic arbitrage, see for e.g. the

Received: January 22, 2014

pioneering works of Kabanov and Kramkov in [7].

This concept of asymptotic arbitrage has been studied since then in slightly different forms by some authors. For examples, after they discussed they subject in a typical case of Urnstein-Uhlenbeck process, Föllmer and Schachermayer in [5] conjectured in a general continuous-time diffusion model the possibility of generating exponential growth profit on investor's wealth in long-term; what in a corresponding discrete-time setting, we proved in a joint work in [9] and called it "asymptotic exponential arbitrage". Moreover, we introduced in [9] a more meaningful version of asymptotic exponential arbitrage by considering a general discrete-time stock prices model, expressed in an exponential form and by showing existence of exponential growth profit on investors' wealth in long-term with the possibility of controlling at a geometrically decaying rate the probability of failing to achieve such a profit.

In this paper, under the modeling settings below, we introduce the new concepts of "asymptotic linear arbitrage with geometrically decaying probability of failure" and "utility-based asymptotic linear arbitrage". The former is similar to the one we just mentioned above, which we do not recall here as it is indeed similarly defined but treated under different settings.

Consider a discrete-time financial market with two assets in trading: a riskless asset (a bank account or a risk-free bond) with fixed interest rate, set to 0 for simplicity, i.e., with prices normalized to $B_t := 1$ for all time $t \in \mathbb{N}$, and a single risky asset (such as stock) whose (discounted) prices S_t, $t \in \mathbb{N}$, is an $\mathbb{R}$-valued process governed by the stochastic difference equation

$$S_{t+1} = S_t + \mu(S_t) + \sigma(S_t)\varepsilon_{t+1}, \ t \in \mathbb{N}. \tag{1}$$

S_0 is assumed constant, $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$, with $\sigma > 0$, are measurable functions determining the drift and volatility of the stock, $(\varepsilon_t)_{t\in\mathbb{N}}$ is an $\mathbb{R}$-valued sequence of *i.i.d* random variables representing the random driving process of the stock prices evolution. We assume that the stock prices process is modeled and integrable in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{N}}$ with $\mathcal{F}_t := \sigma(S_0, S_1, ..., S_t)$, $t \in \mathbb{N}$, is the natural filtration of the stock prices process. $\mathbb{E}$ will always denote the expectation with respect to the probability measure $\mathbb{P}$.

Note that (1) can be thought as the time-discretization of a general diffusion process. In particular, if $\mu(x) := -\alpha x$ with $0 < \alpha < 1$, and $\sigma(x) := 1$ for all $x \in \mathbb{R}$, then we get the discrete-time Ornstein-Uhlenbeck process.

Clearly, the stock prices process S_t in (1) is a (discrete-time) Markov chain in the (uncountable) state space $\mathbb{R}$ (see pp. 211–228 in [1]). Unlike in our joint work [9] and other similar works such as [3], [5], we do not consider it expressed in any exponential form.

Next, in this market model, trading strategies we consider are $\mathbb{R}$-valued $(\mathcal{F}_t)_t$-predictable processes $(\pi_t)_{t\in\mathbb{N}}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$: π_t represents the number of units of the stock an economic agent holds at each time t and it is $\mathcal{F}_{t-1}$-measurable by predictability. Since since π_t is $\mathbb{R}$-valued, then it can be negative; meaning that we allow short-selling of the stock, a more realistic consideration we prohibited from the models in [9].

Given any such trading opportunity π_t, we model the corresponding (discounted) wealth an investor allocates in the stock by an $\mathbb{R}$-valued discrete-time stochastic process V_t^π obeying the (self-financing) dynamics

$$\begin{cases} V_{t+1}^\pi = V_t^\pi + \pi_{t+1}(S_{t+1} - S_t) \text{ for all time } t \geq 1, \\ V_0^\pi := V_0 \geq 0, \text{ is the investor's initial capital.} \end{cases} \tag{2}$$

With these modeling settings, we organize the whole paper as follows. In Section 2 below, we define properly the concept of asymptotic linear arbitrage strategies (with geometrically decaying probability of failure). We state in Theorem 2.4 and prove later the main result on existence of such trading strategies in the models (1) and (2) under suitable conditions. And we check that these conditions hold in two practical examples of discrete-time financial models.

Next, in the third and last section of the paper, we also define the concept of "utility-based asymptotic linear arbitrage", i.e., asymptotic linear arbitrage linked to the concept of expected utilities (see for e.g. [4, Chap. 5]). Classically, an optimal investment for an economic agent with utility function U is the available portfolio π_t with (random) wealth outcome V_t^π for which the expected utility $\mathbb{E}U(V_t^\pi)$ is maximal. In that section, we do not focus on the construction of optimal strategies (which is well discussed in the literature, see for e.g. [5]), but rather on treating the following basic question: Among risk-averse and risk-seeking investors, what type of investor (with a suitable utility function) for which if he/her wealth V_t^π grows linearly fast in the sense of Definition 2.2 above, then his/her expected utility will also increase (at least) linearly fast? In Theorem 3.2 we provide an answer to this question for risk-seeking investors with a suitable class of utility functions.

2. The Concept of Asymptotic Linear Arbitrage

Definition 2.1. Let π_t be any (self-financing) predictable strategy in the models (1) and (2). We say that π_t is an asymptotic linear arbitrage (ALA)

in the wealth model (2), if from zero initial capital V_0, there is a real constant $a > 0$ such that, for all $\epsilon > 0$, there is a time $t_\epsilon \in \mathbb{N}$ satisfying

$$\mathbb{P}(V_t^\pi \geq at) \geq 1 - \epsilon, \text{ for all time } t \geq t_\epsilon. \tag{3}$$

The financial interpretation of this is straightforward: given a "tolerance level" $\epsilon > 0$, there is a treshold time t_ϵ from which an investor starts to generate profit in long-term at a linear growth rate, with probability tending to 1. However, one may need to wait for a long time t_ϵ before starting to realize any such profit in long-term. Therefore we formulate a strengthened version of (3) by connecting the tolerance level ϵ with the running time t.

Definition 2.2. We say that the trading opportunity π_t generates a (strong) asymptotic linear arbitrage (ALA) with geometrically decaying probability of failure (GDP-F) if from zero initial capital V_0, there are real constants $a > 0$, and $c > 0$ such that,

$$\mathbb{P}(V_t^\pi \geq at) \geq 1 - e^{-ct} \text{ for all large enough time } t \geq 1, \tag{4}$$

or equivalently,

$$\mathbb{P}(V_t^\pi < at) < e^{-ct}, \text{ for all large enough time } t \geq 1. \tag{5}$$

The additional financial feature of this definition is that, by (4) the investor's wealth grows linearly fast in long term independently from any treshold time, and by (5) an economic agent can control at a geometrically decaying rate the probability of failing to achieve such a linear growth profit in long-term.

In order to investigate trading strategies π_t that generate ALA with GDP-F in the market models (1) and (2):

First, due to the Markovian structure of the stock prices process S_t, we restrict ourselves to bounded "Markovian strategies" i.e., trading opportunities of the form $\pi_t := \pi(S_{t-1})$, for all $t \in \mathbb{N}$, where $\pi : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function with respect to the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

Next, we assume that the stock prices process S_t in (1) satisfies the so-called "mean-reverting" condition:

$$(MRC): \qquad \limsup_{|x|\to\infty} \frac{|x + \mu(x)|}{|x|} < 1. \tag{6}$$

This is obviously verified for e.g., by the (discrete-time) Ornstein-Uhlenbeck process. It means the stock prices have at most a linear growth and they tend to move about the average price in time.

And we suppose that the following set of conditions holds in the same models (1) and (2):

(A_1) The random variables ε_t's have a common $(a.s.)$ strictly positive density γ with respect to the Lebesgue measure λ on $\mathbb{R}$, and this density is $(a.s.)$ bounded on each compact in $\mathbb{R}$.

(A_2) The drift μ is locally bounded. The volatility σ is positive, bounded away from zero on each compact and is (globally) bounded.

(A_3) And we assume the following integrability property for the law of the ε_t's:

$$\exists\, \kappa > 0 \text{ such that } \mathbb{E}\big(e^{\kappa \varepsilon^2}\big) =: I < \infty, \tag{7}$$

where ε has the distribution as the ε_t's, $t \in \mathbb{N}$. We also assume that $\mathbb{E}\varepsilon = 0$ holds[1].

Under these conditions, to proceed to the statement of the existence theorem, we express first the prices process S_t of the stock from (1) in the form

$$S_{t+1} - S_t = \mu(S_t) + \sigma(S_t)\varepsilon_{t+1} = \sigma(S_t)\big(\varphi(S_t) + \varepsilon_{t+1}\big),$$

where the function φ is defined by $\varphi(x) := \mu(x)/\sigma(x)$, for all $x \in \mathbb{R}$.

Definition 2.3. We call φ the "market price of risk" function for the stock prices S_t.

The quantity $\varphi(S_t)$ bears a straightforward interpretation: since $\mu(S_t)$ represents the average one-step return of of the stock while $\sigma(S_t)$ measures the one-step volatility of this price as driven by the random "noise" ε_t, $\varphi(S_t)$ represents the one-step return of stock per unit volatility.

We may also require the condition below, called "risk-condition", for the market price of risk function φ:

$$(RC): \quad \text{the set } R_0 := \{x \in \mathbb{R} \mid \varphi(x) \neq 0\} \text{ satisfies } \lambda(R_0) > 0. \tag{8}$$

We interpret the set R_0 as representing all states of the stock prices S_t whose market price of risk is not 0. We say that (RC) holds if $\lambda(R_0) > 0$.

Set $R_0^+ := \{x \in \mathbb{R} \mid \varphi(x) > 0\}$ and $R_0^- := \{x \in \mathbb{R} \mid \varphi(x) < 0\}$. If $\mathbf{1}_A$ denotes the indicator function on A for any $A \subseteq \mathbb{R}$, consider the bounded Markovian strategy

$$\pi_t^0 := \pi^0(S_{t-1}) \text{ where } \pi^0(x) := \mathbf{1}_{R_0^+}(x) - \mathbf{1}_{R_0^-}(x) \text{ for all } x \in \mathbb{R}, \tag{9}$$

[1]This is not a restriction of generality. If one had $\mathbb{E}\varepsilon = m$ one could replace $\mu(x)$ by $\mu'(x) := \mu(x) + \sigma(x)m$ and ε_i by $\varepsilon_i' := \varepsilon_i - m$ and in this way we get back to the case $\mathbb{E}\varepsilon = 0$.

which is interpreted as being constructable by a potential long-term arbitrageur as follows: s/he invests all his money in the stock whenever its market price of risk is positive, s/he sells the stock short when the market price of risk is negative, otherwise he puts everything into his bank account. Then we state the first main result of this paper.

Theorem 2.4. *Suppose that the market price of risk function φ satisfies the risk-condition* (*RC*) *in* (8). *Then the bounded Markovian strategy* $\pi_t^0 = \mathbf{1}_{R_0^+}(S_{t-1}) - \mathbf{1}_{R_0^-}(S_{t-1})$ *generates an ALA with GDP-F in the models* (1) *and* (2).

We present the proof at the end of this section after an appropriate preparation.

First, inspecting the dynamics of the investor's wealth process in (2) for any bounded Markovian strategy $\pi_t = \pi(S_{t-1})$, we express it in the functional form

$$V_t^\pi = V_0 + \sum_{n=1}^{t} f(\Phi_n), \text{ for all time } t \geq 1, \tag{10}$$

of the auxiliary process $\Phi_n := (S_{n-1}, S_n)$ of two consecutive values of the stock prices process, where f is the measurable function defined on $\mathbb{R}^2$ by $f(x, y) := \pi(x)(y - x)$. Assume that S_{-1} is an (arbitrary) given initial constant so that the process $\Phi_t = (S_{t-1}, S_t)$ starts at time 0 as well.

We show below a first set of preliminary results derived from the advanced theory of Markov chains presented in [10] and from the ergodic theory of functionals of Markov chains in [8].

Proposition 2.5. *The stochastic process Φ_t is a Markov chain with state space $\mathbb{R}^2$.*

Proof. We derive this from [1] pp. 211-228, where the Markov property of any (discrete-time) Markov chain Y_t in a Polish state space $\mathcal{S}$ is characterized by its evolution in the form $Y_{t+1} = f(Y_t, \xi_{t+1})$, with $(\xi_t)_t$ a sequence of $i.i.d$ random variables independent of Y_0 and valued in some measurable space $\mathcal{S}'$, and $f : \mathcal{S} \times \mathcal{S}' \to \mathcal{S}$ a suitable measurable function. Indeed, using (1), the Markov chain S_t is in the form $S_{t+1} = f(S_t, \varepsilon_{t+1})$ for all time $t \in \mathbb{N}$, with the measurable function f defined by $f(x, y) := x + \mu(x) + \sigma(x)y$ for all $x, y \in \mathbb{R}$. It follows for all time $t \in \mathbb{N}$, that we have $\Phi_{t+1} = (S_t, S_{t+1}) = (S_t; f(S_t, \varepsilon_{t+1})) = F(\Phi_t, \xi_{t+1})$, where $\xi_t := (0, \varepsilon_t)$ and F is the measurable function defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by

$F((x,y);(a,b)) := (y; f(y,b))$. Since the ε_t's are *i.i.d* and independent from S_0, the ξ_t's are also *i.i.d* and independent from Φ_0, showing that the next state Φ_{t+1} is generated from the previous state Φ_t, plus an independent noise ξ_{t+1}, as required. □

Let λ_2 denote the Lebesgue measure on $\mathbb{R}^2$ and $\mathcal{B}(\mathbb{R}^2)$ the Borel σ-algebra on $\mathbb{R}^2$. For $x \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$, let us denote $P(x,A) := \mathbb{P}(S_{t+1} \in A | S_t = x)$, $t \geq 0$, the one-step transition probability kernel of the chain S_t, and $P^t(x,A) := \mathbb{P}(S_t \in A | S_0 = x)$ its t-step transition probability kernel. Also for $z \in \mathbb{R}^2$ and $C \in \mathcal{B}(\mathbb{R}^2)$, let $Q(z,C)$ and $Q^t(z,C)$ denote the corresponding kernels for the chain Φ_t. Then we have the following

Proposition 2.6. *The Markov chain S_t is ψ-irreducible and aperiodic.*

Proof. By the definition of ψ-irreducibility and Propositions 4.2.1 and 4.2.2 on pp. 89-90 in [10], it is enough to show it is λ-irreducible, that is; if $x \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$ such that $\lambda(A) > 0$, then, there is an integer $t \geq 1$ such that $P^t(x,A) > 0$. This clearly follows for $t = 1$ by Assumption (A_1) and by the translation invariance property of the Lebesgue measure λ applied in the last of the equalities below:

$$\begin{aligned} P(x,A) := \mathbb{P}\big(S_{t+1} \in A \mid S_t = x\big) &= \mathbb{P}\big(x + \mu(x) + \sigma(x)\varepsilon_{t+1} \in A\big) \\ &= \int_{(A-x-\mu(x))/\sigma(x)} \gamma(y)\lambda(dy) > 0. \end{aligned} \tag{11}$$

Next, to prove the aperiodicity of S_t, by Definition 5.14 on p.109 in [10] and by Theorem 5.4.4 on p.121 of the same reference, it is enough to show that there exists a subset C in $\mathbb{R}$ with $\lambda(C) > 0$, $n > 0$ and a non-trivial measure ν_n on $\mathcal{B}(\mathbb{R})$ such that

$$P^n(x,A) \geq \nu_n(A) \text{ for all } x \in C,\ A \in \mathcal{B}(\mathbb{R}), \tag{12}$$

and the *g.c.d* (greatest common divisor) of the set E_C is 1, where

$$E_C := \{n \geq 1 : C \text{ satisfies } (12)\}.$$

Indeed, fix any compact subset C in $\mathbb{R}$ with $\lambda(C) > 0$ and let $n := 1$. Using Assumptions (A_1) and (A_2), it follows from (11) that $P^1(x,A) = P(x,A) \geq c\lambda \mathbf{1}_C(A)$ for some constant $c > 0$ and for all $x \in C$, $A \in \mathcal{B}(\mathbb{R})$. Indeed, $c := (\sup\{\sigma(u) : u \in C\})^{-1} \inf\left\{\gamma\Big(\frac{a-u-\mu(u)}{\sigma(u)}\Big) : a \in C, u \in C\right\}$. So, taking $\nu_1 := c\lambda\mathbf{1}_C$, where $\nu_1(dy) := c\lambda(dy \cap C)$, we get $1 \in E_C$, hence $g.c.d(E_C) = 1$. □

Proposition 2.7. *The Markov chain Φ_t is also ψ-irreducible and aperiodic.*

Proof. Similarly to the preceding proof, we show first that Φ_t is irreducible. Using condition (A_1), for all $y \in \mathbb{R}$ the random variable $y + \mu(y) + \sigma(y)\varepsilon_1$ has a λ-*a.e.* positive density, $p_1(u), u \in \mathbb{R}$. By the same argument, for all $u \in \mathbb{R}$ the random variable $y + \mu(y + \mu(y) + \sigma(y)u) + \sigma(y + \mu(y) + \sigma(y)u)\varepsilon_2$ has a λ-*a.e.* positive density $p_2(u, w), w \in \mathbb{R}$ which can be chosen jointly measurable in (u, w). Hence, by independence of $\varepsilon_1, \varepsilon_2$, when $\Phi_0 = (x, y)$, the density of

$$\Phi_2 = (y + \mu(y) + \sigma(y)\varepsilon_1, y + \mu(y + \mu(y) + \sigma(y)\varepsilon_1) + \sigma(y + \mu(y) + \sigma(y)\varepsilon_1)\varepsilon_2)$$

with respect to λ_2 equals $p_1(u)p_2(u, w)$, and this is λ_2-*a.e.* positive. In particular, for all $A, B \in \mathcal{B}(\mathbb{R})$ with $\lambda_2(A \times B) > 0$, setting $(x, y) = z$, we have

$$Q^2(z, A \times B) = \mathbb{P}(\Phi_2 \in A \times B|\Phi_0 = z) = \int_{A\times B} p_1(u)p_2(u, w)\lambda_2(du, dw), \quad (13)$$

which is strictly positive, showing λ_2-irreducibility and hence ψ-irreducibility of the Markov chain Φ_t.

Next for aperiodicity, take any compact rectangle $C := C_1 \times C_2$ such that $\lambda_2(C) > 0$ with C_1 and C_2 intervals in $\mathbb{R}$, there exist constants $c_1, c_2 > 0$ such that, with the measure $\nu_2 := c_1c_2\lambda_2\mathbf{1}_C$ defined on $\mathcal{B}(\mathbb{R}^2)$ by $\nu_2(dy_1, dy_2) = c_1c_2\lambda_2(dy_1\cap C_1, dy_2\cap C_2)$, we have $Q^2((x, y); A\times B) \geq \nu_2(A\times B)$ for all $x \in C_1$, $y \in C_2$ and all $A, B \in \mathcal{B}(\mathbb{R})$, which proves that $2 \in E_C$ (where E_C is defined using the kernel Q instead of P in (12)). Moreover, by an additional similar argument, one gets also $3 \in E_C$, showing that $g.c.d(E_C) = 1$, as required. □

Lemma 2.8. *The random variable ε in (7) of Assumption (A_3) satisfies the following property: for every real number $a \geq 1$, we have,*

$$\mathbb{E}\big(e^{a|\varepsilon|}\big) \leq e^{ca^2} \text{ for some fixed constant } c > 0. \quad (14)$$

Proof. Set $\xi := |\varepsilon|$. Then we have

$$\begin{aligned}
\mathbb{P}\big(e^{a\xi} > x\big) &= \mathbb{P}\Big(\exp\Big(\kappa\Big[\tfrac{\log(e^{a\xi})}{a}\Big]^2\Big) > \exp\Big(\kappa\Big[\tfrac{\log x}{a}\Big]^2\Big)\Big) \\
&\leq I\exp\Big(-\kappa\big(\log(x)/a\big)^2\Big) \text{ by Markov Inequality} \\
&= I\big(\tfrac{1}{x}\big)^{(\kappa/a^2)\log x},
\end{aligned}$$

see (7) for the definition of I. Since the exponent $(\kappa/a^2)\log x > 2$ provided that $x > e^{2a^2/\kappa}$, we have $\mathbb{E}\big(e^{a\xi}\big) = \int_0^\infty \mathbb{P}\big(e^{a\xi} > x\big)dx \leq e^{2a^2/\kappa} + I\int_{\exp(2a^2/\kappa)}^\infty 1/x^2 dx.$

The last integral is less than $\int_1^\infty 1/x^2 dx$, which is finite, thus we conclude the proof by taking for example $c = c_1 + (2/\kappa)$ with $c_1 > 0$ large enough. □

Applying this lemma, we check below the following

Proposition 2.9. *The Markov chain S_t satisfies the "drift condition" $(DV3+)\,(i)$ on p. 6 of [8], which is recalled in the proof below.*

Proof. $(DV3+)\,(i)$ means: the chain S_t is ψ-irreducible, aperiodic and there are (measurable) functions $V, W : \mathbb{R} \to [1, \infty)$, a subset C in $\mathbb{R}$ verifying (12) above for some n, and constants $\delta > 0$, $b < \infty$ such that $\log\big(e^{-V} P\, e^{V}\big)(x) \leq -\delta W(x) + b\mathbf{1}_C(x)$, for all $x \in \mathbb{R}$, with $P\, e^V$ defined by

$$P\, e^V(x) := \int e^{V(y)} P(x, dy), \text{ for all } x \in \mathbb{R}.$$

This is equivalent to requiring that,

$$P\, e^V(x) \leq e^{V(x) - \delta W(x) + b\mathbf{1}_C(x)} \text{ for all } x \in \mathbb{R}. \tag{15}$$

By Proposition 2.6, S_t is ψ-irreducible and aperiodic. Next, define $V(x) = W(x) := 1 + qx^2$, for all $x \in \mathbb{R}$, where $q > 0$ is a small number to be chosen. Consider a compact set $C := [-K, K]$ for a large positive constant K. As in the proof of Proposition 2.6, C satisfies (12).

Since $Pe^V(x) = \mathbb{E}\big(e^{V(S_1)} \mid S_0 = x\big) = \mathbb{E}\big(e^{V(x+\mu(x)+\sigma(x)\varepsilon)}\big)$, it follows from (15) that we need to show for all $x \in \mathbb{R}$,

$$\mathbb{E}\big(e^{1+q(x+\mu(x))^2+2q(x+\mu(x))\sigma(x)\varepsilon+q\sigma^2(x)\varepsilon^2}\big) \leq e^{(1-\delta)V(x)+b\mathbf{1}_C(x)}. \tag{16}$$

To get this, it is sufficient to prove the two conditions below:

Claim 1: for $|x|$ large enough we have

$$\mathbb{E}\big(e^{1+q(x+\mu(x))^2+2q(x+\mu(x))\sigma(x)\varepsilon+q\sigma^2(x)\varepsilon^2}\big) \leq e^{(1-\delta)(1+qx^2)}. \tag{17}$$

Claim 2: for small $|x|$, (that is; for x in any fixed compact), we have

$$\sup_{x\in C} \mathbb{E}\big(e^{1+q(x+\mu(x))^2+2q(x+\mu(x))\sigma(x)\varepsilon+q\sigma^2(x)\varepsilon^2}\big) < G(K), \tag{18}$$

for some positive constant $G(K) < \infty$, and then once this is done, take $b := \log G(K)$.

Proof of Claim 1. Using the mean-reverting condition (MRC) in (6), for $|x|$ large enough, there is a small $\delta > 0$ such that we have $(x + \mu(x))^2 \leq$

$(1-4\delta)x^2$. And since $1 \le \delta(1+qx^2)$ for $|x|$ large, it follows that $e^{1+q(x+\mu(x))^2} \le e^{(1-3\delta)(1+qx^2)}$.

By (A_2) there is $M > 0$ such that, for all x, $\sigma(x) \le M$. If we choose q such that $qM^2 < \kappa/2$, then it is enough to show that $\mathbb{E}\big(e^{2q|x+\mu(x)|M|\varepsilon|+(\kappa/2)\varepsilon^2}\big) \le e^{2\delta qx^2}$. By the Cauchy-Schwarz inequality, this requires to prove that,

$$\sqrt{\mathbb{E}\big(e^{4q|x+\mu(x)|M|\varepsilon|}\big)}\sqrt{\mathbb{E}\big(e^{\kappa\varepsilon^2}\big)} \le e^{2\delta qx^2} \tag{19}$$

By (7), the second term on the left-hand side of (19) is the constant $\sqrt{I}$. This is smaller than $e^{\delta qx^2}$ for large enough $|x|$. So, since again by Condition (MRC), $4q|x+\mu(x)|M \le 4qM|x|$ for $|x|$ large, it follows that we finally have to show $\sqrt{\mathbb{E}\big(e^{4qM|x||\varepsilon|}\big)} \le e^{\delta qx^2}$ for large $|x|$, or, equivalently,

$$\mathbb{E}\big(e^{4qM|x||\varepsilon|}\big) \le e^{2\delta qx^2} \text{ for large } |x|. \tag{20}$$

But applying Lemma 2.8, the left-hand side of (20) is smaller than $e^{16cq^2M^2|x|^2}$ for some fixed constant $c > 0$. Hence, if one chooses q small enough such that $16q^2M^2c < 2\delta q$ and $qM^2 < \kappa/2$ then (20) holds, showing Claim 1.

Proof of Claim 2. By Assumption (A_2), μ is bounded above on any compact $C = [-K, K]$ by some positive constant A. Since μ is bounded on C, the function $x \mapsto (x+\mu(x))^2$ is also bounded on C. We assume it bounded above on that C by some positive constant B. So, with the later choice of q, we have the following estimate applying Cauchy-Schwarz Inequality and (7),

$$\begin{aligned}
&\mathbb{E}\big(e^{1+q(x+\mu(x))^2+2q(x+\mu(x))\sigma(x)\varepsilon+q\sigma^2(x)\varepsilon^2}\big)\\
&\le \mathbb{E}\big(e^{1+qB+2q(K+A)M|\varepsilon|+(\kappa/2)\varepsilon^2}\big)\\
&\le e^{(1+qB)}\sqrt{\mathbb{E}\big(e^{4q(K+A)M|\varepsilon|}\big)}\sqrt{\mathbb{E}\big(e^{\kappa\varepsilon^2}\big)}\\
&= e^{(1+qB)}\sqrt{I}\sqrt{\mathbb{E}\big(e^{4q(K+A)M|\varepsilon|}\big)}.
\end{aligned}$$

We then choose K large enough such that $4q(K+A)M \ge 1$ and we get, by Lemma 2.8, that for all $x \in C = [-K, K]$,

$$\mathbb{E}\big(e^{1+q(x+\mu(x))^2+2q(x+\mu(x))\sigma(x)\varepsilon+q\sigma^2(x)\varepsilon^2}\big) \le e^{(1+qB)}\sqrt{I}\sqrt{e^{16c'q^2(K+A)^2M^2}},$$

for a fixed constant $c' > 0$. This holds for all $x \in C$, hence (18) holds true when taking the supremum over C of the left-hand side of this latter inequality. This completes the proof of the whole proposition. □

Corollary 2.10. *The Markov chain $\Phi_t = (S_{t-1}, S_t)$ also satisfies the drift condition $(DV3+)$ (i) on p. 6 of [8].*

Proof. This follows from the preceding proposition and Proposition 4.1 (v) of [8]. □

Furthermore,

Proposition 2.11. *The Markov chain $\Phi_t = (S_{t-1}, S_t)$ also satisfies the second condition (ii) of $(DV3+)$ on p.6 of [8], recalled in the proof below.*

Proof. We have to show that: there are functions $V, W : \mathbb{R}^2 \to [1, \infty)$, and there is a time $t_0 > 0$ such that for all $r < ||W||_\infty$, there is a measure β_r such that $\beta_r(e^V) := \int_{\mathbb{R}^2} e^{V(x,y)} \beta_r(dx, dy)$ is finite and we have

$$Q_{(x,y)}\left(\Phi_{t_0} \in A \times B, \tau_{C^c_W(r) > t_0}\right) \leq \beta_r(A \times B),$$

for all $(x, y) \in C_W(r) := \{(x, y) : W(x, y) \leq r\}$ and all $A, B \in \mathcal{B}(\mathbb{R})$. Here $\tau_{C^c_W(r)} := min\{t \geq 1 : \Phi_t \in C^c_W(r)\}$ and $C^c_W(r)$ denotes the complement of $C_W(r)$.

Consider then the functions $V(x, y) = W(x, y) := 1 + q(x^2 + y^2)$ for $x, y \in \mathbb{R}$ for a suitable $q > 0$ as in the proof of Proposition 2.9. Since the chain Φ_t starts at time $t = 1$, we choose here $t_0 := 2$, and let $r < \|W\|_\infty = \infty$. Then:

If $0 \leq r < 1$, the statement below logically follows.

Suppose $r \geq 1$, we have $C_W(r) = \{(x, y) : 1 + q(x^2 + y^2) \leq r\} = \{(x, y) : x^2 + y^2 \leq (r - 1)/q\}$ which is the compact disk of radius $(r - 1)/2$ in $\mathbb{R}^2$. So its first and second projections $C_1 := pr_1(C_W(r))$ and $C_2 := pr_2(C_W(r))$ are compact intervals in $\mathbb{R}$.

If $A, B \in \mathcal{B}(\mathbb{R})$ and $(x, y) \in C_W(r)$, then $x \in C_1$ and $y \in C_2$. Hence, setting for simplicity $\Delta := Q_{(x,y)}\big(\Phi_2 \in A \times B, \tau_{C^c_W(r)} > 2\big)$, we obtain that,

$$\begin{aligned}
\Delta &= \mathbb{P}\big(\Phi_2 \in A \times B, \Phi_2 \in C_W(r) \mid \Phi_1 = (x, y)\big) \\
&= \mathbb{P}\big(\Phi_2 \in (A \times B) \cap C_W(r) \mid \Phi_1 = (x, y)\big) \\
&\leq \mathbb{P}\big(S_1 \in A \cap C_1, S_2 \in B \cap C_2 \mid S_0 = x, S_1 = y\big) \\
&= \mathbb{P}\big(x + \mu(x) + \sigma(x)\varepsilon_0 \in A \cap C_1, y + \mu(y) + \sigma(y)\varepsilon_1 \in B \cap C_2\big) \qquad (21) \\
&\leq J_r J'_r \lambda(A \cap C_1)\lambda(B \cap C_2) \\
&= J_r J'_r \lambda_2\big((A \times B) \cap (C_1 \times C_2)\big) \\
&=: \beta_r(A \times B),
\end{aligned}$$

for some constants J_r and J'_r (depending on q), using the independence of ε_0 and ε_1, Fubini's Theorem and Assumptions (A_1), (A_2). β_r so defined is clearly a measure on $\mathbb{R}^2$.

Finally, it is clear that $\beta_r(e^V) = \int_{C_1\times C_2} J_r J_r' e^{1+q(x^2+y^2)} \lambda_2(dx, dy) < \infty$ as integral of a continuous function on a compact of $\mathbb{R}^2$ with respect to λ_2, ending the proof, □

Corollary 2.12. *The Markov chain Φ_t has an invariant probability measure ν equivalent to the Lebesgue measure λ_2 on $\mathbb{R}^2$.*

Proof. By Corollary 2.10 and Proposition 2.11 above, Φ_t satisfies the whole condition $(DV3+)$ (i) and (ii). It follows by Theorem 1.2 in [8] that the Markov chain Φ_t has a unique invariant probability measure, say ν.

Moreover, from the proof of ψ-irreducibility of Φ_t in Proposition 2.7, $\mathbb{P}(\Phi_2 \in \cdot|\Phi_0 = (x, y))$ is λ_2-absolutely continuous for each $(x, y) \in \mathbb{R}^2$, hence we get $\nu \ll \lambda_2$. On the other hand, since the chain Φ_t is ψ-irreducible with ν as its invariant probability measure, from the definition of recurrent and positive chains on pp. 186 and 235 of [10], it follows by Proposition 10.1.1 and Theorem 10.4.9 of the same reference that $\nu \sim \psi$. But, $\psi \gg \lambda_2$ by Proposition 4.2.2 (ii) in [10], so $\nu \gg \lambda_2$, and hence $\nu \sim \lambda_2$, as required. □

Next, after this first set of preliminary results, as indicated in the introduction, we now proceed to the application of classical large deviations techniques from [2]. First, recall the investor's wealth process as in (10) for any given Markovian strategy $\pi_t = \pi(S_{t-1})$,

$$V_t^\pi = V_0 + \sum_{n=1}^{t} f(\Phi_n), \text{ for all time } t \geq 1,$$

with $f(x, y) = \pi(x)(y - x)$, for all $x, y \in \mathbb{R}$.

We have to insure that, for every π_t, the sequence of random variables $(V_t^\pi - V_0)_t = \sum_{n=1}^t f(\Phi_n)$ satisfies the LDP hypotheses, that is; the limit $\Lambda_f(\theta) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{\theta \sum_{n=1}^t f(\Phi_n)})$ for each $\theta \in \mathbb{R}$, exists with Λ_f satisfying the remaining conditions in Gärtner-Ellis' theorem as stated in Theorem 2.3.6, [2].

For that, since we obtained from Corollary 2.10 and Proposition 2.11 that the Markov chain Φ_t is ψ-irreducible, aperiodic and satisfies the conditions $(DV3+)\,(i)$, (ii) in [8], we need to apply ergodic results related to Φ_t from that article. We adopt the notations of [8] below and before checking the conditions under which these results hold. Indeed, considering the functions V, W in the proof of Proposition 2.11, define another function W_0 by $W_0(x, y) := 1 + q(|x| + |y|)$, for $x, y \in \mathbb{R}$, where q is the same as in the definition of V, W in that proposition, we obviously see that

$$\lim_{r\to\infty} \sup_{x,y\in\mathbb{R}} \Big(\frac{W_0(x,y)}{W(x,y)}\mathbf{1}_{W(x,y)>r}\Big) = 0, \tag{22}$$

which is Condition (6) on p. 7 of [8].

For every $\theta \in \mathbb{R}$, we observe that $\theta \sum_{n=1}^{t} f(\Phi_n) = \sum_{n=1}^{t} F_\theta(\Phi_n)$, where $F_\theta = \theta f$. Consider the Banach space $L_\infty^{W_0}$ defined on p. 4 of [8] by $L_\infty^{W_0} := \{h : \mathbb{R}^2 \to \mathbb{C} : \sup_{x,y} \frac{|h(x,y)|}{W_0(x,y)} < \infty\}$, which is equipped with the norm $\|h\|_{W_0} := \sup_{x,y} |h(x,y)|/W_0(x,y)$, for $h \in L_\infty^{W_0}$. Then we have the following

Lemma 2.13. *For all $\theta \in \mathbb{R}$, the function F_θ belongs to the space $L_\infty^{W_0}$.*

Proof. It is enough to show this for $\theta = 1$. Indeed, by the boundedness assumption of π, for some constant $c > 0$, we have $|\pi(x)| \le c$ for all $x \in \mathbb{R}$. If follows that $|F_1(x,y)| \le c|y-x|$ for all $x, y \in \mathbb{R}$. Since clearly $|y-x| \le 1+|x|+|y|$, then we obtain that $|F_1(x,y)| \le c(1+|x|+|y|)$, for all $x, y \in \mathbb{R}$. Hence, taking the supremum over $(x,y) \in \mathbb{R}^2$, we get $\sup_{x,y} |F_1(x,y)|/W_0(x,y) < \infty$; that is $F_1 \in L_\infty^{W_0}$, as required. ☐

Next, consider the sequence of non-linear operators $\Gamma_t : L_\infty^{W_0} \to L_\infty^{V}$ defined as in [8], by setting for all $F \in L_\infty^{W_0}$ and all $(x,y) \in \mathbb{R}^2$,

$$\Gamma_t(F)(x,y) := \frac{1}{t} \log \mathbb{E}_{x,y}\Big(\exp\big(\sum_{n=1}^{t} F(\Phi_n)\big)\Big). \tag{23}$$

where $\mathbb{E}_{x,y}$ means that we have started the chain from $\Phi_0 := (x,y)$ and we compute the expectation accordingly. Then we get,

Proposition 2.14. *Let π_t be any bounded Markovian strategy in the model (2). Then there is an analytic function*

$$\Lambda_f(\theta) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{(S_{-1},S_0)}\big(e^{\theta(V_t^\pi - V_0)}\big),$$

defined for all $\theta \in \mathbb{R}$, such that the average sum $(V_t^\pi - V_0)/t$ satisfies an LDP (large deviations principle) with good convex rate function Λ_f^* (the convex conjugate of Λ_f).

Proof. Since Φ_t satisfies the conditions $(DV3+)$ (i), (ii) in [8] with the previous unbounded W, then by Proposition 3.6, [8], there is a non-linear operator

$\Gamma : L_\infty^{W_0} \to L_\infty^V$ such that the following uniform convergence holds over balls in $L_\infty^{W_0}$,

$$\sup_{\|F-F_0\|_{W_0}\le\delta} \|\Gamma_t(F) - \Gamma(F)\|_V \to 0 \text{ } as \ t \to \infty,$$

for each F_0 and each $\delta > 0$. For every $\theta \in \mathbb{R}$, set $F := F_\theta = \theta g$ and $F_0 := 0$. Since V_t^π depends on g, it follows that for all $\theta \in \mathbb{R}$, the limit

$$\begin{aligned} \Lambda_f(\theta) &:= \Gamma(F_\theta)(S_{-1}, S_0) \\ &= \lim_{t\to\infty} \tfrac{1}{t} \log \mathbb{E}_{(S_{-1},S_0)}\Big(\exp\big(\textstyle\sum_{n=1}^t \theta f(\Phi_n)\big)\Big) \\ &= \lim_{t\to\infty} \tfrac{1}{t} \log \mathbb{E}_{(S_{-1},S_0)}\big(e^{\theta(V_t^\pi - V_0)}\big), \end{aligned} \tag{24}$$

exists in $\mathbb{R}$. Moreover, applying Proposition 4.3 (*ii*) in [8], Λ_f is an analytic function of θ.

Again from (*ii*) of Proposition 4.3, [8], we deduce the second-order Taylor expansion about zero as, $\Lambda_f(\theta) = \Lambda_f(0) + \theta\nu(f) + \frac{1}{2}\theta^2 v_f + O(\theta^3)$ for all $\theta \in \mathbb{R}$, where ν is the invariant measure of Φ_t obtained in Corollary 2.12, the expectation $\nu(f) := \int_{\mathbb{R}^2} f(x,y)\nu(dx,dy)$ is finite, and where $v_f := \lim_{t\to\infty} \mathbb{E}_\nu \sum_{n=1}^t \big(f(\Phi_n) - \nu(f)\big)^2 \ne 0$ is the asymptotic variance given in (37), p. 24 of [8]. Hence $\Lambda_f(\theta)$ is essentially smooth.

So, applying Gärtner-Ellis Theorem 2.3.6 in [2], we conclude that $(V_t^\pi - V_0)/t$ satisfies an (upper) *LDP* estimate with good convex rate function Λ_f^*, as we required. □

Proposition 2.15. *Under the conditions of the preceding proposition, $\nu(f)$ is the unique minimizer of Λ_f^*. Moreover $\Lambda_f^*(x) > 0$ for all $x \ne \nu(f)$.*

Proof. Using (24), we see that $\Lambda_f(0) = 0$. And from the Taylor expansion of Λ_f in the preceding proof, we have $\Lambda_f'(0) = \nu(f)$, so we get by Lemma 2.4 of [6] that $\Lambda_f^*(\nu(f)) = \nu(f) \times 0 - \Lambda_f(0) = 0$. On the other hand, by definition of a conjugate function, we always have $\Lambda_f^*(x) \ge 0 \times x - \Lambda_f(0) = 0$ for all $x \in \mathbb{R}$. It follows that $\nu(f)$ is a global minimizer for Λ_f^*. Since by Proposition 2.14 above, Λ_f is analytic hence differentiable, it follows that its conjugate Λ_f^* is strictly convex on its effective domain which is, in fact, $\mathbb{R}$. This implies that the global minimizer $\nu(f)$ for Λ_f^* is unique. And this uniqueness implies that $\Lambda_f^*(x) > 0$ for all $x \ne \nu(f)$, as required. □

Finally, before the proof of the main theorem, we give the following

Proposition 2.16. *Suppose that the market price of risk function φ in Definition 2.3 satisfies the risk-condition (RC) set in* (8)*. Then the bounded Markovian strategy π_t^0 constructed in* (9) *satisfies*

$$\nu(f) = \mathbb{E}\big(\pi^0(\tilde{S}_0)(\tilde{S}_1 - \tilde{S}_0)\big) > 0, \tag{25}$$

where $(\tilde{S}_0, \tilde{S}_1)$ has distribution ν, the invariant probability measure of Φ_t.

Proof. Since ν is a probability measure on $\mathcal{B}(\mathbb{R}^2)$ and is invariant for the Markov chain $\Phi_t = (S_{t-1}, S_t)$, then there is a pair of $\mathbb{R}$-valued random variables $(\tilde{S}_0, \tilde{S}_1 = \tilde{S}_0 + \mu(\tilde{S}_0) + \sigma(\tilde{S}_0)\varepsilon_1)$ on Ω with distribution ν and such that ε_1 is still independent of $\tilde{S}_0$. For all $x \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\big(\tilde{S}_1 \mid \tilde{S}_0 = x\big) &= \mathbb{E}\big(x + \mu(x) + \sigma(x)\varepsilon_1 \mid \tilde{S}_0 = x\big) \\
&= x + \mu(x) + \sigma(x)\mathbb{E}(\varepsilon_1 \mid \tilde{S}_0 = x) \\
&= x + \mu(x) + \sigma(x)\mathbb{E}(\varepsilon_1) \text{ by independence of } \varepsilon_1 \text{ from } \tilde{S}_0 \\
&= x + \sigma(x)\varphi(x) \text{ by Assumption } (A_3).
\end{aligned}$$

Since (A_2) implies $\sigma > 0$, it follows that if $x \in R_0$ (the set defined in (8)), then we have

$$\mathbb{E}(\tilde{S}_1 \mid \tilde{S}_0 = x) \neq x. \tag{26}$$

Consider our constructed strategy π_t^0 given by the function $\pi^0(x) := \mathbf{1}_{R_0^+}(x) - \mathbf{1}_{R_0^-}(x)$, for all $x \in \mathbb{R}$. By Corollary 2.12, ν has a λ_2-*a.e.* positive density with respect to λ_2, hence its $\tilde{S}_0$-marginal, denoted by η, has a λ-*a.e.* positive density $\ell(x)$. Therefore,

$$\begin{aligned}
\nu(f) &= \textstyle\int_{\mathbb{R}} \mathbb{E}\big(\pi^0(x)(\tilde{S}_1 - x) \mid \tilde{S}_0 = x\big)\eta(dx) \\
&= \textstyle\int_{R_0} \mathbb{E}\big(\tilde{S}_1 - x \mid \tilde{S}_0 = x\big)\ell(x)\lambda(dx) \\
&= \textstyle\int_{R_0} \operatorname{sgn}(\mathbb{E}\big(\tilde{S}_1 - x \mid \tilde{S}_0 = x\big))\mathbb{E}\big(\tilde{S}_1 - x \mid \tilde{S}_0 = x\big)\ell(x)\lambda(dx),
\end{aligned}$$

which is strictly positive. Hence $\nu(f) > 0$, showing the result. □

Proof of Theorem 2.4. Proposition 2.14 says that the average sum $(V_t^{\pi^0} - V_0)/t$ satisfies an LDP with good rate function Λ_f^* and Proposition 2.16 above says $\nu(f) > 0$. Next by Proposition 2.15, $\nu(f)$ is the unique minimizer of Λ_f^*, and by strict convexity, Λ_f^* is decreasing on $(-\infty, \nu(f)]$. Hence applying the upper LDP inequality (2.3.7) of Gärtner-Ellis Theorem 2.3.6 in [2], we get,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\frac{V_t^{\pi^0} - V_0}{t} < \nu(f)/2\right) \leq - \inf_{x\in(-\infty,\nu(f)/2]} \Lambda_f^*(x).$$

But

$$-\inf_{x\in(-\infty,\nu(f)/2]} \Lambda_f^*(x) = -\Lambda_f^*(\nu(f)/2).$$

These imply, under the defining *ALA* hypothesis $V_0 = 0$, that

$$\mathbb{P}\big(V_t^{\pi^0} \geq \nu(f)t/2\big) \geq 1 - e^{-t\Lambda_f^*(\nu(f)/2)}$$

for all large time t.

To complete the proof of the theorem, it remains to check that $\Lambda_f^*(\nu(f)/2) > 0$, which clearly follows by Proposition 2.15 again since $\nu(f)/2 \neq \nu(f)$, □

Remark 2.17. The additional research advance in the subject of asymptotic arbitrage theory provided by this new result is that the self-financing strategy π_t^0 generating *ALA* with *GDP-F* is explicitly constructed unlike in other works, as in [5], treating existence of earlier forms of asymptotic arbitrage.

Example 2.18. (The Discrete-Time Ornstein-Uhlenbeck Process) Consider the discrete-time Ornstein-Uhlenbeck (O-U) process,

$$S_{t+1} = \alpha S_t + \varepsilon_{t+1}, \text{ for all time } t \geq 1, \tag{27}$$

where $0 < |\alpha| < 1$ and S_0 are constants and ε_t are *i.i.d* $\mathcal{N}(0,1)$. S_t is also known as a stable auto-regressive process $AR(1)$.

In this stock prices model, the drift and volatility functions are identified as $\mu(x) = (\alpha-1)x$ and $\sigma(x) = 1$, for all $x \in \mathbb{R}$, and are clearly measurable. Hence the market price of risk function is $\varphi(x) = (\alpha-1)x$, for all $x \in \mathbb{R}$. The mean-reverting condition in (6) and all the remaining conditions of Theorem 2.4 trivially hold. From (8) we find $R_0 = \mathbb{R}\setminus\{0\} \equiv \mathbb{R}^*$, $R_0^+ = \mathbb{R}_-^*$ and $R_0^- = \mathbb{R}_+^*$. Obviously, $\lambda(R_0) = \infty > 0$. It follows that the corresponding constructed strategy $\pi_t^0 = \mathbf{1}_{\mathbb{R}_-^*}(S_{t-1}) - \mathbf{1}_{\mathbb{R}_+^*}(S_{t-1})$ produces *ALA* with *GDP-F* in the investor's wealth (2) for this discrete-time O-U model of stock prices.

Example 2.19. (A Cox-Ingersoll-Ross Type Process) In Mathematical Finance the process described by the stochastic differential equation

$$dZ_t = -\beta Z_t dt + \sigma\sqrt{|Z_t|}dW_t \tag{28}$$

is often is called the Cox-Ingersoll-Ross (CIR) process and is used to model stochastic volatility or the short rate in bond markets. Here W_t is Brownian motion. We present a slight modification of the discretization of this model here. The modifications are necessary, since the volatility of Z_t is neither bounded above nor below. For that, let us define the stock prices process by

$$S_{t+1} = \alpha S_t + \sigma \min\{\max\{\sqrt{|S_t|}, M\}, \eta\}\varepsilon_t, \quad t \geq 1, \tag{29}$$

where $|\alpha| < 1$, $\sigma > 0$, $0 < \eta < M$ are given constants and ε_t are as in the preceding example.

It is easy to check that the CIR type process S_t also satisfies the conditions of Theorem 2.4.

3. Utility-Based Asymptotic Linear Arbitrage

For this section, the stock prices process S_t, predictable (self-financing) strategies π_t and the corresponding wealth process V_t^π are still assumed relative to the same models and the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ of the introductory Section 1.

Consider the concept of expected utility of investors' wealth discussed for e.g. in [4, Chap. 5]. If $U : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then $U(V_t^\pi)$ represents the measure/level of satisfaction at time t for an investor using strategy π_t on his/her amount of wealth V_t^π with respect to the risk of losses. And $\mathbb{E}U(V_t^\pi)$ is the expected level of such a satisfaction. U is called a utility function and it is strictly increasing because investors usually prefer more money than less. Moreover, utility functions are assumed either concave for risk-averse investors, or convex for risk-seeking investors, or linear for risk-neutral investors in the market. Usually, it is rare to see investors behaving in a risk-neutral way i.e., being indifferent between preferring a random (risky) outcome on their investments and a certain (riskless) amount of wealth. Hence, as announced at the end of the introduction, we introduce below the concept of utility-based asymptotic linear arbitrage only for risk-averse or risk-seeking investors.

Definition 3.1. Let U be any utility function (convex or concave). We say that a trading strategy π_t generates a utility-based asymptotic linear arbitrage with respect to U (abbreviated by *U-ALA* w.r.t. U), if starting from zero initial capital V_0 corresponding to zero or negative initial utility level $U(V_0)$, the expected utility $\mathbb{E}U(V_t^\pi)$ increases (at least) linearly fast in long-term, i.e., $\mathbb{E}U(V_t^\pi) \geq b + ct$ for all large enough time $t \geq 1$, for some constants b and $c > 0$.

For the result of the section stated below, we consider only risk-seeking investors with the class of convex utility functions $U_\alpha : \mathbb{R} \to \mathbb{R}$, for a fixed real constant $\alpha > 0$, defined by $U_\alpha(x) := e^{\alpha x} - 1$, for all $x \in \mathbb{R}$. Similarly to the coefficient of constant risk-aversion (CARA) defined in [4, Chap. 5], α represents here the level of risk-seeking for those investors: the higher α is, the more an investor takes risk and may hence get higher satisfaction. This is typical to investors known as speculators in financial markets. Then we have the following

Theorem 3.2. *Let π_t be any trading strategy in the models* (1) *and* (2). *If π_t is an ALA (with GDP-F), then π_t also generates U-ALA w.r.t U_α.*

Proof. First, to the investor's initial capital $V_0 = 0$, it corresponds the initial utility $U_\alpha(V_0) = e^0 - 1 = 0$. Next, by definition of ALA, there are a constant $a > 0$ and a time $t_{1/2}$ such that we have $\mathbb{P}(V_t^\pi \geq at) \geq 1/2$ for all time $t \geq t_{1/2}$. It follows by monotonicity and convexity of U_α that

$$\begin{aligned} \mathbb{E}U_\alpha(V_t^\pi) &\geq -1 + \mathbb{E}U_\alpha(at)\mathbf{1}_{\{V_t^\pi \geq at\}} && (30) \\ &= -1 + U_\alpha(at)\mathbb{P}(V_t^\pi \geq at) \\ &\geq -1 + (1/2)(e^{\alpha a t} - 1) \\ &\geq \frac{1}{2}(-3 + \alpha a t), && (31) \end{aligned}$$

for all large enough time t, as required. □

To conclude this section, let us explain how in practice this easily proved result may connect long-term arbitrageurs' investment performances with market speculators' level of satisfaction. Indeed, if a speculator investor risks higher by investing an amount of money on the stock S_t and chooses a utility function U_α, moreover if, as guaranteed by the existence Theorem 2.4, s/he manages to construct an ALA with GDP-F strategy in the market models (1) and (2), then while his/her wealth grows linearly fast (with probability tending to 1), his/her expected level of satisfaction increases also (at least) linearly fast in long-term.

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