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MORE ON THE MAXIMUM AREA POLYGONS IN A PLANAR POINT SET

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Abstract: A finite set of points in the plane is described as in convex position if it forms the set of vertices of a convex polygon. Let P be a set of n points in convex position in the plane, this work studies the ratio between the maximum area of convex $(n-2)$-polygons with vertices in P and the area of the convex hull of P, and the ratio between the maximum area of convex $(n-1)$-polygons with vertices in P and the area of the convex hull of P respectively.

AMS Subject Classification: 52C10

Key Words: convex position, convex hull, affine transformation

1. Introduction

A finite set of points in the plane is described as in *convex position* if it forms the set of vertices of a convex polygon. Let P be a finite set of points in convex position in the plane, hence any subset of P is also a point set in convex position. Denote the area of the convex hull of $Q \subset P$ by $S(Q)$. For the sake of convenience we may call a subset $Q \subset P$ a polygon if Q forms the vertices of a polygon. Let

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$$f_k(P) =: \max\Big\{\frac{S(Q)}{S(P)} : Q \subset P, |Q| = k, P \text{ is in convex position}\Big\},$$

$$f_k^{conv}(n) =: \min\{f_k(P) : |P| = n,\ P \text{ is in convex position}\}.$$

In 1992, Fleischer et al. [1] showed that in the study of motion-planning problems in robotics by using heuristics, the largest area polygons in a planar point set play an important role. Hosono et al. [2] mainly studied $f_3^{conv}(n)$. Du and Ding studied $f_4^{conv}(n)$ and $f_5^{conv}(n)$ respectively in [3] and [4]. In this work we evaluate $f_{n-1}^{conv}(n)$ and $f_{n-2}^{conv}(n)$.

2. Main Results

Theorem 1. $f_{n-1}^{conv}(n) \geq \dfrac{1}{2 - f_{n-2}^{conv}(n-1)}$.

Proof. Let P be a convex n-gon with vertices A_1, A_2, $\cdots$, A_n in clockwise order. Suppose $(n-1)$-gon $Q = A_1A_2\cdots A_{n-1}$ is the one which has the maximum area of all the $(n-1)$-gons of P. By an affine transformation, assume that $A_1 = (0, 0)$, $A_2 = (0, 1)$, $A_{n-1} = (1, 0)$, $A_{n-2} = (a, b)$ $(a > 0,\ b > 0)$. See Figure 1.

Let f be the line through A_1 and A_{n-2}, and let f' be the parallel line through A_{n-1}. Similarly, let g be the line through A_2 and A_{n-1}, and let g' be the parallel line through A_1. For Q has the maximum area of all the $(n-1)$-gons of P, A_n must lie completely above f' and g'. Denote $B = f' \cap g'$, then $B = (\frac{b}{a+b}, \frac{-b}{a+b})$ and $A_n \in \triangle A_1BA_{n-1}$, hence P is always contained in the convex n-gon $P_1 = A_1A_2\cdots A_{n-1}B$.

Let $\mathbb{T} = \{\triangle A_iA_{i+1}A_{i+2} | i = 1, 2, \cdots, n-1\}$ (the addition in the subscript is in modulo $(n-1)$), that is, $\mathbb{T}$ is the set of all the triangles formed by three consecutive vertices of Q. Without loss of generality, let $\triangle A_2A_3A_4$ be the triangle of $\mathbb{T}$ which has the minimum area. Hence $(n-2)$-gon $Q_1 = A_1A_2A_4\cdots A_{n-1}$ is the one which has the maximum area of all the $(n-2)$-gons of Q.

Suppose $S(\triangle A_2A_3A_4) = \alpha$, then $S(Q_1) = S(Q) - \alpha$.

By the definition of $f_{n-2}^{conv}(n-1)$, $\frac{S(Q_1)}{S(Q)} \geq f_{n-2}^{conv}(n-1)$, that is, $\frac{S(Q)-\alpha}{S(Q)} \geq f_{n-2}^{conv}(n-1)$, thus

$$\frac{\alpha}{S(Q)} \leq 1 - f_{n-2}^{conv}(n-1). \tag{1}$$

Let $A_n = (x_0, y_0)$. Since Q has the maximum area of all the $(n-1)$-gons

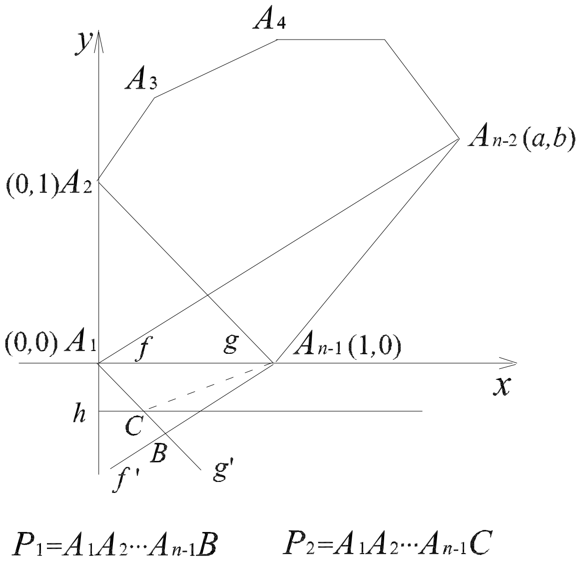


Figure 1

of P, then

$$S(\triangle A_2A_3A_4) \geq S(\triangle A_1A_{n-1}A_n) \Longrightarrow \alpha \geq \frac{-y_0}{2} \Longrightarrow y_0 \geq -2\alpha.$$

So A_n lies above the horizontal line $h : y = -2\alpha$.

Case 1. Suppose B lies above the line h, then $\frac{b}{a+b} \leq 2\alpha$. Notice that $P \subset P_1$ and so $S(P) \leq S(P_1)$, where $S(P_1) = S(Q) + S(\triangle A_1BA_{n-1}) = S(Q) + \frac{b}{2(a+b)}$.

Hence by (1),

$$\frac{S(P)}{S(Q)} \leq \frac{S(P_1)}{S(Q)} = 1 + \frac{\frac{b}{2(a+b)}}{S(Q)} \leq 1 + \frac{\alpha}{S(Q)} \leq 2 - f^{conv}_{n-2}(n-1),$$

$$\frac{S(Q)}{S(P)} \geq \frac{1}{2 - f_{n-2}^{conv}(n-1)}\,.$$

Case 2. Suppose B lies below the line h, then $\frac{b}{a+b} > 2\alpha$. So $S(P) \leq S(P_2)$, where $P_2 = A_1 A_2 \cdots A_{n-1} C$ is a n-gon with $C = g' \cap h$. Since $g' : y = -x$, $h : y = -2\alpha$, $C = (2\alpha, -2\alpha)$, then $S(P_2) = S(Q) + S(\triangle A_1 A_{n-1} C) = S(Q) + \alpha$. Hence by (1),

$$\frac{S(P)}{S(Q)} \leq \frac{S(P_2)}{S(Q)} = \frac{S(Q)+\alpha}{S(Q)} = 1 + \frac{\alpha}{S(Q)} \leq 2 - f_{n-2}^{conv}(n-1),$$

$$\frac{S(Q)}{S(P)} \geq \frac{1}{2 - f_{n-2}^{conv}(n-1)}\,.$$

From the above argument, we obtain that for any n-point set P in convex position we have $f_{n-1}(P) \geq \frac{1}{2 - f_{n-2}^{conv}(n-1)}$ and hence

$$f_{n-1}^{conv}(n) \geq \frac{1}{2 - f_{n-2}^{conv}(n-1)}. \qquad \square$$

Theorem 2. $f_{n-2}^{conv}(n) \geq \frac{1}{3 - 2 f_{n-3}^{conv}(n-2)}$.

Proof. Let P be a convex n-gon with vertices $A_1,\ A_2,\ \cdots\ A_n$ in clockwise order. Suppose $(n-2)$-gon Q is the one which has the maximum area of all the $(n-2)$-gons of P. Here the set of vertices of Q is a subset of $\{A_1,\ A_2,\ \cdots\ A_n\}$. We have only two types of $(n-2)$-gon Q:

Type I: Vertices of Q are non-consecutive in $\{A_1,\ A_2,\ \cdots\ A_n\}$;

Type II: Vertices of Q are consecutive in $\{A_1,\ A_2,\ \cdots\ A_n\}$.

See Figure 2 for two types of Q, where $n = 8$, and $Q's$ in Figure 2 (a), (b), (c) are of type I, and Q in Figure 2 (d) is of type II. For Q of type I it suffices to prove the theorem for Q as shown in Figure 2 (a).

Assume $Q = A_1 A_3 A_4 A_5 A_6 A_7$, $P_1 = A_1 A_2 A_3 A_4 A_5 A_6 A_7$, $P_2 = A_1 A_3 A_4 A_5 A_6 A_7 A_8$, then Q is also the one which has the maximum area of all the hexagons of P_1 and of P_2. By Theorem 1, we have

$$\frac{S(P)}{S(Q)} = \frac{S(P_1)+S(P_2)-S(Q)}{S(Q)} \leq 2\big(2 - f_5^{conv}(6)\big) - 1 = 3 - 2 f_5^{conv}(6).$$

Thus $\frac{S(Q)}{S(P)} \geq \frac{1}{3 - 2 f_5^{conv}(6)}\,.$

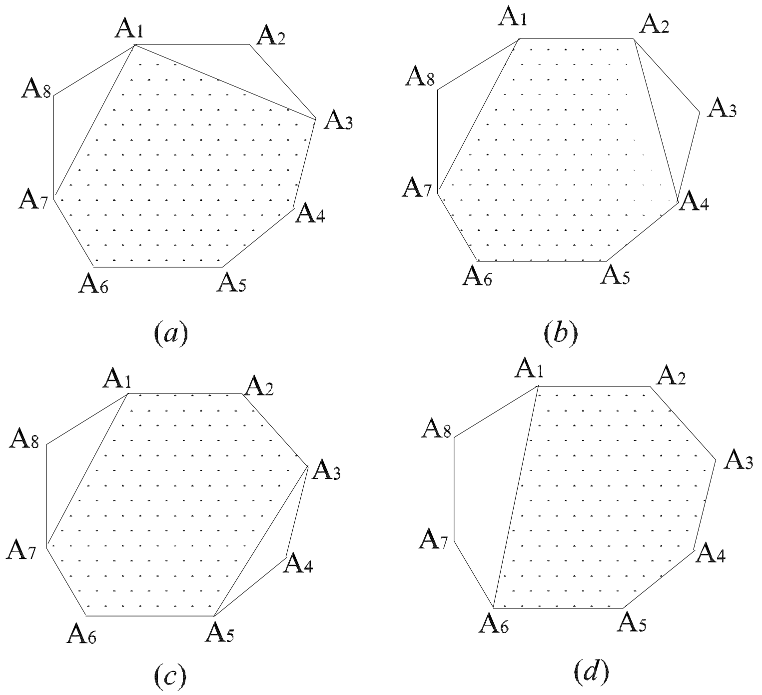


Figure 2

For other values of n the proof is similar, and we can get

$$\frac{S(P)}{S(Q)} = \frac{S(P_1') + S(P_2') - S(Q)}{S(Q)} \leq 2\big(2 - f_{n-3}^{conv}(n-2)\big) - 1 = 3 - 2f_{n-3}^{conv}(n-2).$$

Thus $\dfrac{S(Q)}{S(P)} \geq \dfrac{1}{3 - 2f_{n-3}^{conv}(n-2)}$.

Now we prove the theorem when Q is of type II, as shown in Figure 2 (d) for $n = 8$ and Figure 3 for all possible values of n. Q is formed by $(n-2)$ consecutive vertices of P. Without loss of generality, let $Q = A_1A_2 \cdots A_{n-2}$. Assume (by an affine transformation) that $A_1 = (0,0)$, $A_2 = (0,1)$, $A_{n-2} = (1,0)$, $A_{n-3} = (a,b)$ $(a > 0,\ b > 0)$. See Figure 3.

Let f be the line through A_1 and A_{n-3}, and let f' be the parallel line through A_{n-2}. Similarly, let g be the line through A_2 and A_{n-2}, and let g' be the parallel line through A_1. Denote $B = f' \cap g'$, then $B = (\frac{b}{a+b}, \frac{-b}{a+b})$. Similar

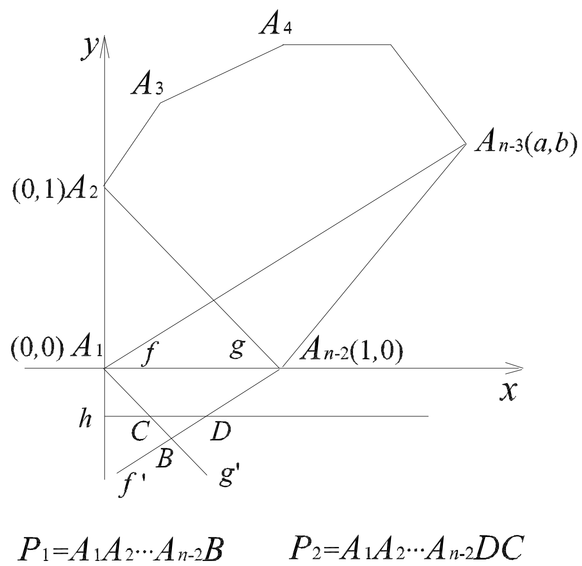


Figure 3

to the proof of Theorem 1, here A_{n-1}, $A_n \in \triangle A_1BA_{n-2}$ and P is always contained in the convex $(n-1)$-gon $P_1 = A_1A_2 \cdots A_{n-2}B$.

Let $\mathbb{T} = \{\triangle A_iA_{i+1}A_{i+2} | i = 1, 2, \cdots, n-2\}$ (the addition in the subscript is in modulo $(n-2)$). Without loss of generality, let $\triangle A_2A_3A_4$ be the triangle of $\mathbb{T}$ which has the minimum area. Hence $(n-3)$-gon $Q_1 = A_1A_2A_4 \cdots A_{n-2}$ is the one which has the maximum area of all the $(n-3)$-gons of Q.

Suppose $S(\triangle A_2A_3A_4) = \alpha$, then $S(Q_1) = S(Q) - \alpha$.

Similarly to the proof of Theorem 1, A_{n-1}, A_n lies above the horizontal line $h : y = -2\alpha$, and $\dfrac{S(Q_1)}{S(Q)} \geq f^{conv}_{n-3}(n-2)$, that is, $\dfrac{S(Q)-\alpha}{S(Q)} \geq f^{conv}_{n-3}(n-2)$, thus

$$\frac{\alpha}{S(Q)} \leq 1 - f^{conv}_{n-3}(n-2). \tag{2}$$

Case 1. Suppose B lies above the line h, then $\frac{b}{a+b} \leq 2\alpha$. By the same argument as in case 1 of Theorem 1, then $\frac{S(P)}{S(Q)} \leq \frac{S(P_1)}{S(Q)} \leq 1 + \frac{\alpha}{S(Q)} \leq 2 - f_{n-3}^{conv}(n-2) < 3 - 2f_{n-3}^{conv}(n-2)$, thus $\frac{S(Q)}{S(P)} > \frac{1}{3 - 2f_{n-3}^{conv}(n-2)}$.

Case 2. Suppose B lies below the line h, then $\frac{b}{a+b} > 2\alpha$. So P must be contained in the n-gon $P_2 = A_1A_2 \cdots A_{n-2}DC$, where $C = g' \cap h$, $D = f' \cap h$ and $C = (2\alpha, -2\alpha)$.

$$S(P_2) = S(Q) + S(A_1A_{n-2}DC) < S(Q) + 2S(\triangle A_1CA_{n-2}) = S(Q) + 2\alpha,$$

$$\big(\because S(\triangle CDA_{n-2}) < S(A_1DA_{n-2}) = S(A_1CA_{n-2})\big).$$

Hence by (2), we get $\frac{S(P)}{S(Q)} \leq \frac{S(P_2)}{S(Q)} < \frac{S(Q)+2\alpha}{S(Q)} = 1 + \frac{2\alpha}{S(Q)} \leq 3 - 2f_{n-3}^{conv}(n-2)$, thus $\frac{S(Q)}{S(P)} \geq \frac{1}{3 - 2f_{n-3}^{conv}(n-2)}$.

From the above argument, we obtain that for any n-point set P in convex position we have $f_{n-2}(P) \geq \frac{1}{3 - 2f_{n-3}^{conv}(n-2)}$ and hence $f_{n-2}^{conv}(n) \geq \frac{1}{3 - 2f_{n-3}^{conv}(n-2)}$. □

Lemma 1. *Let P_n be the set of vertices of a regular n-gon, and let $r_k(n) =: f_k(P_n)$, then*

$$r_k(n) = \frac{k \sin \frac{2\pi}{k}}{n \sin \frac{2\pi}{n}} \quad when\ n \equiv 0 \mod k;$$

$$r_k(n) = \frac{(k-1) \sin \frac{\lfloor \frac{n}{k} \rfloor 2\pi}{n} + \sin \frac{\lceil \frac{n}{k} \rceil 2\pi}{n}}{n \sin \frac{2\pi}{n}} \quad when\ n \equiv 1 \mod k;$$

$$r_k(n) = \frac{(k-2) \sin \frac{\lfloor \frac{n}{k} \rfloor 2\pi}{n} + 2 \sin \frac{\lceil \frac{n}{k} \rceil 2\pi}{n}}{n \sin \frac{2\pi}{n}} \quad when\ n \equiv 2 \mod k;$$

$$\cdots\cdots$$

$$r_k(n) = \frac{2 \sin \frac{\lfloor \frac{n}{k} \rfloor 2\pi}{n} + (k-2) \sin \frac{\lceil \frac{n}{k} \rceil 2\pi}{n}}{n \sin \frac{2\pi}{n}} \quad when\ n \equiv (k-2) \mod k;$$

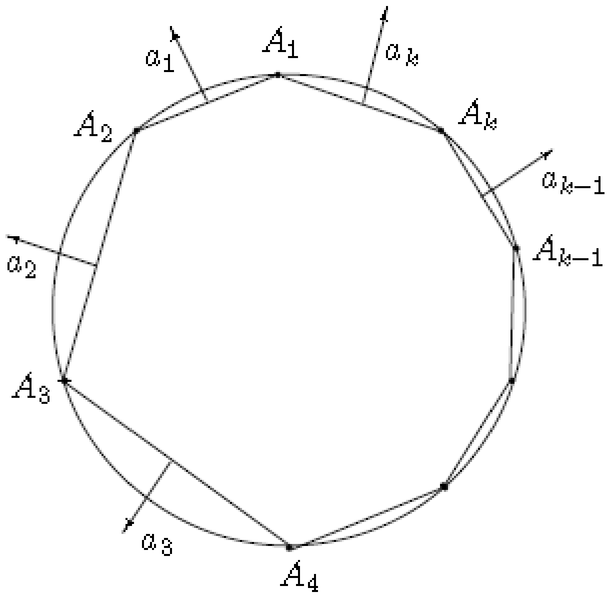


Figure 4

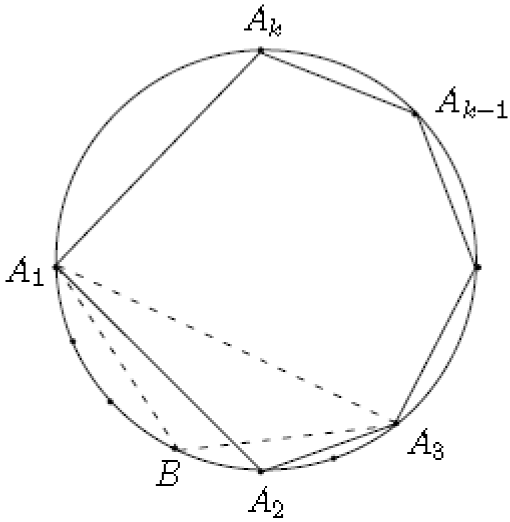
$$r_k(n) = \frac{\sin\frac{\lfloor\frac{n}{k}\rfloor 2\pi}{n} + (k-1)\sin\frac{\lceil\frac{n}{k}\rceil 2\pi}{n}}{n\sin\frac{2\pi}{n}} \quad \textit{when } n \equiv (k-1) \textit{ mod } k.$$

Proof. Suppose that the maximum area k-gon $P = A_1A_2\cdots A_k$ with vertices in P_n divides the boundary of the convex hull of P_n into k chains A_1A_2, A_2A_3, $\cdots$, $A_{k-1}A_k$ and A_kA_1, with a_1, a_2, $\cdots$, a_{k-1} and a_k edges, respectively, as shown in Figure 4.

For any two of edge numbers a_1, a_2, $\cdots$, a_k, say, a_i and a_j $(i < j)$, the number of edge numbers between them is $m = j - i - 1$ $(0 \le m \le k-2)$, we prove that a_i and a_j differ at most by 1 by induction on m.

For $m = 0$, that is, for any two adjacent edge numbers, the assertion is true. If not, say, for a_1 and a_2 we have $a_1 - a_2 \ge 2$. See Figure 5.

Let B be the nearest point of P_n to A_2 in clockwise order. Observe that since $a_1 - a_2 \ge 2$, the numbers of points of P_n on $\widehat{A_1A_2}$ is at least two more than that on $\widehat{A_2A_3}$. Then $S(\triangle A_1BA_3) > S(\triangle A_1A_2A_3)$, and the area of k-gon $A_1BA_3\cdots A_k$ is greater than the area of k-gon P, contradicting the choice of P.



$a_1 = 4$, $a_2 = 2$

Figure 5

Suppose the conclusion true for the case when the number of edge numbers between a_i and a_j is less than m, and consider the case when it equals m.

Suppose on the contrary, say, for a_1 and a_{m+2} we have $a_1 - a_{m+2} \geq 2$. Therefore, by the induction hypothesis, we only need to consider the case $a_1 = t,\ a_2 = a_3 = \cdots = a_{m+1} = t-1,\ a_{m+2} = t-2$.

Let B_2, B_3, $\cdots$, B_{m+2} be the nearest point of P_n to A_2, A_3, $\cdots$ and A_{m+2} in clockwise order, respectively. Then the area of k-gon $A_1B_2B_3\cdots B_{m+2}A_{m+3}\cdots A_k$ is greater than the area of k-gon P, contradicting the choice of P.

For example, let $m = 3, t = 5$ and let B_2, B_3, B_4, B_5 be the nearest point of P_n to A_2, A_3, A_4 and A_5 in clockwise order, respectively. See Figure 6. Recall that P_n is the set of vertices of a regular $n-$gon, $B_5A_5//A_4A_6$, so $S(\triangle A_4B_5A_6) = S(\triangle A_4A_5A_6)$, replace A_5 by B_5 in k-gon $A_1A_2\cdots A_k$ and denote the new k-gon by P_1, $S(P_1) = S(P)$. Similarly, $S(\triangle A_3B_4B_5) = S(\triangle A_3A_4B_5)$, so we can replace A_4 by B_4 in k-gon P_1 and obtain another new k-gon P_2 with $S(P_2) = S(P)$. Replace A_3 by B_3 in k-gon P_2 and we obtain the third new k-gon P_3 with $S(P_3) = S(P)$. At last, we replace A_2 by B_2 in k-gon P_3 and obtain the k-gon $P_4 = A_1B_2B_3B_4B_5A_6\cdots A_k$. Obviously $S(\triangle A_1B_3B_2) > S(\triangle A_1B_3A_2)$,

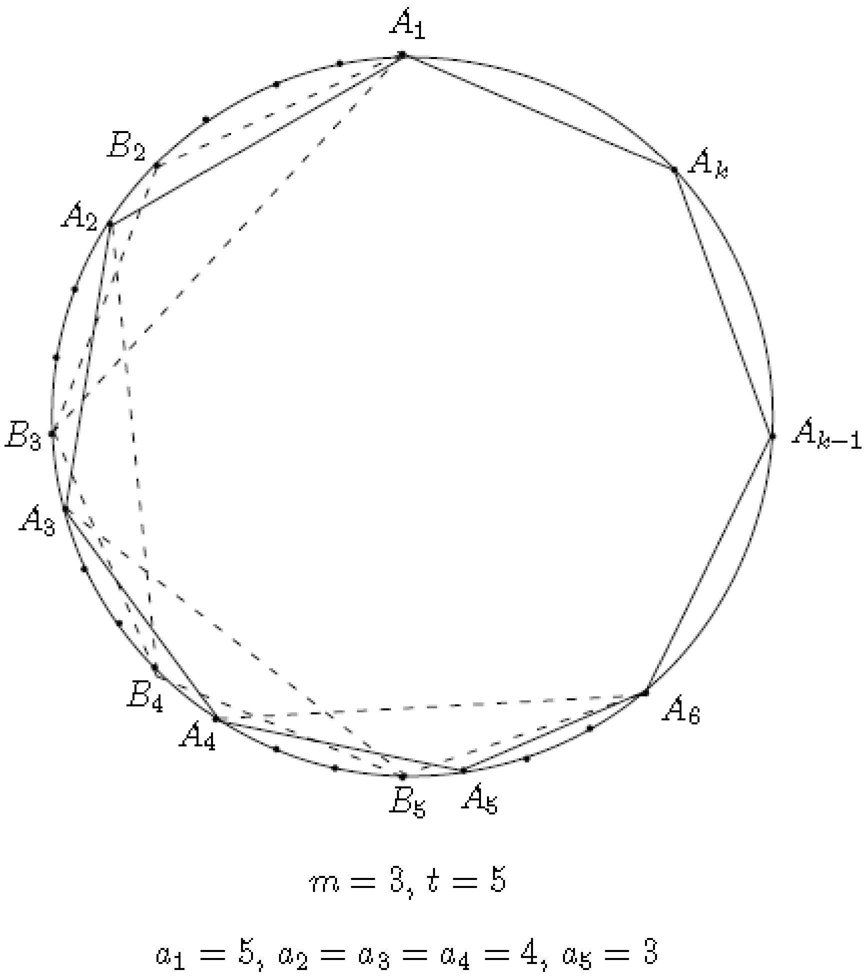


Figure 6

and hence $S(P_4) > S(P_3) = S(P)$, contradicting the choice of P.

Therefore, we conclude that the maximal area k-gon splits the boundary into k chains whose numbers of edges are $\{ t, t, \cdots, t, t\}, \{ t, t, \cdots, t, t{+}1 \}, \{t, t, \cdots, t+1, t+1\}, \cdots, \{t, t+1 \cdots, t+1, t+1\}$, when $n \equiv 0, 1, \cdots, k-1 \mod k$, respectively. An easy computation can lead to the claimed formulas. $\square$

Notice that each $r_k(n)$ is a decreasing function. Thus we can deduce that

$$\lim_{n\to\infty} r_k(n) = \frac{k}{2\pi}\sin\frac{2\pi}{k}.$$

Lemma 2. *([5]) Let B be a compact convex body in the plane and B_k be a largest area k-gon inscribed in B. Then $area(B_k) \geq area(B)\frac{k}{2\pi}\sin\frac{2\pi}{k}$, where equality holds if and only if B is an ellipse.*

From Theorem 1, Theorem 2, Lemma 1 and Lemma 2, the following results can be easily obtained:

Theorem 3. *For planar point sets in convex position of size $n \geq k \geq 3$ we have*

$$\frac{k}{2\pi}\sin\frac{2\pi}{k} \leq f_k^{conv}(n) \leq r_k(n).$$

Theorem 4. *For every $n \geq 5$, we have*

1. $$\frac{1}{2 - f_{n-2}^{conv}(n-1)} \leq f_{n-1}^{conv}(n) \leq 1 - \frac{2(1-\cos\frac{2\pi}{n})}{n};$$

2. $$\frac{1}{3 - 2f_{n-3}^{conv}(n-2)} \leq f_{n-2}^{conv}(n) \leq 1 - \frac{4(1-\cos\frac{2\pi}{n})}{n}.$$

where $r_{n-1}(n) = 1 - \dfrac{2(1-\cos\frac{2\pi}{n})}{n}$, $r_{n-2}(n) = 1 - \dfrac{4(1-\cos\frac{2\pi}{n})}{n}$.

3. Acknowledgments

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