International Journal of Applied Mathematics

Volume 26 No. 6 2013, 695-699
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v26i6.5

A NEW RANDOM APPROACH TO THE LEBESGUE INTEGRAL

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Abstract: In this paper a new method is given for generating the Lebesgue integral. For the corresponding random Riemann sums it is shown that they converge to the Lebesgue integral in probability.

AMS Subject Classification: 60Hxx, 60Bxx, 60Gxx
Key Words: Lebesgue integral, Riemann sums, random Riemann sums

1. Introduction

The concept of random Riemann sums is introduced in [2] and [3] in the following manner.

Denote the interval $[0,1)$ by I and let I be equipped by Borel σ-algebra. Let m be the Lebesgue measure on I. By a partition $\mathcal{P}_0$ of I we mean a finite sequence, $x_0, x_1, ..., x_n$ of elements of I such that $0 = x_0 < x_1 < ... < x_n = 1$. The norm of $\mathcal{P}_0$ with respect to the arbitrary measure μ on I is $\mid \mathcal{P}_0 \mid_\mu := \max\{\mu(I_k) \ : I_k = [x_{k-1}, x_k),\ 1 \le k \le n\}$.

For each $I_k \in \mathcal{P}_0$, let $t_k \in I_k,\ 1 \le k \le n$, be a random variable with uniform distribution in the interval (x_{k-1}, x_k), t_k's being independent.

Received: November 25, 2013
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Let $f : I \longrightarrow \mathbb{R}$ be a Lebesgue integrable function. The random Riemann sum of f on $\mathcal{P}_0$ is

$$\mathcal{S}_{\mathcal{P}_0}(f) = \sum f(t_k)m(I_k).$$

In [3] some results are proved for Lebesge measure m. As an example, Proposition 2.1. of [3], can be mentioned which is equivalent to the following

Theorem 1. *For any* $\epsilon > 0$, *and any sequence of partitions* $\mathcal{P}_n$, $n \geq 1$, *if* $\lim_{n\longrightarrow\infty} \mid \mathcal{P}_n \mid_m = 0$, *then*

$$P(\mid S_{\mathcal{P}_n}(f) - \int_I f dm \mid > \epsilon) \longrightarrow 0.$$

In this paper the sequence of partitions based on which the random Riemann sums are defined is randomized.

2. Notations

Throughout the paper we assume a sequence $\{\mathcal{P}_n\}_{n\geq 1}$ of partitions of I such that for each $n \geq 1$, $\mathcal{P}_n$, consists of a finite sequence of mutually independent and distinct r.v.s, each with uniform distribution in I. Moreover $\mathcal{P}_n$s, $n \geq 1$, form a mutually independent sequence of random vectors. We show by construction that a randomization mechanism which yields the desired random elements exists.

3. Random Sequence

Lemma 1. *There is a probability space* $(\Omega_0, \mathcal{B}_0, P_0)$ *on which a random vector* $(T_1, X_1, T_2, ..., X_n, T_{n+1})$ *can be defined such that* $X_1 < X_2 < ... < X_n$, *and* $X_1, X_2, ..., X_n$, *are the corresponding ordered statistics of a random sample from Uniform distribution in* I *([1]) and given* $(X_1, ..., X_n)$, T_i*'s are independent and* T_i *is uniformly distributed on* $[X_{i-1}, X_i), 1 \leq i \leq n$ *letting* $X_0 = 0$ *and* $X_{n+1} = 1$.

Proof. We have

$$f_{(X_1,...,X_n)}(x_1, ..., x_n) = \begin{cases} n!, & \text{if } 0 \leq x_1 < x_2 < ... \leq x_n < 1; \\ 0, & \text{if otherwise,} \end{cases}$$

and hence if we take

$$g = f_{(T_1,X_1,T_2,...,X_n,T_{n+1})}(t_1, x_1, t_2, ..., x_n, t_{n+1})$$

and $h = f_{(X_1,...,X_n)}(x_1, ..., x_n)$, then we have

$$g = hf_{T_1|X_1=x_1}(t_1)f_{T_2|X_2=x_2,X_3=x_3}(t_2)...f_{T_n|X_n=x_n}(t_n),$$

i.e.

$$g = \begin{cases} \frac{n!}{x_1(x_2-x_1)...(x_n-xn-1)(1-x_n)}, & \text{if } 0 \le t_1 < x_1 \le t_2 < ... < x_n \le t_{n+1} < 1; \\ 0, & \text{if otherwise.} \end{cases}$$

Now let $(\Omega_0, \mathcal{B}_0, P_0)$ be s.t. Ω_0 is the set of elements of I^{2n+1} with distinct coordinates and $\mathcal{B}_0$ the Borel σ-algebra in it. Define for $A \in \mathcal{B}_0$,

$$P_0(A) = \int_{A\cap\{0<T_1<X_1<T_2<...<X_n<T_{n+1}<1\}} gdt_1dx_1dt_2....dx_ndt_{n+1}. \qquad \square$$

Lemma 2. *Let $k_1, k_2, ...$ be an increasing sequence of natural numbers. There is a probability space $(\Omega, \mathbf{B}, P)$ for which each realization of an outcome yields a sequence $\{A_n\}_{n\ge1}$, where $A_n, n \ge 1$, is a strictly increasing sequence like $0 = x_0{}^{(n)}, t_1^{(n)}, x_1^{(n)}, t_2^{(n)}, ..., x_{k_n}^{(n)}, t_{k_n+1}^{(n)}, x_{k_n+1}^{(n)} = 1$ s.t. for each $n, x_1^{(n)}, x_2^{(n)}, ..., x_{k_n}^{(n)}$ are the corresponding ordered statistics of a random sample from uniform distribution in I and for each n, given $(x_1^{(n)}, x_2^{(n)}, ..., x_{k_n}^{(n)})$, $t_i^{(n)}$ s are independent each having uniform distribution in $[x_{i-1}^{(n)}, x_i^{(n)})$. Moreover A_ns, $n \ge 1$, form a mutually independent sequence of random vectors.*

Proof. According to the previous lemma for each $n \ge 1$, there is a probability space $(\Omega_n, \mathbf{B}_n, P_n)$ which yields the random vector

$$(t_1^{(n)}, x_1^{(n)}, t_2^{(n)}, ..., x_{k_n}^{(n)}, t_{k_n+1}^{(n)}).$$

Now let $(\Omega, \mathbf{B}, P)$ be s.t. $\Omega = \Omega_1\,\Omega_2\,...\Omega_n\,...$ and $\mathbf{B}$ the Borel σ-algebra in it and take $P = P_1 \otimes P_2 \otimes ... \otimes P_n \otimes ....$ $\square$

Lemma 3. *For the sequence of partitions $\{\mathcal{P}_n\}_{n\ge1}$, if $\mathcal{P}_n$ is constituted of points $0, X_1^{(n)}, X_2^{(n)}, ..., X_{k_n}^{(n)}, 1$, where $X_i^{(n)}$'s and and k_js being as described in the above lemma, then $\mid \mathcal{P}_n \mid$ tends to zero with probability 1.*

Proof. It is sufficient to prove that, a.s., $\bigcup_{n\geq 1}\mathcal{P}_n$ is dense in I. For fixed arbitrary sub-interval (a,b) of I, let G_n be the event of the sequence $\{\mathcal{P}_i\}_{i\geq 1}$, having at least one point in (a,b), in the n-th term for the first time. The assertion will be proved if we show that $P(\bigcup_{n\geq 1} G_n) = 1$. We have

$$P(\bigcup_{n\geq 1} G_n) = \sum_{n\geq 1} P(G_n) = \sum_{n\geq 1}(1-(b-a))^{k_1+k_2\ldots+k_{n-1}}(1-(1-(b-a))^{k_n}).$$

It is clear that the above series tends to one. So $\bigcup_{n\geq 1}\mathcal{P}_n$ is dense in I and the truth of result is obvious. □

For partition $\{\mathcal{P}_n\}_{n\geq 1}$, define

$$Y_k = f(t_k^{(n)})(x_k^{(n)} - x_{k-1}^{(n)})|(X_k^{(n)} = x_k, X_{k-1}^{(n)} = x_{k-1}),\ k\geq 1,$$

and $S_{\mathcal{P}_n}(f) = \sum Y_k$. We have

$$E(f(t_k^{(n)})(x_k^{(n)} - x_{k-1}^{(n)})|(X_k^{(n)} = x_k, X_{k-1}^{(n)} = x_{k-1})) = E(f(t_k^{(n)})(x_k^{(n)} - x_{k-1}^{(n)}))$$
$$= \int_{I_k} f dm,$$

and so

$$E(S_{\mathcal{P}_n}(f)) = \sum \int_{I_k} f dm = \int_I f dm.$$

The main theorem is the following, based on Lemmas 1,2,3 in this paper and Proposition 2.1. in [3] which is coming, in the sequel.

Theorem 2. *Suppose $f : I \longrightarrow \mathbb{R}$ is a Lebesgue integrable function. For the sequence of random partitions $\{\mathcal{P}_n\}_{n\geq 1}$, if $\mid \mathcal{P}_n \mid$ tends to zero with probability 1, when $n\longrightarrow\infty$, we have*

$$P(\mid S_{\mathcal{P}_n}(f) - \int_I f dm \mid > \epsilon) < \epsilon\ ,\ forall \epsilon > 0.$$

Remark 1. In the same manner, after the same conditions other results in [3] can be seen to hold naturally.

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