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APPROXIMATE ANALYTICAL AND NUMERICAL SOLUTIONS TO FRACTIONAL NEWELL-WHITEHEAD EQUATION BY FRACTIONAL COMPLEX TRANSFORM

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Abstract: The aim of this paper is by using the fractional complex transform and the optimal homotopy analysis by method (OHAM) to find the analytical approximate solutions for time-space nonlinear partial fractional Newell-Whitehead equations. Fractional complex transformation is proposed to convert time-space nonlinear partial fractional differential Newell-Whitehead equation to nonlinear partial differential equations. Also, we use the optimal homotopy analysis method (OHAM) to the obtained nonlinear PFDEs. This optimal approach has general meaning and can be used to get the fast convergent series solution of the different type of nonlinear partial fractional differential equations. The results reveal that this method is very effective and powerful to obtain the approximate solutions.

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1. Introduction

The transform method is an important method to solve mathematical problems. Many useful transforms for solving various problems appeared in the literature, such as the traveling wave transform, the Laplace transform, the Fourier transform, other classical integral transforms, and the local fractional integral transforms (see [3]). Recently, it was suggested to convert fractional order differential equations with local fractional derivative, and the resultant equations can be solved by some advanced calculus [5]. The fractional complex transform was first proposed in Refs. [6] and [7].

Our objective is to obtain analytical solutions of the following time and space fractional derivatives nonlinear differential equations by Fractional Complex Transform (FCT) with the help of OHAM, and to determine the effectiveness of FCT in solving these kinds of problems.

The time and space fractional derivatives Newell-Whitehead equation (see [14] and [11]) is:

$$D_t^{\alpha} u - D_x^{2\beta} u - u + u^3 = 0, \tag{1}$$

where α and β are the parameters standing for the order of the fractional time and space derivatives, respectively, and satisfying $0 < \alpha, \beta \leq 1$, and $x > 0$.

2. Preliminaries and Notations

In this section, we mention some basic definitions of fractional calculus theory which can be used further in this work. Local fractional derivative to $f(x)$ order α in interval $[a, b]$ is defined by [13] and [2]:

$$D^{\alpha} f(x_0) = \frac{d^{\alpha} f}{dx^{\alpha}} |_{x=x_0} = \lim_{x \longrightarrow x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^{\alpha}}, \tag{2}$$

where $\Delta^{\alpha}(f(x) - f(x_0)) = \Gamma(\alpha + 1)\Delta(f(x) - f(x_0))$.

Also, the inverse of local fractional derivative to of $f(x)$ order α in interval [a,b] is defined by [2] and [15] as follows:

$$_aI_b^{\alpha} f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t)(dt)^{\alpha} = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t \longrightarrow 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\alpha}, \tag{3}$$

where $\Delta t_j = t_{j-1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, ...\}$, $j = 0, 1, ..., N - 1, t_0 = a, t_N = b$ are the partition of the interval $[a, b]$.

3. The Optimal Homotopy Analysis Method (OHAM)

For more clarifications about the basic ideas of the OHAM for nonlinear partial differential equations, it is better to see the following nonlinear partial differential equation:

$$N[u(x,t)] = 0, \tag{4}$$

where N is a nonlinear operator for this problem, x and t denotes the independent variables, and $u(x,t)$ is an unknown function:

By using the HAM, we first construct zero-order deformation equation

$$(1-q)\mathcal{L}(\phi(x,t;q) - u_0(x,t)) = qhH(t)N[\ \phi(x,t,q)], \tag{5}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $u_0(x,t)$ is an initial guess, at $q = 0$ and $q = 1$, we have

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t). \tag{6}$$

By considering a Taylor series expression of $\phi(x,t,q)$ with respect to q in the form

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{7}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\,|_{q=0}, \tag{8}$$

the initial guess, the auxiliary parameter h and the auxiliary function $H(t)$ are selected such that the series (7) is convergent at $q = 1$, then we have from (7)

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{9}$$

We give the definition of the vector

$$\vec{u_n}(t) = \{u_0(x,t), u_1(x,t), u_2(x,t), ..., u_n(x,t)\}. \tag{10}$$

Differentiating (5) m times with respect to q, then setting $q = 0$ and dividing then by $m!$, we have the m^{th}-order deformation equation

$$\mathcal{L}(u_m(x,t) - \varkappa_m u_{m-1}(x,t)) = hH(t)\mathcal{R}_m(\vec{u_{m-1}}), \tag{11}$$

where

$$\mathcal{R}_m(\vec{u_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(x,t;q)]}{\partial q^{m-1}}\,|_{q=0}, \tag{12}$$

and

$$\varkappa_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{13}$$

Applying the integral operator on both sides of (11), we get

$$u_m(x,t) = \varkappa_m u_{m-1}(x,t) + h\int_0^t H(t)\mathcal{R}_m(\vec{u_{m-1}})dt, \tag{14}$$

the m^{th}- order deformation Eq. (11) is linear and thus can be easily solved, especially by means of symbolic computation software such as *Mathematica*.

S.J. Liao [8], Yabushita et al. [9] and Mohamed S. Mohamed et al. [10] and [4], suggested the so-called optimization method to find out the optimal convergence control parameters by minimum of the square residual error integrated in the whole region having physical meaning. Their method depends on the square residual error. Let $\triangle(h)$ denote the square residual error of the governing equation (4) and express as

$$\triangle(h) = \int_\Omega (N[\widetilde{u_n}(t)])^2 d\Omega, \tag{15}$$

where

$$\widetilde{u_m}(t) = u_0(t) + \sum_{k=1}^{m} u_k(t), \tag{16}$$

the optimal value of h is given by a nonlinear algebraic equation as:

$$\frac{d\triangle(h)}{dh} = 0. \tag{17}$$

4. The Fractional Complex Transform

The following nonlinear partial fractional differential equation is given

$$f(u, u_t^{(\alpha)}, u_x^{(\beta)}, u_y^{(\gamma)}, u_z^{(\lambda)}, u_t^{(2\alpha)}, u_x^{(2\beta)}, u_y^{(2\gamma)}, u_z^{(2\lambda)}, ...) = 0,\ 0 < \alpha,\ \beta, \gamma\ ,\ \lambda\ \leq 1. \tag{18}$$

where

$$u_t^{(\alpha)} = \frac{\partial^\alpha u(x,y,z,t)}{\partial t^\alpha}, \quad u_x^{(\beta)} = \frac{\partial^\beta u(x,y,z,t)}{\partial x^\beta},$$
$$u_y^{(\gamma)} = \frac{\partial^\gamma u(x,y,z,t)}{\partial y^\gamma}, \quad u_z^{(\lambda)} = \frac{\partial^\lambda u(x,y,z,t)}{\partial z^\lambda}$$

denote the local fractional derivatives with respect to t, x, y, z, respectively. The fractional complex transform requires that

$$T = \frac{wt^\alpha}{\Gamma(1+\alpha)}, X = \frac{px^\beta}{\Gamma(1+\beta)}, \quad Y = \frac{ky^\gamma}{\Gamma(1+\gamma)}, \quad Z = \frac{lz^\lambda}{\Gamma(1+\lambda)}, \tag{19}$$

where w, p, k, and l are unknown constants. Using the basic properties of the fractional derivative and the above transforms, we can convert the fractional derivatives into the following classical partial derivatives:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = w\frac{\partial u}{\partial T}, \ \frac{\partial^\beta u}{\partial x^\beta} = p\frac{\partial u}{\partial X}, \frac{\partial^\gamma u}{\partial y^\gamma} = k\frac{\partial u}{\partial Y}, \ \frac{\partial^\lambda u}{\partial t^\lambda} = l\frac{\partial u}{\partial Z}.$$

Therefore, we can easily convert the nonlinear partial fractional differential equations into nonlinear partial differential equations which can be solved by using the optimal homotopy analysis.

5. Applications

To mention the rate of force of this method, we use the complex transformations and the optimal homotopy analysis to find the approximate series solutions of the following the time-space fractional Newell-Whitehead equation of the form:

$$D_t^\alpha u - D_x^{2\beta} u - u + u^3 = 0, \tag{20}$$

where $0 < \alpha, \beta \leq 1$ is a parameter describing the order of the fractional time derivative. The exact solution to Eq. (20) at $\alpha = \beta = 1$ and subject to the initial condition

$$u(x,0) = \frac{\sinh(\frac{x}{\sqrt{2}})}{1+\cosh(\frac{x}{\sqrt{2}})}, \tag{21}$$

was derived in [14] and is given as:

$$u(x,t) = \frac{e^{\frac{x}{\sqrt{2}}} - e^{\frac{-x}{\sqrt{2}}}}{e^{\frac{x}{\sqrt{2}}} + e^{\frac{-x}{\sqrt{2}}} + 2e^{-\frac{3t}{2}}}. \tag{22}$$

To apply FCT to eq. (20), we use the above transformations, so we have the following partial differential equation:

$$w\frac{\partial u}{\partial T} - p^2\frac{\partial^2 u}{\partial x^2} - u + u^3 = 0. \tag{23}$$

Thus we get

$$\frac{\partial u}{\partial T} + \frac{1}{w}(-p^2\frac{\partial^2 u}{\partial X^2} - u + u^3) = 0. \tag{24}$$

Now, we solve eq. (24) using the OHAM, choosing the linear operator

$$\mathcal{L}[\phi(X,T;q)] = \frac{\partial \phi(X,T;q)}{\partial T}, \tag{25}$$

with property $\mathcal{L}[c] = 0$, where c is a constant. We define a nonlinear operator as

$$N[\phi(X,T;q)] = \frac{\partial \phi(X,T;q)}{\partial T} + \frac{1}{w}(-p^2\frac{\partial \phi^2(X,T;q)}{\partial X^2} - \phi(X,T;q) + \phi^3(X,T;q)). \tag{26}$$

We construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}(\phi(X,T;q) - u_0) = qhH(t)N[\phi(X,T;q)].$$

For $q = 0$ and $q = 1$, we can write

$$\begin{aligned}\phi(X,T;0) =& u_0 = u(X,0),\\ \phi(X,T;1) =& u(X,T).\end{aligned} \tag{27}$$

Thus, we obtain the m^{th}- order deformation equations

$$\mathcal{L}(u_m(X,T) - \varkappa_m u_{m-1}(X,T)) = hH(t)\mathcal{R}_m(\vec{u_{m-1}}), \tag{28}$$

where

$$\mathcal{R}_m(\vec{u_{m-1}}) = \frac{\partial u_{m-1}}{\partial T} + \frac{1}{w}(-p^2\frac{\partial^2 u_{m-1}}{\partial X^2} - u_{m-1} + \sum_{i=0}^{m-1}\sum_{j=0}^{i} u_i u_{i-j}\ u_{m-1-i}).$$

The auxiliary function can be determined uniquely when $H(t) = 1$. Now the solution of the m^{th}-order deformation equations (28) for $m \geq 1$ become

$$\begin{aligned}u_m(x,t) =& \varkappa_m u_{m-1}(x,t) + h\mathcal{L}^{-1}\mathcal{R}_m(\vec{u_{m-1}})\\ =& \varkappa_m u_{m-1}(x,t) + h\int_0^T \mathcal{R}_m(\vec{u_{m-1}})dT,\\ T =& \frac{t^\alpha}{\Gamma(1+\alpha)} \text{ and } X = \frac{x^\beta}{\Gamma(1+\beta)}.\end{aligned} \tag{29}$$

For simplicity we set $w = 1$, and $p = 1$ with the subject initial condition

$$u(x,0) = \frac{\sinh(\frac{x}{\sqrt{2}})}{1+\cosh(\frac{x}{\sqrt{2}})}, \tag{30}$$

$$\begin{aligned}
u\,(x,t) =& Tanh[\frac{x^{\beta}}{2\sqrt{2}\Gamma(1+\beta)}] \\
&- \frac{3h(2+h)(2+h(2+h))t^{\alpha}Sech[\frac{x^{\beta}}{2\sqrt{2}\Gamma(1+\beta)}]^2 Tanh[\frac{x^{\beta}}{2\sqrt{2}\Gamma(1+\beta)}]}{4\Gamma(1+\alpha)} \\
&- \frac{9h^2(6+h(8+3h)t^{2\alpha}Sech[\frac{x^{\beta}}{2\sqrt{2}\Gamma(1+\beta)}]^2 Tanh[\frac{x^{\beta}}{2\sqrt{2}\Gamma(1+\beta)}]^3)}{16\Gamma(1+\alpha)^2} \\
&- \frac{27h^4t^{\alpha}}{2048\Gamma(1+\alpha)4} \\
&(3 - 20Cosh[\frac{x^{\beta}}{\sqrt{2}\Gamma(1+\beta)}] + Cosh[\frac{\sqrt{2}x^{\beta}}{\sqrt{2}\Gamma(1+\beta)}]Sech[\frac{x^{\beta}}{2\sqrt{2}\Gamma(1+\beta)}]^6 \\
&Tanh[\frac{x^{\beta}}{2\sqrt{2}\Gamma(1+\beta)}]^3) + \frac{t^{3\alpha}}{\Gamma(1+\alpha)^3}(\frac{-9}{512}h^3(4+3h)Sech[\frac{x^{\beta}}{2\sqrt{2}\Gamma(1+\beta)}]^7 \\
&(-9\mathrm{Sin}h[\frac{3x^{\beta}}{2\sqrt{2}\Gamma(1+\beta)}] + \mathrm{Sin}h[\frac{5x^{\beta}}{2\sqrt{2}\Gamma(1+\beta)}])) \\
&- \frac{t^{3\alpha}}{\Gamma(1+\alpha)^3}(\frac{63}{256}h^3(4+3h) \\
&Sech[\frac{x^{\beta}}{2\sqrt{2}\Gamma(1+\beta)}]^6 Tanh[\frac{x^{\beta}}{2\sqrt{2}\Gamma(1+\beta)}]) + ...
\end{aligned} \tag{31}$$

When $h = -1$, $\alpha = 1$, and $\beta = 1$ we obtain the same solution as the solution obtained by [14] and [11]. According to the h-curves, it is easy to discover the valid region of h. We used 5-terms in evaluating the approximate solution $u\,(x,t) = \sum_{i=0}^{4} u_i(x,t)$. Note that the solution series contains the auxiliary parameter h which provides us with a simple way to adjust and control the convergence of the solution series. Therefore, it is straightforward to choose an appropriate range for h which ensure the convergence of the solution series. We stretch the h-curve of $u''(1,1))$ in Figure 5.1, which shows that the solution series is convergent when $-0.8 \leq h \leq 0.5$.

As mentioned in Section 3, the optimal value of h is determined by the minimum of $d\triangle 4$, corresponding to the nonlinear algebraic equation $\frac{d\triangle 4}{dh} =$

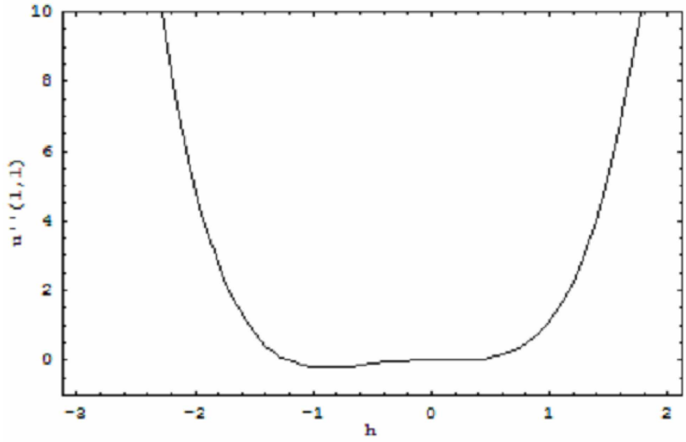

Figure 5.1: The h-curve of $u''(1,1)$ at the 5-th order of approximation, when $H(x,t) = 1$, $\alpha = \beta = 1$

0, our calculations showed that, $\frac{d\triangle 4}{dh}$ has its minimum value at $h\ |_{\text{optimal}}= -0.688674$.

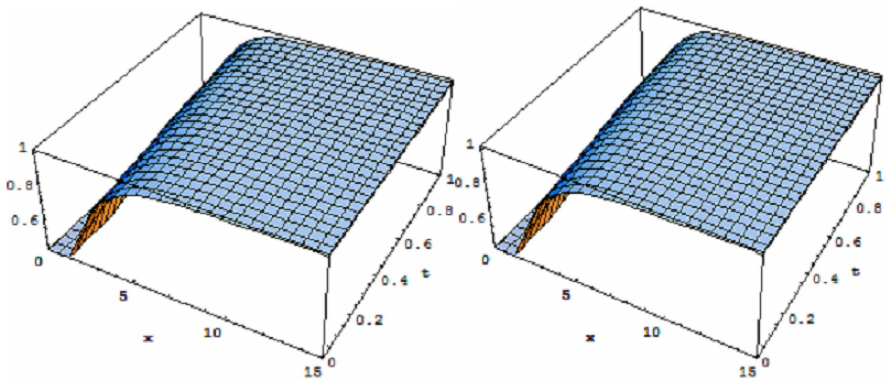

Figure 5.2: (a) Exact solution; (b) HAM solution with $h = -0.6887$, $\alpha = \beta = 1$

Figures 5.4–5.8: The exact solution (22) is compared with the approximate solution (31) at $h = -0.688674$, and $x = 1$ for different values of α and β.

Also the optimal values of h for different parameters are mentioned in Table 5.1.

In summary, from Figures 5.1–5.8, we deduce the behavior of the approx-

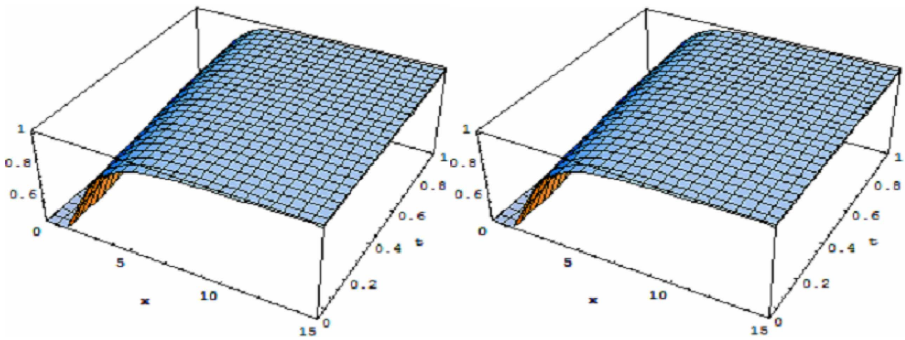

Figure 5.3: (a) HAM solution with $\beta = 1$, $h = -0.688674$, $\alpha = 0.05$; (b) HAM solution with $\alpha = 0.5$

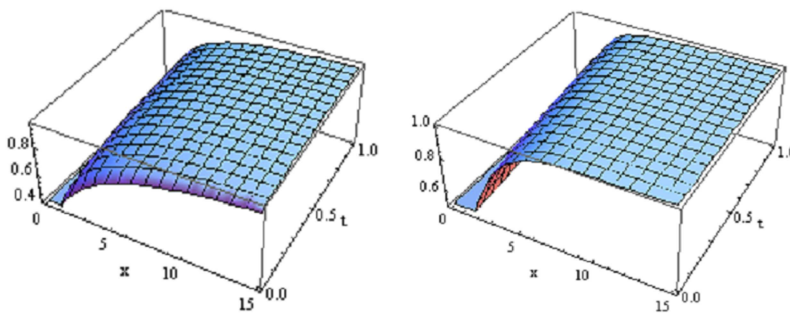

Figure 5.4: (a) HAM solution with $h = -0.688674$, $\alpha = \beta = 0.5$; (b) HAM solution with $\alpha = \beta = 0.095$

n	w	p	Optimal value of h	Minimum value
5	1	1.0	-0.6887	0.000030
4	1	0.5	-0.5000	0.000800
3	-1	-0.5	-0.5000	0.000007
2	-1	0.5	-0.5000	0.000300

Table 5.1: The approximate solutions of (31) when $\alpha = \beta = 1$ and $x = 0.2$ at the Optimal h.

imate solutions is the same behavior of the exact solution at some different values α and β. Consequently, we deduce that the approximate solution is rapidly convergent series as the exact solutions. The approximations given

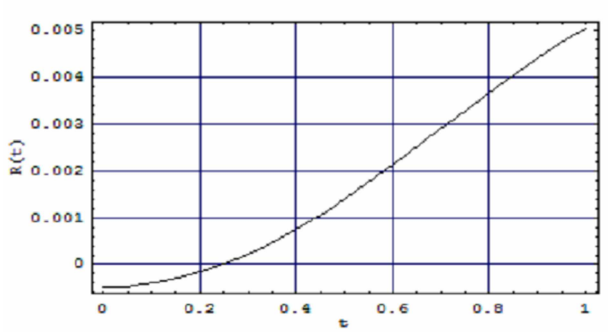

Figure 5.5: The residual of the 5-th order approximation for $h = -0.688674$

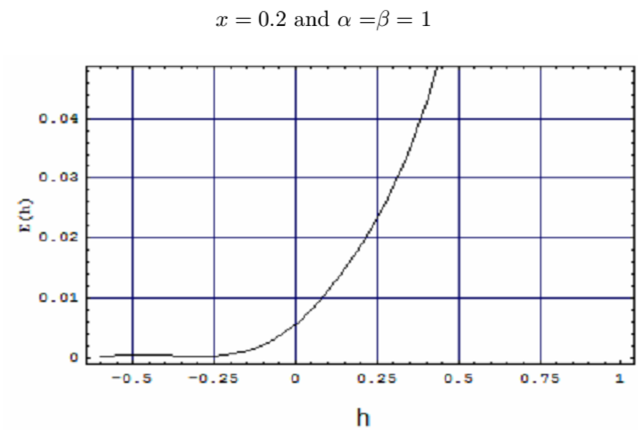

Figure 5.6: Square resifual error for the 5-th order approximation for $h = -0.688674$, $x = 0.2$, $t = 0.2$, and $\alpha = \beta = 1$

by an OHAM converge much faster than the normal HAM in general. The example considered in this paper suggests that the OHAMs with one or two convergence-control parameters are computationally most efficient and can give accurate enough approximations.

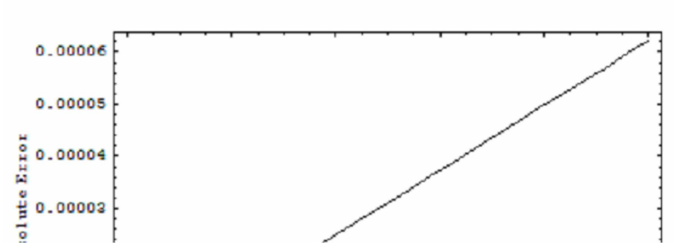

Figure 5.7

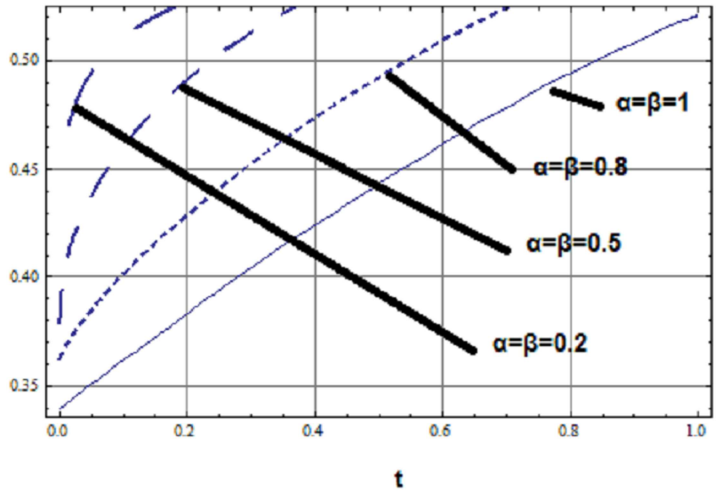

Figure 5.8: The exact solution (22) is compared with the approximate solution (31) at $h = -0.688674$ and $x = 1$ for different values of α and β. Also the optimal values of h for different parameters are mentioned in Table 5.1

6. Conclusions

In this paper, the fractional complex transform is very simple and the use of this method does not need the knowledge of fractional calculus. The optimal homotopy analysis method has been successfully applied for solving nonlinear fractional Newell-Whitehead equation. The results obtained by using the OHAM presented here agree well with the results obtained by [14] and [11].

The results show that OHAM is powerful mathematical tool for solving nonlinear fractional differential equations having wide applications in engineering. *Mathematica* has been used for computations in this paper.

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