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SOME CONSIDERATIONS ABOUT COHOMOLOGY OF FINITE GROUPS

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Abstract: In this work we present some considerations about cohomology of finite groups. In the first part we use the restriction map in cohomology to obtain some results about subgroups of finite index in a group. In the second part, we use Tate cohomology to present an application of the theory of groups with periodic cohomology in topology.

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1. Introduction

The theory of cohomology of groups provides a significant interaction between Algebra and Topology and it was very important in the creation of an important area of mathematics: the Homological Algebra. Moreover, that theory is closely related with the theory of ends of groups and group pairs, and those invariants have an interpretation in the graph theory when G is finitely generated, more specifically, a Cayley graph.

An important invariant for a group pair $(G, \mathcal{S})$ with $\mathcal{S}$ a family of subgroup of G is the number given by the dimension of the kernel of the restriction map $res^G_{\mathcal{S}} : H^1(G, M) \to H^1(\mathcal{S}, M)$, for specific $\mathbb{Z}_2G$-modules M, which has been

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studied by Andrade and Fanti in [1], [2], [4] and Andrade et al. in [3]. In Section 2 we introduce the restriction map and we present some results in group theory by using that map.

In Section 3 we work with the theory of cohomology of finite groups. This theory arises in various contexts in Topology and Algebra. One of the classic results in the area is the proof that any finite group which acts freely on a sphere must be periodic (equivalently, have all its abelian subgroups cyclics).

Homology and cohomology of finite groups have *similar* properties and Tate (see Brown [5]) discovered an ingenious way to exploit similarities between H_* an H^* for G a finite group. An illustration of the usefulness of Tate cohomology theory is the theory of groups with periodic cohomology. If we know that a group G has periodic cohomology, then the task of computing $H^*(G)$ is obviously enormously simplified. Here we present an application of that in Topology.

2. The Restriction Map in Cohomology of Groups

In this section we give the definition of the restriction map in cohomology and some applications in the theory of groups.

Let G be a group, S a subgroup of G and M a RG-module, with R a commutative ring with unit. We recall here the definition of (co)homology of G with coefficients in M. For details, see Brown [5].

Definition 1. Let G be a group. A RG-projective resolution of a RG-module M is an exact sequence of RG-modules:

$$\cdots \to F_n \stackrel{\partial_n}{\to} F_{n-1} \to \cdots \to F_1 \stackrel{\partial_1}{\to} F_0 \stackrel{\varepsilon}{\to} M \to 0$$

in which each F_i is projective. The map $F_0 \stackrel{\varepsilon}{\to} M$ is called augmentation map and we denote the projective resolution by $F \twoheadrightarrow M$.

Definition 2. Let G be a group, M a RG-module and $F \twoheadrightarrow R$ a projective resolution of R over RG, with R viewed as trivial RG-module. The *homology groups* of G with coefficients in M are, for all $n \in \mathbb{Z}$, defined by

$$H_n(G; M) = H_n(F \otimes_G M).$$

The *cohomology groups* of G with coefficients in M are, for all $n \in \mathbb{Z}$, defined by

$$H^n(G; M) = H^n(Hom_G(F, M)).$$

Definition 3. Let G be a group an M a RG-module. The map $res_S^G : H^1(G, M) \to H^1(S, M)$ induced by the canonical inclusion map $i : S \to G$ is called the restriction map in the level 1 of cohomology, or simply, restriction map.

In this work the ring R will be $\mathbb{Q}$, $\mathbb{Z}$ or $\mathbb{Z}_2$. Now we present some results about cohomology of finite group by using the restriction map.

Proposition 1. *Let G be a group, S a finite subgroup of G and $M = \mathbb{Q}(G/S)$ the free $\mathbb{Q}G$-module generated by the set G/S. Then*

$$res_S^G : H^1(G, \mathbb{Q}(G/S)) \to H^1(S, \mathbb{Q}(G/S))$$

is the null map. Namely, in the conditions of the proposition, if $res_S^G \neq 0$, then S (hence G) is infinite.

Proof. Let $\mid S \mid = m$ and consider the homomorphism

$$\begin{array}{rccc} \phi_m : & \mathbb{Q}(G/S) & \to & \mathbb{Q}(G/S), \\ & \alpha & \to & m.\alpha \end{array}$$

where $m.\alpha = \alpha + \cdots + \alpha($ m-times$)$.

Since $\mathbb{Q}(G/S)$ is free, if $m\alpha = 0$ then $\alpha = 0$. Thus ϕ_m is a monomorphism. Furthermore, ϕ_m is an epimorphism, since given $\beta \in \mathbb{Q}(G/S)$, there exists $\alpha = (1/m)\beta \in \mathbb{Q}(G/S)$ such that $\phi_m(\alpha) = \beta$. Hence ϕ_m is an isomorphism and so m is invertible in the $\mathbb{Q}(G/S)$. By Brown [5], Corollary III.10.2, we have $H^1(S, \mathbb{Q}(G/S)) = 0$.

Therefore, res_S^G is the null map. □

Remark 1. In Andrade and Fanti [1], Corollary 3.6, a proof of Proposition 1 is given when $\mid S \mid$ is odd and $R = \mathbb{Z}_2$. Hence, it is shown in Proposition 1 that when $R = \mathbb{Q}$, the result can be extended for all $\mid S \mid$.

Theorem 4. *Let G be a group and S a subgroup of G. If $[G : S] < \infty$, then $res_S^G : H^1(G, \mathbb{Z}_2(G/S)) \to H^1(S, \mathbb{Z}_2(G/S))$ is a monomorphism. Thus, under the conditions of the theorem, if res_S^G is not injective then $[G : S] = \infty$.*

Proof. Denote $Hom(\mathbb{Z}_2(G/S), \mathbb{Z}_2)$ by $\overline{\mathbb{Z}_2(G/S)}$. By Shapiro's Lemma (see Brown [5], III.6.2) we have an isomorphism

$$s : H^1(G, \overline{\mathbb{Z}_2(G/S)}) \to H^1(S, \mathbb{Z}_2).$$

Let $\alpha : S \to G$ be the inclusion map and let $\pi : \overline{\mathbb{Z}_2(G/S)} \to \mathbb{Z}_2$ be the canonical projection defined by $\pi(f) = f(1S) \in \mathbb{Z}_2$, for all $f \in \overline{\mathbb{Z}_2(G/S)}$. In the cohomology, we have

$$H^1(G, \overline{\mathbb{Z}_2(G/S)}) \overset{\alpha^*}{\to} H^1(S, \overline{\mathbb{Z}_2(G/S)}) \overset{\pi^*}{\to} H^1(S, \mathbb{Z}_2)$$

and, by Brown [5], p. 80, $\pi^* \circ \alpha^* = s$ (Shapiro's isomorphism). It follows that α^* is a monomorphism.

Now, since $[G : S] < \infty$, we have an isomorphism

$$\varphi : \mathbb{Z}_2(G/S) \to \overline{\mathbb{Z}_2(G/S)}$$

(see Brown [5], III.5.9) which provides the following commutative diagram:

$$\begin{array}{ccc} H^1(G, \mathbb{Z}_2(G/S)) & \overset{res_S^G}{\to} & H^1(S, \mathbb{Z}_2(G/S)) \\ \varphi_G^* \downarrow\simeq & \circlearrowleft & \simeq\downarrow \varphi_S^* \\ H^1(G, \overline{\mathbb{Z}_2(G/S)}) & \overset{\alpha^*}{\to} & H^1(S, \overline{\mathbb{Z}_2(G/S)}). \end{array}$$

Hence, for all $[f] \in \ker res_S^G$, $res_S^G([f]) = 0 \Rightarrow \varphi_S^*(res_S^G([f])) = 0 \Rightarrow (\alpha^* \circ \varphi_G^*)([f]) = 0 \Rightarrow [f] = 0$, since $(\alpha^* \circ \varphi_G^*)$ is a monomorphism.

Therefore $\ker res_S^G = 0$ and so, res_S^G is a monomorphism. □

Remark 2. The reciprocal of Theorem 4 is not true since there exist group pairs (G, S) for which res_S^G is a monomorphism and $[G : S] = \infty$ as shown in the following example.

Example 1. Let $G = < a > \oplus < b > \oplus < c > \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ be and let $S = < c > \simeq \mathbb{Z}$. The subgroup S is normal in G and, by MacLane [6], p.355, we have the exact sequence:

$$0 \to H^1(G/S; (\mathbb{Z}_2(G/S)^S) \overset{p}{\to} H^1(G; \mathbb{Z}_2(G/S) \overset{res_S^G}{\to} H^1(S; \mathbb{Z}_2(G/S)).$$

Since $S \lhd G$, the S-action in $\mathbb{Z}_2(G/S)$ is trivial and so, $\mathbb{Z}_2(G/S)^S = \mathbb{Z}_2(G/S)$. It follows that $\ker res_S^G = H^1(G/S; \mathbb{Z}_2(G/S))$. By using the invariant end defined in Scott and Wall [7], we have

$$e(G/S) = 1 + \dim H^1(G/S; \mathbb{Z}_2(G/S)) = 1, \qquad (*)$$

which provides $\ker res_S^G = H^1(G/S; \mathbb{Z}_2(G/S)) = 0$.

To see $(*)$, we can observe that, since G/S is finitely generated, $e(G/S)$ measures the maximum number of unlimited connected components of the Cayley graph $\Gamma_{G/S}$ when we remove compacts subsets K of $\Gamma_{G/S}$. In other words,

$$e(G/S) = sup\{n(K), K \text{ compact subset of } \Gamma_{G/S}\}$$

where $n(K)$ is the number of unlimited connected components of $\Gamma_{G/S} - K$.

Since $G/S = \mathbb{Z} \oplus \mathbb{Z}$, the Cayley graph is

$$\Gamma_{G/S} = \bigcup_{n \in \mathbb{Z}} (\{n\} \times \mathbb{R}) \cup (\mathbb{R} \times \{n\}) \subset \mathbb{R}^2$$

and if we remove a compact K of $\Gamma_{G/S}$, we have only one unlimited connected component.

3. Tate Cohomology and Periodic Cohomology of groups

In this section we give some properties about groups with periodic cohomology and an application in Topology. First, we introduce some notations and definitions.

Let M be a $\mathbb{Z}G$-module and consider the submodule of M

$$A = < gm - m \mid g \in G \text{ and } m \in m > .$$

Then

$$M_G = M/A \quad \text{and} \quad M^G = \{m \in M \mid gm = m, \forall g \in G\}.$$

If G is finite, i.e. $G = \{t_1, t_2, \ldots, t_n\}$, we define $N : M \to M$ by $N(m) = (\sum_{i=1}^{n} t_i)m$. It is easy to show that N induces the map

$$\overline{N} : M_G \to M^G$$

given by $\overline{N}(\overline{m}) = (\sum_{i=1}^{n} t_i)m$. $\overline{N}$ is called norm map.

Definition 5. Let G be a finite group and M a $\mathbb{Z}G$-module. The Tate cohomology of G with coefficients in M is defined by

$$\widehat{H}^i(G,M) = \begin{cases} H^i(G,M), & i > 0 \\ coker\overline{N}, & i = 0 \\ ker\overline{N}, & i = -1 \\ H_{-i-1}(G,M) & i < -1 \end{cases}$$

where $\overline{N} : M_G \longrightarrow M^G$ is the norm map.

Example 2. If G is a finite group with $|G| = n$, then

$$\widehat{H}^0(G,\mathbb{Z}) = \mathbb{Z}_n \text{ and } \widehat{H}^{-1}(G,\mathbb{Z}) = \{0\}.$$

In fact, since the G-action in $\mathbb{Z}$ is trivial, we have $\overline{N} : \mathbb{Z} \longrightarrow \mathbb{Z}$ with $\overline{N}(r) = (\sum_{i=0}^{n-1} t_i)r = nr$, for all $r \in \mathbb{Z}$. Therefore,

$$\widehat{H}^0(G,\mathbb{Z}) = coker\overline{N} = \mathbb{Z}/Im\overline{N} = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$$

and

$$\widehat{H}^{-1}(G,\mathbb{Z}) = \ker\overline{N} = \{0\}.$$

Remark 3. There is a cup product in Tate cohomology,

$$\begin{array}{rcl} \widehat{H}^p(G,M) \otimes \widehat{H}^q(G,N) & \longrightarrow & \widehat{H}^{p+q}(G, M \otimes N) \\ u \otimes v & \longmapsto & u \cup v \end{array}$$

with formal properties analogous to the properties of the cup product for ordinary cohomology $H^*(G,\mathbb{Z})$. In particular, the cup product has an identity element $1 \in \mathbb{Z}/|G|\mathbb{Z} = \widehat{H}^0(G,\mathbb{Z})$ and is associative. Thus $\widehat{H}^*(G,\mathbb{Z})$ is a *graded ring with idendity.* Furthermore, $\widehat{H}^*(G,M)$ is a module over $\widehat{H}^*(G,\mathbb{Z})$, for any M.

Definition 6. A finite group G is said to have periodic cohomology if for some $d \neq 0$, there is an element $u \in \widehat{H}^d(G,\mathbb{Z})$ which is invertible in the ring $\widehat{H}^*(G,\mathbb{Z})$.

Remark 4. In the conditions of the previous definition, cup product with u provides a periodic isomorphism

$$u \cup - : \widehat{H}^n(G, M) \xrightarrow{\approx} \widehat{H}^{n+d}(G, M)$$

for all $n \in \mathbb{Z}$ and all $\mathbb{Z}G$-module M. In particular, taking $n = 0$ e $M = \mathbb{Z}$, we see that $\widehat{H}^d(G, \mathbb{Z}) \approx \mathbb{Z}/|G|.\mathbb{Z}$ and that u generates $\widehat{H}^d(G, \mathbb{Z})$.

If we know that a group G has periodic cohomology, then the task of computing $\widehat{H}^*(G)$ is obviously enormously simplified. The following result gives us a criterion for deciding when G has periodic cohomology.

Theorem 7. *(Brown [5], VI.9.1) The following conditions are equivalent:*
(i) G has periodic cohomology.
(ii) There exist integers n and d, with $d \neq 0$, such that, for all $\mathbb{Z}G$-modules M, $\widehat{H}^n(G, M) \approx \widehat{H}^{n+d}(G, M)$.
(iii) For some $d \neq 0$, $\widehat{H}^d(G, \mathbb{Z}) \approx \mathbb{Z}/|G|.\mathbb{Z}$.
(iv) For some $d \neq 0$, $\widehat{H}^d(G, \mathbb{Z})$ contains an element u of order $|G|$.

Example 3. Let $G =< t >\simeq \mathbb{Z}_n$ be a finite cyclic group of order n. By, Brown [5], I.6, we have a projective resolution of period $d = 2$,

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

and, it follows from this resolution that

$$H_i(G; \mathbb{Z}) = \begin{cases} \mathbb{Z}, \text{ if } i = 0 \\ \mathbb{Z}_n, \text{ if } i \text{ is odd} \\ 0, \text{ if } i \text{ is even} \end{cases} \quad \text{and} \quad H^i(G; \mathbb{Z}) = \begin{cases} \mathbb{Z}, \text{ if } i = 0 \\ \mathbb{Z}_n, \text{ if } i \text{ is even} \\ 0, \text{ if } i \text{ is odd} \end{cases}.$$

By using this and Example 2, it follows that

$$\widehat{H}^i(G, \mathbb{Z}) = \begin{cases} \mathbb{Z}_n, & \text{for } i \text{ even} \\ \{0\} & \text{for } i \text{ odd} \end{cases}.$$

Therefore, by Theorem 7,(iii), G has periodic cohomology with period $d = 2$.

Example 4. If G is a finite group which acts freely on a CW-complex X homeomorphic to an odd dimensional sphere S^{2k-1}, then G has periodic cohomology.

Indeed, by Brown [5], I.6.2, G admits a periodic resolution of period $d = 2k$ and thus condition (ii) of the previous theorem is true. Hence, G has periodic cohomology.

An application of the cohomology of groups in topology is provided by the next theorem.

Theorem 8. *Let G be a group which has a subgroup with periodic cohomology. If Y is a $K(G,1)$-complex then Y has infinite dimension. In particular, Y cannot be a manifold.*

Proof. Let H be a subgroup of G with periodic cohomology. Thus, by Theorem 7, there exists an integer d such that $\widehat{H}^n(G,\mathbb{Z}) \approx \widehat{H}^{n+d}(G,\mathbb{Z}), \forall\, n \in \mathbb{Z}$. In particular, we can assume $d > 0$, and from the definition of Tate cohomology we have

$$H_n(H,\mathbb{Z}) \simeq H_{n+d}(H,\mathbb{Z}), \forall n > 0. \qquad (*)$$

Suppose that Y is a $K(G,1)$-complex of finite dimension m. Hence, the augmented cellular chain complex of the universal cover $\widetilde{Y}$ of Y:

$$C_*(\widetilde{Y}):\ 0 \to C_m(\widetilde{Y}) \to \cdots \to C_1(\widetilde{Y}) \xrightarrow{\partial_1} C_0(\widetilde{Y}) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Since H is a subgroup of G, then $C_*(\widetilde{Y})$ is also a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$. Thus, $H_k(H) = H_k(\mathbb{Z} \otimes_{\mathbb{Z}H} C_*(\widetilde{Y})) = 0$ if $k > m$, which contradicts $(*)$. Therefore, Y has infinite dimension and can not be a manifold. □

Corollary 9. *Let G be a group which has a torsion element and let Y be a $K(G,1)$-complex. Then Y is infinite dimensional. In particular, Y cannot be a manifold.*

Proof. If t is a torsion element of G of order n, then $H =< t >\simeq \mathbb{Z}_n$, and by Example 3, H has periodic cohomology. By Theorem 8, Y can not be a manifold. □

Example 5. Consider $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$ where p is a prime. It can be shown, by using Kunneth formula, that G does not have periodic cohomology. However, $H = \mathbb{Z}_p$ is a subgroup of G with periodic cohomology. Therefore, the complex $K(\mathbb{Z}_p \oplus \mathbb{Z}_p, 1)$ is infinite dimensional.

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