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CHARACTERIZATION OF THE DUAL SPACE OF L^1 AND LEBESGUE DECOMPOSITION FOR NON-σ-FINITE MEASURE SPACES

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Abstract: The purpose of this paper is to determine the dual space of the space $\mathcal{L}$ of all Daniell integrable functions and to prove the Lebesgue decomposition theorem in general measure spaces. In the measure theory, it is well known that the dual space $(L^1)^*$ can be identified with essentially bounded function space $L^\infty = L^\infty(\Omega, \Sigma, \mu)$ when μ is σ-finite, and that the non-σ-finite measure μ fails the Lebesgue decomposition. We show, in general, that the element of $(L^1)^*$ consists of a particular family of measurable functions. We call this family "folder", and the folder enables us to determine the dual space of L^1 and to formulate the general Lebesgue decomposition theorem.

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1. Introduction

In [25], the author proved the Radon-Nikodym Theorem in general measure theory without σ-finiteness or localizability. When the measure μ is not necessarily localizable, the Radon-Nikodym derivative fails to be a function, but forms a particular family of functions, which is called *folder*. Moreover, in [25] we have proved that the localizability is equivalent to that all folders are represented by "one" certain measurable function. In this paper, we apply the

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results in [25] with Daniell scheme to characterizing of the dual space of L^1 of the set of all integrable functions and to prove the Lebesgue decomposition theorem, in general (not necessarily localizable) measure spaces.

It is well known that the set of all bounded linear functionals on L^1 (dual space $(L^1)^*$) can be identified with L^∞ when the underlying measure space is σ-finite. Moreover, Segal [27] proved that this identification is true if and only if the measure μ is localizable. This argument can be seen in Rao [22] from the view point of measure theory and in Zaanen [33] from the view point of Daniell integral. In [3, 12, 15, 16, 18, 26], the authors studied the scalar-valued function spaces. It should be noted that Fedorova [12] considered by using Daniell-type integration. Kakutani [15] considered the dual space of L^∞ and characterized the condition for $(L^\infty)^* = L^1$ when we do not admit the choice of axiom. In [4, 14, 17], they studied the dual space of the set of Banach space-valued functions. The general Lebesgue decomposition theorem has been studied in various contexts. In [5, 7, 23, 31] the authors considered the decomposition of additive set functions defined on a certain group, or measures taking values in a certain group, but all measures are assumed bounded. On the other hand, we are interested in general σ-additive measures on an arbitrary set taking values in positive real number but unbounded.

It should be noted that there are various schemes called Daniell integral ([1, 6, 10, 11, 17, 19, 22, 28, 32, 33]) and these schemes are not equivalent each to other. The essential difference is in measurability. In particular, we adopt the scheme that the whole space is not necessarily measurable set. It will propose a new method to study measure spaces such that they are σ-rings and that they are not always σ-finite. A a consequence, we shall newly formulate the Radon-Nikodym derivative. So $(L^1)^*$ can be identified with the space of essentially bounded folders, see Theorem 5.4.

The paper is organized as follows. In Section 2, we describe the Daniell integral and its elementary properties. In Section 3, we introduce the new concept of the folder, and describe its essential properties. The proofs and details can be found in Saito [25]. In Section 4, we consider the elementary properties of the signed Daniell integral and the signed Radon-Nikodym Theorem. In Section 5, we determine the dual space of the Daniell integrable function space. In Section 6, we prove the Lebesgue Decomposition theorem for general Daniell integral by using folders. In Section 7, we apply our results to the localizable measure spaces and obtain another proof of the classical results of [27].

2. Summary of Daniell Scheme

A vector space $\mathcal{H}$ consisting of all $\mathbb{R}$-valued functions on a set $\Omega(\neq \emptyset)$ is said to be an *elementary function space* if $\mathcal{H}$ is closed under taking absolute value. The functions in $\mathcal{H}$ are called elementary. The set $\mathcal{H}$ is also called a *vector lattice* or a *Riesz space*, if it is a partially ordered vector space closed under taking pointwise maxima, minima of functions h, k, denoted by $h \vee k$, $h \wedge k$, respectively.

A $\mathbb{R}$-valued linear functional $\int$ on $\mathcal{H}$ satisfying:

(1) non-negativity: $\mathcal{H} \ni h \geq 0 \Rightarrow \int h \geq 0$,

(2) continuity: $h_n \searrow 0 \Rightarrow \int h_n \to 0$,

is said to be an *elementary integral* or a *Daniell integral* [11, 28, 30, 32]. The triplet $(\Omega, \mathcal{H}, \int)$ is called a *Daniell system*.

We denote by $\mathcal{H}^+$ the class of all $(-\infty, \infty]$-valued functions f which can be expressed as the pointwise limit of a sequence of the monotone increasing elementary functions, see [28, 30, 32]. Here, we understand that any function in $\mathcal{H}^+$ assumes its value in $\overline{\mathbb{R}}$. We define the integral of $f \in \mathcal{H}^+$ by $\int f = \lim \int h_n$, where $\{h_n\}_{n=1}^{\infty}$ is a sequence of the monotone increasing elementary functions. The integral on $\mathcal{H}^+$, for which we still write $\int$, is an $\overline{\mathbb{R}}$-valued functional.

Remark 2.1. Obviously, the couple $(\mathcal{H}^+, \int)$ extends the elementary integral $(\mathcal{H}, \int)$.

A function $f \in \mathcal{H}^+$ is said to be *integrable* if $\int f < \infty$ and we denote the set of all such f by $\mathcal{H}^+_{\text{int}}$. A subset $Z \subset \Omega$ is said to be a *null set*, if it is realized as a subset of $\{f = +\infty\}$ for some $f \in \mathcal{H}^+_{\text{int}}$ (see [28, 32]). A subset of a null set, and a countable union of null sets are still null sets. When a given property holds on Ω except on a null set, we say that the property holds *almost everywhere* on Ω, or "a.e." for short. For example, we can verify $f \in \mathcal{H}^+_{\text{int}}$ takes in $\mathbb{R}$ almost everywhere.

An $\overline{\mathbb{R}}$-valued φ, defined a.e. on Ω, is said to be *measurable* if it is an a.e. limit of a sequence of elementary functions, see [28]. The set of all measurable functions is denoted by $\mathcal{M}$. (Here, $f \in \mathcal{M}$ takes values in $\overline{\mathbb{R}}$, and $\mathcal{H}^+ \subset \mathcal{M}$.) A subset $D \subset \Omega$ is said to be *measurable*, or more precisely *Daniell measurable* if $I(D) \in \mathcal{M}$ and we denote the set of all such D by $\mathcal{D}$. The set of all measurable sets forms a σ-ring (in general it is not necessarily Ω is in $\mathcal{D}$). We note that this definition is essentially different from any other definition in [28, 32] and so on.

A function $\varphi \in \mathcal{M}$ is said to be in $\mathcal{L}^+$ if it can be represented as $\varphi = f - g$ a.e. for some $f \in \mathcal{H}^+$ and $g \in \mathcal{H}^+_{\text{int}}$, and we define $\int \varphi := \int f - \int g \in (-\infty, \infty]$.

Remark 2.2. Obviously, $\mathcal{H}^+ \subset \mathcal{L}^+ \subset \mathcal{M}$. The integral $\int$ on $\mathcal{L}^+$ is an extension of $\int$ on $\mathcal{H}^+$. The space $\mathcal{L}^+$ is not a vector space and the extended integral $\int$ on $\mathcal{L}^+$ is not linear. But as far as we ignore the difference on a null set, $\mathcal{L}^+$ is closed under addition, multiplication by non-negative constants, $\vee, \wedge$ and taking limits for increasing sequence of $\mathcal{L}^+$. The extended integral $\int$ is closed under addition, and it has non-negative homogeneity, and continuity of increasing sequence of $\mathcal{L}^+$.

We use frequently the fact that any non-negative measurable function $\varphi \in \mathcal{M}$ is in $\mathcal{L}^+$ [28, p.115]. Further, for any non-negative $\varphi \in \mathcal{L}^+$, there exist $f \in \mathcal{H}^+$ and $g \in \mathcal{H}^+_{\text{int}}$ such that $\varphi = f - g$ a.e. Since we can choose $g_n \in \mathcal{H}$ with $g_n \nearrow g$, it follows $\varphi_n := f - g_n \in \mathcal{H}^+$ and $\varphi_n \searrow \varphi$ a.e.

If the integral of $\varphi \in \mathcal{L}^+$ is finite, φ is said to be an *integrable* function [28, 32], and the set of all such functions is denoted by $\mathcal{L}$. We deduce $\mathcal{H} \subset \mathcal{H}^+_{\text{int}} \subset \mathcal{L} \subset \mathcal{L}^+$. As far as we ignore the difference on a null set, $\mathcal{L}$ has a linear structure and the integral $\int$ on $\mathcal{L}$ is a real-valued linear functional. Further, the Monotone Convergence Theorem and the Dominated Convergence Theorem remain valid for $\mathcal{L}$. We will use the fact that any $\varphi \in \mathcal{L}$ is finite almost everywhere.

In addition, throughout of this paper we always suppose that

$$h \in \mathcal{H} \Rightarrow h \wedge 1 \in \mathcal{H},$$

the so-called *the Stone condition*, see [32]. This condition guarantees the measurability of the product of measurable functions.

Remark 2.3. The above procedure is called a *Daniell scheme*. Several types of the Daniell scheme are described in [6, 19, 28, 30, 32], with different contents and constructions, and are not equivalent one to another. The scheme we adopted is almost the same as adopted in [28], however, the Stone condition in [28] includes the assumption of σ-finiteness of the whole space Ω.

Next, for any measurable function φ,

$$\operatorname*{ess.sup}_{x \in \Omega} |\varphi(x)| \quad \text{or} \quad \|\varphi\|_\infty$$

denotes the greatest lower bound of all numbers C such that $|\varphi| \leq C$ almost everywhere. A function $\varphi \in \mathcal{M}$ is essentially bounded if $\|\varphi\|_\infty < \infty$ and all such functions is denoted by $\mathcal{L}^\infty$. And for $\varphi \in \mathcal{L}$ we write, as usual,

$$\|\varphi\|_1 := \int |\varphi|.$$

The next proposition is often referred to as the semi-finiteness of $\int$; see [13, 22, 33].

Proposition 2.4. *For any* $0 \leq \varphi \in \mathcal{L}^+$ *satisfying* $\int \varphi > 0$, *there exists* $\psi \in \mathcal{L}$ *such that* $0 \leq \psi \leq \varphi$ *and* $\int \psi > 0$.

Here for the sake of convenience for readers we recall the proof.

Proof. By the definition of $\varphi \in \mathcal{L}^+$, there exist $f \in \mathcal{H}^+, g \in \mathcal{H}^+_{\text{int}}$ such that $\varphi = f - g$ a.e. Since $f \in \mathcal{H}^+$, we may find a sequence $h_n \in \mathcal{H}$, $n \in \mathbb{N}$ such that $h_n \nearrow f$. Now, defining $\psi_n := h_n - g$, we learn this is integrable and hence so is positive part ψ_n^+. Since φ is assumed non-negative, $0 \leq \psi_n^+ \nearrow \varphi$ almost everywhere. The Monotone Convergence Theorem gives $\int \psi_n^+ \nearrow \int \varphi > 0$ and we can find a sufficient large integer n_0 such that $\int \psi_{n_0}^+ > 0$. This $\psi_{n_0}^+$ is the desired function. □

3. Folder and Its Properties

In this section, we introduce the notion of folders and summarize its elementary properties. The proofs can be found in [25].

Definition 3.1. A subset $E \subset \Omega$ is said to be an *elementary measurable set* if $I(E) \in \mathcal{H}^+$ and the set of all elementary measurable sets is denoted by $\mathcal{E}$.

Remark 3.2. Since $\mathcal{H}^+$ is closed under countably many $\vee$ and finitely many $\wedge$, we deduce that $\mathcal{E}$ is closed under countable union and finite intersection. Further, all elementary measurable sets are measurable with respect to all elementary integrals on $\mathcal{H}$, that is, this definition of elementary measurability doesn't depend on the integral $\int$.

The following proposition follows from Proposition 2.2 in [25].

Proposition 3.3. *For any $\varphi \in \mathcal{M}$, there exists $E_0 \in \mathcal{E}$ such that $\{\varphi \neq 0\} \subset E_0$.*

Keeping in mind the definition of $\mathcal{E}$, we now recall the notion of folders, which play an important role in this paper.

Definition 3.4. (1) Let $(f_E)_{E\in\mathcal{E}}$ be a family of measurable functions. We call it a *folder*, if

$$f_F I(E) = f_{E\cap F} \text{ a.e.} \tag{3.1}$$

for any $E, F \in \mathcal{E}$, and write $\langle f\rangle := (f_E)_{E\in\mathcal{E}}$. Each f_E is called a *file*.

(2) Let $\langle f\rangle, \langle g\rangle$ be folders. Then, we say that $\langle f\rangle =$ (or $\leq$)$\langle g\rangle$ a.e. if $f_E =$ (or $\leq$)g_E a.e. for all $E \in \mathcal{E}$.

(3) We say $\langle f\rangle$ is a *complete* folder if there exists $E_0 \in \mathcal{E}$ such that $f_F = f_{E_0\cap F}$ a.e. holds for any $F \in \mathcal{E}$. The file f_{E_0} is called a complete file of the folder $\langle f\rangle$.

Remark 3.5. (1) The mapping from $E \in \mathcal{E}$ to the indicator function $I(E) \in \mathcal{M}$ is clearly a folder. We denote this folder by $\langle I\rangle$ and call it the *indicator folder*.

(2) We say that $\mathcal{H}$ is σ-finite if $1 \in \mathcal{H}^+$ (cf. [32]). This condition is equivalent to $\Omega \in \mathcal{E}$, so that if $\mathcal{H}$ is σ-finite then all folders are complete because we can choose the complete file as h_Ω whenever we are given a folder $\langle h\rangle$.

(3) Let φ be a measurable function, and let $\langle h\rangle$ be a folder. Then $\mathcal{E} \ni E \mapsto \varphi h_E \in \mathcal{M}$ is also a folder. We denote this folder by $\varphi\langle h\rangle$. In particular, if we put $\varphi = I(F)$ $(F \in \mathcal{E})$, then we have $I(F)\langle h\rangle = h_F\langle I\rangle$ a.e.

(4) We combine the observation above with Definition 3.4. We suppose that $\langle f\rangle$ is a complete folder and that $E_0 \in \mathcal{E}$ satisfies $f_F = f_{E_0\cap F}$ a.e. for any $F \in \mathcal{E}$. By the definition of completeness, $f_{E_0\cap F} = f_{E_0} I(F)$ a.e. holds, and this implies the complete folder satisfies $\langle f\rangle = f_{E_0}\langle I\rangle = I(E_0)\langle f\rangle$ a.e.

Proposition 3.6. *For any $\varphi \in \mathcal{M}$, there exists $E_0 \in \mathcal{E}$ such that*

$$\varphi\langle h\rangle = \varphi h_{E_0}\langle I\rangle \text{ a.e.},$$

that is, $\varphi\langle h\rangle$ is a complete folder. Moreover, if there is another $\widetilde{E_0} \in \mathcal{E}$ satisfying the above condition, then $\varphi h_{E_0} = \varphi h_{\widetilde{E_0}}$ a.e. holds.

Proof. By Proposition 3.3, we can find $E_0 \in \mathcal{E}$, so that $\{\varphi \neq 0\} \subset E_0$, and hence we have $\varphi h_E = \varphi I(E_0) h_E = \varphi h_{E_0 \cap E}$ a.e. for any $E \in \mathcal{E}$. The second assertion is obvious. □

Now we describe how to define the linear functional when we are given a folder $\langle f \rangle$.

Definition 3.7. We say a measurable folder $\langle h \rangle$ is a *density folder*, if for every $f \in \mathcal{H}$, $f\langle h \rangle$ is integrable.

Remark 3.8. The following assertions are significant but not obvious. The proofs are in [25].

(1) Given a density folder $\langle h \rangle$ and $f \in \mathcal{H}$, the folder $f\langle h \rangle$ is complete by Proposition 3.6, where its complete file is fh_{E_0}; there exists $E_0 \in \mathcal{E}$ such that $\{f \neq 0\} \subset E_0$. Note that E_0 depends on f. Now we define the integral of the folder $f\langle h \rangle$ against $\int$ by $\int(f\langle h \rangle) := \int(fh_{E_0})$. We can show that it does not depend the choice of $E_0 \in \mathcal{E}$ containing the carrer of f.

(2) We can show that the files h_E of density folder $\langle h \rangle$ is finite a.e. on Ω.

(3) If the density $\langle h \rangle$ is non-negative, then $P : \mathcal{H} \to \mathbb{R}$ is a Daniell integral on $\mathcal{H}$. Moreover, if a set $Z \subset \Omega$ is null, then Z is P-null, i.e., $P \ll \int$. As usual, we say that P *is absolutely continuous with respect to* $\int$, if the above condition holds.

Theorem 3.9 (Theorem 3.1 (2), [25]). *Let $\langle h \rangle, \langle h' \rangle$ be two non-negative density folders. If*

$$\int f\langle h \rangle = \int f\langle h' \rangle, \quad \text{for all } f \in \mathcal{H},$$

then $\langle h \rangle = \langle h' \rangle$ a.e.

We can yield the new formulation of the Radon-Nikodym Theorem with Daniell integral:

Theorem 3.10 ([25]). *Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system satisfying the Stone condition, and Q be any integral on $\mathcal{H}$ such that $Q \ll \int$. Then there exists a non-negative density folder $\langle h \rangle$, such that for any $f \in \mathcal{L}(\int + Q)$,*

$$Q(f) = \int f\langle h \rangle. \tag{3.2}$$

This $\langle h \rangle$ is determined a.e.-uniquely.

Let $\langle h \rangle$, $\langle k \rangle$ be two folders. Then the mappings $\mathcal{E} \ni E \mapsto h_E \pm k_E$ and $E \mapsto h_E k_E$ satisfy the axiom of folder. Therefore, we denote these folders as:

$$\begin{aligned} \langle h \pm k \rangle \quad &\text{or} \quad \langle h \rangle \pm \langle k \rangle, \\ \langle hk \rangle \quad &\text{or} \quad \langle h \rangle \langle k \rangle. \end{aligned}$$

The following properties are obvious:

Proposition 3.11. (1) *For any $f, g \in \mathcal{H}$, $(f+g)\langle h \rangle = f\langle h \rangle + g\langle h \rangle$ a.e.*
(2) *For any $f \in \mathcal{H}$, $f\langle h+k \rangle = f\langle h \rangle + f\langle k \rangle$ a.e.*

4. Signed Integral

In this section, we describe the property of the signed Daniell integral, which is a functional having linearity and continuity. The proofs can be found in [28].

Let $\Phi : \mathcal{H} \to \mathbb{R}$ be a linear mapping. For positive elementary functions, we define the total variation $|\Phi|$, positive variation Φ^+, negative variation Φ^- as follows:

$$\begin{aligned} |\Phi|(h) &:= \sup\{\Phi(k) \ ; \ |k| \le h, \ k \in \mathcal{H}\} \\ \Phi^+(h) &:= \sup\{\Phi(k) \ ; \ 0 \le k \le h, \ k \in \mathcal{H}\} \\ \Phi^-(h) &:= \sup\{-\Phi(k) \ ; \ 0 \le k \le h, \ k \in \mathcal{H}\}. \end{aligned}$$

We say Φ has finite variation if $|\Phi|(h)$ is finite for any positive elementary functions h.

Theorem 4.1. *If Φ has finite variation then $|\Phi|, \Phi^+$ and Φ^- can be extended uniquely to the non-negative linear mapping on $\mathcal{H}$ and*

$$\Phi = \Phi^+ - \Phi^- \tag{4.1}$$

holds. This decomposition is essentially minimum, in the sense that if there exists any other decomposition $\Phi = \Psi_1 - \Psi_2$, then $\Phi^+ \le \Psi_1, \Phi^- \le \Psi_2$ hold for any non-negative elementary functions. We call this decomposition (Φ^+, Φ^-) Jordan Decomposition.

Proof. Suppose that Φ has finite variation. We decompose $\mathcal{H} \ni h = h^+ - h^-$, and define $|\Phi|h := |\Phi|h^+ - |\Phi|h^-$. Then the linearity and non-negativity are obviously valid for all $h \in \mathcal{H}$. Moreover,

$$\Phi^+ := \frac{|\Phi| + \Phi}{2}, \quad \text{and} \quad \Phi^- := \frac{|\Phi| - \Phi}{2} \tag{4.2}$$

has the desired properties.

We prove the minimality of the decomposition. Let $\Phi = \Psi_1 - \Psi_2$ be a general decomposition of non-negative linear functionals. For any $h \in \mathcal{H}$ and any $k \in \mathcal{H}$ with $|k| \leq h$, we have

$$-\Psi_2 k^+ \leq \Phi k^+ \leq \Psi_1 k^+, \quad -\Psi_2 k^- \leq \Phi k^- \leq \Psi_1 k^-,$$

and hence

$$\Phi k \leq \Psi_1 k^+ + \Psi_2 k^- \leq (\Psi_1 + \Psi_2)h, \quad \text{for } |k| \leq h.$$

Taking supremum over all such k, we obtain $|\Phi|h \leq (\Psi_1 + \Psi_2)h$ for $0 \leq h \in \mathcal{H}$. Combining the definition $\Phi h = (\Psi_1 - \Psi_2)h$ and (4.2), we have $\Phi^+ \leq h \leq \Psi_1 h$ and $\Phi^- h \leq \Psi_2 h$. □

Definition 4.2. We say a linear map $\Phi : \mathcal{H} \to \mathbb{R}$ is *a signed Daniell integral* if $\Phi(h_n) \to 0$ for any sequences of elementary functions with $h_n \searrow 0$.

Theorem 4.3. *If Φ is a signed Daniell integral on $\mathcal{H}$, then Φ has finite variation and the Jordan decomposition (Φ^+, Φ^-) is non-negative and continuous, i.e., Φ^+, Φ^- are Daniell integrals.*

The proof can be found in [28], Theorem 2.11.6.

Definition 4.4. Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system with the Stone condition and Q be a signed Daniell integral. We say Q is absolutely continuous with respect to $\int$ if any $\int$-null set is $|Q|$-null set. This is written by $Q \ll \int$ analogously to the non-negative integral.

We extend the domain of signed integral Q. If $f \in \mathcal{H}^+_{\text{int}}(|Q|)$, then $Q^+(f)$, $Q^-(f)$ is finite and hence we can define

$$Q(f) := Q^+(f) - Q^-(f).$$

For $f \in \mathcal{L}(|Q|)$, since $f = f_1 - f_2$ with $f_1, f_2 \in \mathcal{H}^+_{\text{int}}(|Q|)$, we define

$$Q(f) := Q(f_1) - Q(f_2).$$

The Radon-Nikodym Theorem still holds for the signed Daniell integrals.

Theorem 4.5. *Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system with the Stone condition, and Q be a signed Daniell integral on $\mathcal{H}$ such that $Q \ll \int$. Then there exists a density folder $\langle h \rangle$, such that for any $f \in \mathcal{L}(\int + |Q|)$,*

$$Q(f) = \int f \langle h \rangle. \tag{4.3}$$

This $\langle h \rangle$ is determined a.e.-uniquely.

Proof. Since Q is a signed Daniell integral on $\mathcal{H}$, Q has finite variation and $Q^{\pm}$ is a Daniell integral by Theorem 4.3. Moreover, by Theorem 4.1 $Q = Q^+ - Q^-$ holds on $\mathcal{H}$.

We will show $Q^{\pm} \ll \int$. If Z is an $\int$-null set, then it is a $|Q|$-null set by the assumption. There exists $f \in \mathcal{H}^+_{\text{int}}(|Q|)$ such that $Z \subset \{f = +\infty\}$. Since $f \in \mathcal{H}^+$, $|Q|(f) = Q^+(f) + Q^-(f)$ holds. But the left-hand is finite, so $f \in \mathcal{H}^+_{\text{int}}(Q^{\pm})$. This means Z is a $Q^{\pm}$-null set.

By Theorem 3.10, there exist unique non-negative density folders $\langle h^{\pm} \rangle$ such that

$$Q^{\pm}(f) = \int f \langle h^{\pm} \rangle, \quad \text{for all } f \in \mathcal{L}(\textstyle\int + Q^{\pm}).$$

We observe $\mathcal{L}(\int + Q^+) \cap \mathcal{L}(\int + Q^-) = \mathcal{L}(\int + |Q|)$. By Remark 3.8 (2), each file h_E of $\langle h \rangle$ is finite $\int$-a.e., so that we can take difference of each side and obtain

$$\begin{aligned} Q^+(f) - Q^-(f) &= \int f \langle h^+ \rangle - \int f \langle h^- \rangle \\ &= \int f (\langle h^+ \rangle - \langle h^- \rangle). \end{aligned}$$

Therefore, $\langle h \rangle := \langle h^+ \rangle - \langle h^- \rangle$ is obviously a density folder, and $Q(f) = \int f \langle h \rangle$ holds for any $f \in \mathcal{L}(\int + |Q|)$. The uniqueness of $\langle h \rangle$ follows from that of $\langle h^{\pm} \rangle$. □

5. The Dual Space of $\mathcal{L}$

Definition 5.1. We say that $\langle h \rangle$ is an *essentially bounded folder* if

$$\sup_{E \in \mathcal{E}} \|h_E\|_\infty = \sup_{E \in \mathcal{E}} (\operatorname{ess.sup}_{x \in E} |h_E(x)|) < \infty.$$

We denote this by $\|\langle h \rangle\|_\infty$, and the set of all such folders is denoted by $\mathcal{L}^\infty$.

We first consider the elementary estimation of norm inequalities.

Lemma 5.2. *Let* $\langle h\rangle \in \mathcal{L}^\infty$. *It follows*

$$\|\langle h\rangle\|_\infty = \sup\left\{\left|\int f\langle h\rangle\right|; f\in\mathcal{L}, \|f\|_1 = 1\right\}.$$

Proof. Let $f\in\mathcal{L}$. We choose $E_0\in\mathcal{E}$ such that $\{f\neq 0\}\subset E_0$, then

$$\begin{aligned}\left|\int f\langle h\rangle\right| &= \left|\int f h_{E_0}\right| \\ &\leq \int |f||h_{E_0}| \leq \|\langle h\rangle\|_\infty \int |f| = \|\langle h\rangle\|_\infty \|f\|_1.\end{aligned}$$

Now, taking supremum over all $f\in\mathcal{L}$ with $\|f\|_1 = 1$, we have $\sup\limits_{\|f\|_1=1}\left|\int f\langle h\rangle\right| \leq \|\langle h\rangle\|_\infty$.

To show the converse, let $\alpha := \|\langle h\rangle\|_\infty > 0$. Since $\|\langle h\rangle\|_\infty = \sup_{E\in\mathcal{E}} \|h_E\|_\infty$, for any a with $0<a<\alpha$, there exists $E_a\in\mathcal{E}$ such that $a < \|h_{E_a}\|_\infty$. We deduce $I(h_{E_a} > a)\in\mathcal{L}^+$ and $\int I(h_{E_a}>a) > 0$. By Proposition 2.4 there exists $g_{E_a}\in\mathcal{L}$ such that

$$0\leq g_{E_a}\leq I(h_{E_a}>a) \text{ and } 0<\int g_{E_a}. \tag{5.1}$$

We define $f_{E_a} := (\int g_{E_a})^{-1} g_{E_a}\cdot \mathrm{sgn} h_{E_a}$, then $f_{E_a}\in\mathcal{L}$ and $\|f_{E_a}\|_1 = 1$. By (5.1), we deduce

$$\{g_{E_a}\neq 0\}\subset\{h_{E_a}>a\}\subset\{h_{E_a}>0\}\subset E_a.$$

Hence

$$\begin{aligned}\left|\int f_{E_a}h_{E_a}\right| &= \frac{1}{\int g_{E_a}}\int g_{E_a}|h_{E_a}| \\ &= \frac{1}{\int g_{E_a}}\int g_{E_a} I(g_{E_a}\neq 0) I(h_{E_a}>a)|h_{E_a}| > a.\end{aligned}$$

Now, since $\{f_{E_a}\neq 0\} = \{g_{E_a}\neq 0\}\subset E_a$, we see $\int f_{E_a}h_{E_a} = \int f_{E_a}\langle h\rangle$. Moreover, taking supremum over all elements such that $\|f\|_1 = 1$, we see that for any a with $0<a<\alpha$, there exists $E_a\in\mathcal{E}$ such that

$$\sup_{\|f\|_1=1}\left|\int f\langle h\rangle\right| \geq \left|\int f_{E_a}h_{h_{E_a}}\right| > a,$$

which yields the inverse inequality. $\square$

Lemma 5.3. *Let $\langle h\rangle$ be a density folder. If there exists $0 < C < \infty$ such that for any $f \in \mathcal{L}$*

$$\left|\int f\langle h\rangle\right| \leq C\|f\|_1,$$

then it follows that $\langle h\rangle \in \mathcal{L}^\infty$.

Proof. Let us fix $E \in \mathcal{E}$ and its corresponding file h_E. We shall prove $h_E(x) \leq C$ a.e. $x \in \Omega$ by the use of the reduction to contradiction. For any $\varepsilon > 0$, putting $F_{E,\varepsilon} := \{|h_E| > C+\varepsilon\}$, then we have $0 \leq I(F_{E,\varepsilon}) \in \mathcal{L}^+$. Now we assume that $\int I(F_{E,\varepsilon}) > 0$ (if not, we have nothing to prove). By Proposition 2.4, there exists $g_E \in \mathcal{L}$ such that $0 \leq g_E \leq I(F_E)$ and $\int g_E > 0$. Defining $\varphi_E := g_E \cdot (\mathrm{sgn} h_E)$, we see $\varphi_E \in \mathcal{L}$. Since

$$\{\varphi_E \neq 0\} \subset \{g_E \neq 0\} \subset F_{E,\varepsilon} \subset \{h_E > 0\} \subset E$$

as observed in the proof of Lemma 5.2, we deduce that

$$\begin{aligned}(C+\varepsilon)\|g_E\|_1 &\leq \left|\int g_E|h_E|\right| = \left|\int \varphi_E h_E\right| \\ &= \left|\int \varphi_E\langle h\rangle\right| \leq C\|\varphi_E\|_1 = C\|g_E\|_1.\end{aligned}$$

It follows that $\|g_E\|_1 = 0$ and it contradicts $\int g_E > 0$. This means that $\|h_E\| \leq C$ for any $E \in \mathcal{E}$. The proof is complete. □

Theorem 5.4. *Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system satisfying the Stone condition. Then there exists a one-to-one linear and norm preserving mapping τ between essentially bounded folders space $\mathcal{L}^\infty$ and the dual space $\mathcal{L}^*$; the correspondence is given by*

$$\tau(\langle h\rangle)f = \int f\langle h\rangle \quad (f \in \mathcal{L}).$$

Proof. Let $\langle h\rangle \in \mathcal{L}^\infty$. Defining

$$T_{\langle h\rangle}f := \int f\langle h\rangle \quad (f \in \mathcal{L}), \tag{5.2}$$

then $T_{\langle h\rangle}$ is linear and $|T_{\langle h\rangle}f| = |\int f\langle h\rangle| \leq \|\langle h\rangle\|_\infty\|f\|_1$, hence $\|T_{\langle h\rangle}\| \leq \|\langle h\rangle\| < \infty$, so $T_{\langle h\rangle} \in \mathcal{L}^*$. Moreover, from equation (5.2) and Lemma 5.2 we have

$\|T_{\langle h\rangle}\| = \|\langle h\rangle\|_\infty$. It is shown that τ is the isometry from $\mathcal{L}^\infty$ to $\mathcal{L}^*$, This immediately implies that τ is injective.

Therefore, it suffices to prove that this mapping is surjective. Let $T \in \mathcal{L}^*$. Defining

$$Q(g) := Tg \quad \text{for } g \in \mathcal{L},$$

we see that Q is a signed Daniell integral on $\mathcal{L}$. Indeed, the linearity is obvious. If $\mathcal{L} \ni g_n \searrow 0$, then $|Q(g_n)| = |Tg_n| \le \|T\|\|g_n\|_1 \to 0$ by the Dominated Convergence Theorem, so that Q is a signed Daniell integral on $\mathcal{L}$. We next prove $Q \ll \int$. Let Z be an $\int$-null set. Then there exists $f \in \mathcal{H}^+_{\text{int}}$ such that $Z \subset \{f = +\infty\}$. Since $f \in \mathcal{H}^+_{\text{int}} \subset \mathcal{L}$, we have $Q(f) < \infty$ and $|Q|(f) < \infty$ because $Q(f) = Q^+(f) - Q^-(f) < \infty$ and $|Q|(f) = Q^+(f) + Q^-(f) < \infty$ holds for $f \in \mathcal{L}$ by the Jordan Decomposition. This means $f \in \mathcal{H}^+_{\text{int}}(|Q|)$, and hence Z is Q-null.

Using Theorem 4.5, we can uniquely construct a density folder $\langle h\rangle$ such that $Q(f) = \int f\langle h\rangle$ holds for $f \in \mathcal{L}(\int + |Q|)$. However, since $\mathcal{L} \subset \mathcal{L}(\int + |Q|)$ by the definition of Q, we have

$$Tf = Q(f) = \int f\langle h\rangle, \quad \text{for all } f \in \mathcal{L}.$$

Since $|\int f\langle h\rangle| \le \|T\|\|f\|_1$, by Lemma 5.3 it follows $\langle h\rangle \in \mathcal{L}^\infty$. It means that τ is surjective. □

6. Lebesgue Decomposition

Definition 6.1. Let $\int, Q$ be Daniell integrals on $\mathcal{H}$. We say that $\int$ and Q are mutually singular if the $(\int + Q)$-measurable indicator folder $\langle Z\rangle$ satisfies $\int I(Z_E \cap E) = Q(I(Z^c_E \cap E)) = 0$ for any $E \in \mathcal{E}$, denoted $Q \perp \int$.

Theorem 6.2. (1) *The integral $\int : \mathcal{H} \to \mathbb{R}$ is zero if and only if $\int I(E) = 0$ for any $E \in \mathcal{E}$.*

(2) *If $Q \perp \int$ and $Q \ll \int$ then $Q = 0$.*

Proof. We first note that, in general, an elementary integral $\int$ on $\mathcal{H}$ is zero if and only if $\int f = 0$ for any $f \in \mathcal{H}^+$. Indeed, the sufficiency is clear. To prove the necessity, choosing $h_n \in \mathcal{H}$, so that $h_n \nearrow f$, we obtain $\int f = \lim_{n\to\infty} \int h_n = 0$.

(1) The necessity is clear. To prove the sufficiency, let f be a positive function in $\mathcal{H}^+$. Defining

$$s_n := \frac{1}{2^n}\sum_{k=1}^{\infty} I\left(f > \frac{k}{2^n}\right),$$

we find $s_n \in \mathcal{H}^+$ and $0 \leq s_n \nearrow f$. Since $I(f > k/2^n) \in \mathcal{H}^+$ for each $n, k \in \mathbb{N}$, we see $Q(I(f > k/2^n)) = 0$ by assumption. The Monotone Convergence Theorem gives us $Q(s_n) = 0$ and also gives $0 = Q(s_n) \nearrow Q(f) = 0$. For general $f \in \mathcal{H}$, we apply the same argument to f^+, f^- separately.

(2) Suppose that there exists an $(\int +Q)$-measurable folder $\langle Z\rangle$ such that $Q(I(Z_E^c \cap E)) = \int I(Z_E \cap E) = 0$ for any $E \in \mathcal{E}$. By absolute continuity, we see $Q(I(Z_E \cap E)) = 0$. Therefore,

$$Q(I(Z_E^c \cap E)) + Q(I(Z_E \cap E)) = 0$$

and hence $Q(I(E)) = 0$ for any E. By (1), we obtain $Q = 0$. □

Keeping in mind that the notion of folder plays a key role in our result, we formulate and prove the Lebesgue decomposition theorem in our setting.

Theorem 6.3. *Let $(\Omega, \mathcal{H})$ be an elementary space satisfying the Stone condition, and let $\int, Q$ be Daniell integrals. Then Q can be uniquely expressed as $Q = Q_a + Q_s$ where $Q_a \ll \int$ and $Q_s \perp \int$.*

Proof. Since we see $Q \ll (\int +Q)$, it follows from Theorem 3.10 that there exists a non-negative $(\int +Q)$-density $\langle g\rangle$ such that

$$Q(f) = \left(\int +Q\right) f\langle g\rangle \tag{6.1}$$

for any $f \in \mathcal{L}^+(\int +Q)$.

We first prove $\langle g\rangle \leq \langle I\rangle$ $(\int +Q)$-*a.e.* and $\langle g\rangle < \langle I\rangle$ $\int$-a.e. For every $E \in \mathcal{E}$, we can choose $E_n \in \mathcal{E}_0$ so that $E_n \nearrow E$. Noting that $\{g_E > 1\} \subset E$ $(\int +Q)$-*a.e.*, we substitute $f := I(E_n)I(g_E > 1) \in \mathcal{L}(\int +Q)$ for the equation (6.1). Then we have

$$\begin{aligned} Q(I(E_n)I(g_E > 1)) &= \int I(E_n)I(g_E > 1)g_E + Q(I(E_n)I(g_E > 1)g_E) \\ &\geq \int I(E_n)I(g_E > 1) + Q(I(E_n)I(g_E > 1)). \end{aligned}$$

Since $Q(I(E_n)I(g_E > 1)) < \infty$, we obtain $\int I(E_n)I(g_E > 1) = 0$, and hence $I(E_n)I(g_E > 1) = 0$ a.e. Letting $n \to \infty$, we find $I(g_E > 1) = 0$ a.e. Again, substituting it for the equation (6.1), we obtain $Q((1 - g_E)I(g_E > 1)) = 0$, and hence $I(g_E > 1) = 0$ Q-*a.e.*. To show that $\langle g \rangle < \langle I \rangle$ a.e., substituting $f = I(g_E = 1)$ in (6.1) and applying the same argument, we can deduce that $\{g_E = 1\}$ is null.

Next, the family $(I(g_E = 1))_{E \in \mathcal{E}}$ is obviously an $(\int +Q)$-density folder, so we denote it by $\langle G \rangle$. Now, Since $f\langle g \rangle \in \mathcal{L}(\int +Q)$ for any $f \in \mathcal{L}(\int +Q)$, we have

$$\begin{aligned} Q(f) &= \left(\int +Q\right) f\langle g \rangle = \int f\langle g \rangle + Q(f\langle g \rangle) \\ &= \int f\langle g \rangle + \left(\int +Q\right) f\langle g^2 \rangle \\ &= \int f(\langle g \rangle + \langle g^2 \rangle) + Q(f\langle g^2 \rangle) \\ &= \int f(\langle g \rangle + \langle g^2 \rangle + \cdots + \langle g^n \rangle) + Q(f\langle g^n \rangle). \end{aligned}$$

Since $\langle g^n \rangle \searrow \langle G \rangle$ $(\int +Q)$-*a.e.* and

$$\langle 0 \rangle \le \langle g \rangle + \langle g^2 \rangle + \cdots + \langle g^n \rangle \nearrow \quad (\textstyle\int +Q)\text{-}a.e.,$$

we can denote this limit folder by $\langle h \rangle$. It follows that $\langle h \rangle$ is $(\int +Q)$-measurable and takes value in $[0, \infty]$. In fact, $\langle h \rangle$ takes real values almost everywhere. For any non-negative function $f \in \mathcal{L}(\int +Q)$, note that $f\langle g^n \rangle \searrow f\langle G \rangle$ $(\int +Q)$-*a.e.* and

$$\langle 0 \rangle \le f(\langle g \rangle + \langle g^2 \rangle + \cdots + \langle g^n \rangle) \nearrow f\langle h \rangle \quad (\textstyle\int +Q)\text{-}a.e.,$$

applying the Monotone Convergence Theorem and the Dominated Convergence Theorem to Q and $\int$, we obtain

$$Q(f) = \int f\langle h \rangle + Q(f\langle G \rangle). \tag{6.2}$$

For general $f \in \mathcal{L}(\int +Q)$, we apply the same argument to f^+, f^- separately. This equation is valid for $f \in \mathcal{L}^+(\int +Q)$ because $f^- \in \mathcal{L}(\int +Q)$. If we take $f \in \mathcal{H}$ in (6.2), we deduce $\langle h \rangle$ is an $\int$-density.

We define $Q_a(f) := \int f\langle h \rangle$, $Q_s(f) := Q(f\langle G \rangle)$ for any $f \in \mathcal{H}$. Since $\langle h \rangle$ is an $\int$-density and Q_a, Q_s are non-negative, we see $Q_a \ll \int$ and $Q_s \ll Q$ by Remark 3.8 (3). To prove $Q_s \perp \int$, noting that $\langle G \rangle$ is an $(\int +Q_s)$-measurable folder because $Q_s \ll Q$, we can easily see that $\langle G \rangle$ is a $(\int + Q_s)$-measurable folder satisfying the definition of $Q_s \perp \int$.

Finally, we will show the uniqueness of the decomposition. Suppose that $Q = Q_1 + Q_2$ for some Q_1 and Q_2 with $Q_1 \ll \int, Q_2 \perp \int$. Then we have $Q_1 + Q_2 = Q_a + Q_s$. We define a signed Daniell integral $\lambda : \mathcal{H} \to \mathbb{R}$ to be

$$\lambda(f) := Q_1(f) - Q_a(f) = Q_s(f) - Q_2(f), \quad \text{for } f \in \mathcal{H}.$$

By Theorem 4.3, we obtain the Jordan Decomposition $\lambda = \lambda^+ - \lambda^-$. For non-negative $h \in \mathcal{H}$,

$$\begin{aligned} \lambda^+(h) &= \sup\{\lambda(k) : 0 \leq k \leq h,\ k \in \mathcal{H}\} \\ &= \sup\{Q_1(k) - Q_a(k) : 0 \leq k \leq h,\ k \in \mathcal{H}\} \leq Q_1(h), \end{aligned}$$

by the non-negativity of Q_a. Similarly, we have $\lambda^-(h) \leq Q_a(h)$. Therefore, we obtain

$$|\lambda|(h) = \lambda^+(h) + \lambda^-(h) \leq Q_1(h) + Q_a(h) = (Q_1 + Q_a)(h), \tag{6.3}$$

for all non-negative $h \in \mathcal{H}$. (6.3) remains valid for non-negative $f \in \mathcal{H}^+$, and similarly we have $|\lambda| \leq Q_2 + Q_s$ for non-negative $f \in \mathcal{H}^+$. Combining these results, we have

$$|\lambda| \ll Q_a + Q_1,\ |\lambda| \ll Q_2 + Q_s. \tag{6.4}$$

By (6.4) and $Q_a + Q_1 \ll \int$, we obtain $|\lambda| \ll \int$.

We shall next show $|\lambda| \perp \int$. By the assumption of $Q_s \perp \int$ and $Q_2 \perp \int$, there exist a $(Q_s + \int)$-measurable folder $\langle Z_s \rangle$ and a $(Q_2 + \int)$-measurable folder $\langle Z_2 \rangle$ such that

$$\int I(Z_{s,E} \cap E) = Q_s(I(Z_{s,E}^c \cap E)) = 0, \text{ and } \int I(Z_{2,E} \cap E) = Q_2(I(Z_{2,E}^c \cap E)) = 0,$$

respectively. We note that $Z_{s,E}$ and $Z_{2,E}$ are both $\int$-measurable and $\int$-null sets. Defining $Z_E := Z_{s,E} \cup Z_{2,E}$, we see that $\langle Z \rangle := (I(Z_E))_{E \in \mathcal{E}}$ is obviously a $\int$-measurable and $\int$-null folder. Moreover, we recall $|\lambda| \ll \int$, so that $\langle Z \rangle$ is $(|\lambda| + \int)$-measurable and $(|\lambda| + \int)$-null folder. Since

$$Z_E^c \cap E \subset Z_{s,E}^c \cap E, \quad Z_E^c \cap E \subset Z_{2,E}^c \cap E,$$

and the right-hand-sides are Q_s-null and Q_2-null, respectively. It follows that $Z_E^c \cap E$ is a $(Q_s + Q_2)$-null set. By the fact that $\lambda \ll Q_s + Q_2$, we verify that $Z_E^c \cap E$ is a $|\lambda|$-null set. It means that $|\lambda| \perp \int$, and hence by Theorem 6.2 (2), we have $\lambda = 0$. This completes the proof of $Q_1 = Q_a,\ Q_2 = Q_s$. □

7. Applications

In this section, we apply our results of Sections 5 and 6 to general measure spaces and localizable measure spaces. We recall some notions, see [25].

Definition 7.1. Let $(\Omega, \mathcal{H}, \int)$ be a Daniell system with the Stone condition.

(1) A function $f : \Omega \to \overline{\mathbb{R}}$ is said to be *locally (Daniell) measurable*, if fh is Daniell measurable for all $h \in \mathcal{H}$.

(2) A folder $\langle h \rangle$ is said to be *weakly complete*, if there exists a locally measurable function f_0 such that

$$\langle h \rangle = f_0 \langle I \rangle \ a.e.$$

By definition, all complete folders are weakly complete. We call f_0 *weakly complete file.*

We fix a complete measure space $(\Omega, \mathcal{F}_0, \mu)$. Let $(\Omega, \mathcal{H}(\mathcal{F}_0), \int d\mu)$ be a Daniell system induced by $(\Omega, \mathcal{F}, \mu)$, where $\mathcal{H}(\mathcal{F}_0)$ is the set of all $\mathcal{F}_0$-simple functions and $\mathcal{F}_0$ is the set of all μ-finite sets in $\mathcal{F}$. A functional $\int d\mu$ is an elementary integral defined by $\int h \, d\mu := \sum_{k=1}^{n} a_k I(A_k)$ for $h \in \mathcal{H}(\mathcal{F}_0)$, $a_k \in \mathbb{R}, A_k \in \mathcal{F}_0$.

Since the measure space is complete, each null set obtained by Daniell scheme is also μ-null set and the converse is true. We see that $\mathcal{E}_0 = \mathcal{F}_0$, and $\mathcal{E} = \{$all countable unions of elements of $\mathcal{F}_0\}$, i.e., $\mathcal{E}$ is the set of all σ-finite sets in $\mathcal{F}$. Further, all Daniell measurable functions are $\mathcal{F}$-measurable, and all $\mathcal{F}$-measurable functions having σ-finite carrier are Daniell measurable. The set $\mathcal{D}$ of all the Daniell measurable sets is a σ-ring generated by the union of the elements of $\mathcal{E}$ and the null sets.

Let $(\Omega, \mathcal{F}, \mu)$ be a complete localizable measure space (cf. [13, 27, 22, 33]). We induce the Daniell system $(\Omega, \mathcal{H}(\mathcal{F}_0), \int)$ in the same way as above. For any non-negative folder $\langle h \rangle = (h_E)_{E \in \mathcal{E}}$, let $\mathcal{A} := \{h_E : E \in \mathcal{E}\} \subset \mathcal{M}$. Since $\mathcal{A}$ is the subset of $\mathcal{F}$-measurable functions, there exists an essential supremum f_0 for $\mathcal{A}$ by the localizability of μ (cf. [22, 33]). It is not difficult to verify that

$$h_E = f_0 I(E) \text{ a.e. for all } E \in \mathcal{E}.$$

The essential supremum f_0 is $\mathcal{A}$-measurable but not Daniell measurable. However, we can obtain the following characterization; see [25]:

Theorem 7.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete measure space. Then the measure μ is localizable if and only if all non-negative folder $\langle h \rangle$ is weakly complete, and its weakly complete file f_0 is $\mathcal{F}$-measurable.*

The non-negativity in the above theorem can be eliminated, because the usual argument is available to $\langle h \rangle = \langle h^+ \rangle - \langle h^- \rangle$, where $\langle h^{\pm} \rangle = (h_E^{\pm})_{E \in \mathcal{E}}$.

Now, we obtain the following results:

Corollary 7.3. *Let $(\Omega, \mathcal{F}, \mu)$ be a localizable measure space. Then there exists a one-to-one linear and norm preserving mapping τ between essentially bounded "function" space L^∞ and the dual space $(L^1)^*$; the correspondence is given by*

$$\tau(g)f = \int fg, \quad \text{for } f \in L^1.$$

Corollary 7.4. *Let $(\Omega, \mathcal{F}, \mu)$ be a localizable measurable space, and let ν be a signed measure on $\mathcal{F}$. Then ν can be uniquely expressed as $\nu = \nu_a + \nu_s$ where $\nu_a \ll \mu$ and $\nu_s \perp \mu$. Moreover, each measure can be expressed as follows: there exists a unique $\mathcal{F}$-measurable function h such that*

$$\nu_a(E) = \int_E h\,\mu, \quad \text{for any } E \in \mathcal{F}_0,$$

and there exists $Z \in \mathcal{F}$ such that

$$\nu_s(Z) = \mu(Z^c) = 0.$$

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