

ORDER CLASSES OF SYMMETRIC GROUPS

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Abstract: The order of an element x in a finite group G is the smallest positive integer k , such that x^k is the group identity. The set of all possible such orders joint with the number of elements that each order referred to, is called the order classes of G . In this paper, the order classes of symmetric groups is derived.

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1. Introduction

Symmetric groups allow access to specific and detailed group theories, such as Lagrange, Able, Galois, invariant theory and the representation theory of Lie groups. On the other hand, Cayley's Theorem asserts that: "Any finite group G of order n is isomorphic to a subgroup of S_n ", in addition to its applications in the arts and particularly in architecture.

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Let x be an elements in a finite group G . Let \sim be the relation: “ $y \in G$ has the same order as x ”. Then \sim is an equivalence relation. For all $x \in G$, the set of all $y \in G$ which have the same order as x is the order class of x , and denoted by $OC_G(x)$.

The order of $x \in G$ denoted by $o(x)$. Then $OC_G(x) = \{y \in G \mid o(y) = o(x)\}$. Each order class has a unique order, let $O = \{o(x) \mid x \in G\}$ be the set of all available orders for the finite group G . Then, there is a one-one mapping between the set of all order classes $OC_G = \{OC_G(x) \mid x \in G\}$ and $O = \{o(x) \mid x \in G\}$. Hence, we can write the order classes of a finite group G as O instead of OC_G . The main interest in the obtained classes are: the order of each class which is an element of O , and the number of elements of each class. These classifications will represent the order classes of a group G in term of a collection of order pairs $[o(x), |OC_G(x)|]$, which is denoted by G . Thus:

$$G = \{[j, Y_j] \mid j \in O \text{ and } Y_j \text{ is the number of } x \in G \text{ such that } o(x) = j\}.$$

This paper introduces the definition of symmetric groups and the classification of its elements, some preliminary results and remarks related to their constructions. Then, it presents new results of obtaining and classifying the order classes of symmetric groups.

2. Preliminaries

To keep this research self contained, this section will introduce the definition of symmetric groups and some preliminary results to describe the elements structure, which will be used in later sections.

Definition 1. (see [4]) A group G which elements are permutations (one-one mapping of a set onto itself) on a finite set S equipped with composition operation (\circ) is called a permutation group or symmetric group; that is $G = \{\sigma \mid \sigma \text{ is a permutation on } S\}$, such that $(\sigma \circ \tau)(s) = \sigma(\tau(s)) \in S$ for all $s \in S$. The group (G, \circ) is denoted by $Sym(S)$.

Let $S = \{x_1, x_2, \dots, x_n\}$ be a finite set, define the map ϕ as follows:

$$\phi: S \rightarrow \mathbb{N}, \text{ with } \phi(x_i) = i \text{ for } i = 1, 2, \dots, n.$$

Then ϕ is a bijection. Loosely speaking, the action of any permutation $\sigma \in Sym(S)$ on an element $x_i \in S$ is the same action of σ on i the index of x_i , in

this case the symmetric group $Sym(S)$ will be renamed by S_n , the group of all permutations on a set of n elements.

The order of the group S_n is $n!$, each element of S_n is represented in a form of 2-rows; the first row is $(1, 2, \dots, n)$ and the second row assigns each i with $\sigma(i)$ for all $i = 1, 2, \dots, n$ as follows:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}, \quad \sigma(i) \in \{1, 2, \dots, n\}$$

for all $i = 1, 2, \dots, n$ and $\sigma(i) = \sigma(j)$ if and only if $i = j$.

Throughout the composition of two permutations σ and τ will be denoted by $\sigma\tau$ instead of $\sigma \circ \tau$, and the composition of the same permutation σ for r -times by σ^r .

A shorter notation for the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

is the cycle representation:

$$(1 \ \sigma(1) \ \sigma^2(1) \cdots \sigma^{n-1}(1))$$

where $\sigma^n(1) = 1$. The k -cycle is a cycle of length k i.e. $(x_1 \ x_2 \ \cdots \ x_k)$ consists of k distinct elements such that $x_1 = \sigma(x_k)$, $x_i = \sigma(x_{i-1})$ for $i = 2, 3, \dots, k$ and $\sigma(x) = x$ for all $x \in S \setminus \{x_1, x_2, \dots, x_k\}$.

The identity element of S_n is the cycle of length 0 and denoted by $()$ where

$$e = () = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

The inverse of the permutation $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$ is

$$\sigma^{-1} = \begin{pmatrix} \sigma(1) & \sigma(2) & \cdots & \sigma(n) \\ 1 & 2 & \cdots & n \end{pmatrix}$$

Theorem 1. (see [1]) *If $n \geq 3$, then S_n is non abelian group.*

Two cycles $(x_1 \ x_2 \ \cdots \ x_m)$ and $(y_1 \ y_2 \ \cdots \ y_k)$ are said to be disjoint cycles, if and only if $x_i \neq y_j$ for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k$.

Lemma 1. (see [3]) *If $\sigma, \tau \in S_n$ are disjoint cycles, then $\sigma\tau = \tau\sigma$.*

Theorem 2. (see [1]) *Any permutation of a finite set is either a cycle or can be written as a composition of pairwise disjoint cycles, except for the order in which the cycles are written, and the inclusion or omission of the 0-cycle (the identity), this can be done in only one way.*

3. Order Classes of Symmetric Groups

The order classes for symmetric groups from previous studies were obtained by an ordinary way, that is, by finding the order of each element in the group, then counting the number of elements for each order class. Unfortunately, this process can take a lot of time as the size of symmetric group grows rapidly. Actually, there is no exact formula in terms of n to find all possible orders and the size of each order class for symmetric groups. In [2], they proved that $\log o(x)$, $x \in S_n$ has a normal distribution with mean $\frac{1}{2} \log n$ and variance $\frac{1}{3} \log^3 n$.

Throughout this research, the cycle representation for all permutation elements will be used. This research needs to classify the elements of $G = S_n$ in terms of orders by locating all achievable orders. Moreover, counting the elements of each order. This classification need to combine all elements in G in cycles composition structure.

Definition 2. A permutation $x \in S_n$ is said to be simple if and only if x is either a cycle or can be written as a composition of pairwise disjoint cycles all of the same length.

Let C_k denote the set of all simple elements of G , with cycle length $k \geq 2$. Then:

$$C_k = \left\{ x = \sigma^r \mid \sigma \text{ is a cycle of length } k \text{ and } r = 1, 2, \dots, \left\lfloor \frac{n}{k} \right\rfloor \right\}.$$

Remark 1. (see [3]) Let $x \in S_n$, such that $x = \sigma_1 \sigma_2 \cdots \sigma_r$, with $\sigma_i \in G$ is a cycle of length k_i , $i = 1, 2, \dots, r$. Then

$$o(x) = \text{lcm} \{k_i \mid i = 1, 2, \dots, r\}.$$

Lemma 2. Let $C_k \subseteq S_n$ be the class of all simple permutations of length k , with $k = 2, 3, \dots, n$. Then $o(x) = k$ for all $x \in C_k$.

Proof. Let $x \in C_k$, then there exists a k -cycle $\sigma \in S_n$, such that $x = \sigma^r$, for some $r \in [1, \lfloor \frac{n}{k} \rfloor]$. Using Remark 1, it follows that $o(x) = \text{lcm}\{ \underbrace{k, k, \dots, k}_{r\text{-times}} \} = k$. \square

Let (k) indicate all cycles of length k regardless of the permutations structure. That is, (2) will denote all cycles of length 2, and $(2)(3)$ denotes all cycles induced by composition of disjoint cycles of lengths 2 and 3. Note that $(2)(3)$ and $(3)(2)$ give permutations of length 5 which not the same of (5) ; all have the same lengths but different structures.

Lemma 3. *Let C_k be the class of all simple permutations of order k in S_n . Then*

$$|C_k| = \sum_{r=1}^{\lfloor \frac{n}{k} \rfloor} \frac{n!}{r!k^r(n-rk)!}.$$

Proof. Let $C_k \subset S_n$. Then there exists a k -cycle $\sigma \in S_n$ such that:

$$C_k = \left\{ x = \sigma^r \mid \sigma \text{ is a cycle of length } k \text{ and } r = 1, 2, \dots, \left\lfloor \frac{n}{k} \right\rfloor \right\}.$$

So,

$$C_k = \bigcup_{r=1}^{\lfloor \frac{n}{k} \rfloor} \{x \in S_n \mid x = \sigma^r\}.$$

Hence

$$|C_k| = \sum_{r=1}^{\lfloor \frac{n}{k} \rfloor} |\{x \in S_n \mid x = \sigma^r\}|.$$

Using the counting principle and permutations to count the number of all possible such arrangements for the k -cycle permutations σ , it follows

$$|\{x \in S_n \mid x = \sigma^r\}| = \frac{n!}{r!k^r(n-rk)!}.$$

Therefore:

$$|C_k| = \sum_{r=1}^{\lfloor \frac{n}{k} \rfloor} \frac{n!}{r!k^r(n-rk)!}.$$

\square

Example 1. In S_{15} , then

$$\begin{aligned} C_2 &= \{x \in S_n \mid x = (2)^r, r = 1, 2, \dots, 7\} \\ &= \{(2), (2)^2, (2)^3, (2)^4, (2)^5, (2)^6, (2)^7\} \end{aligned}$$

and $o(x) = 2$ for all $x \in C_2$. Moreover

$$\begin{aligned} |C_2| &= \sum_{r=1}^7 \frac{15!}{r!2^r(15-2r)!} \\ &= 1054095 + 75075 + 675675 + 2837835 + 4729725 + 2027025 \\ &= 10349535. \end{aligned}$$

Similarly,

$$\begin{aligned} C_3 &= \{x \in S_n \mid x = (3)^r, r = 1, 2, \dots, 5\} \\ &= \{(3), (3)^2, (3)^3, (3)^4, (3)^5\} \end{aligned}$$

and

$$\begin{aligned} |C_3| &= \sum_{r=1}^5 \frac{15!}{r!3^r(15-3r)!} \\ &= 910 + 200200 + 11211200 + 112112000 + 44844800 \\ &= 168369110. \end{aligned}$$

Definition 3. Let $G = S_n$, then N_k (none-simple permutations) denotes the set of all classes χ , each class χ contains all permutations $x \in G$ of order k , where x is a composition of multi pairwise disjoint k_i -cycles not all of the same length, with $2 \leq k_i \leq n$ for all i .

Example 2. Let $G = S_9$, then

$$\begin{aligned} N_{10} &= \{(2)(5), (2)^2(5)\} \\ N_6 &= \{(2)(3), (2)^2(3), (2)^3(3), (2)(3)^2, (2)(6), (3)(6)\} \end{aligned}$$

Lemma 4. Let $\chi \in N_k$ be the class of all permutations

$$x = (k_1)^{s_1}(k_2)^{s_2} \dots (k_r)^{s_r}, \text{ with } \text{lcm}\{k_1, k_2, \dots, k_r\} = k.$$

Then:

$$|\chi| = \frac{n!}{\left(n - \sum_{i=1}^r k_i s_i\right)!} \prod_{i=1}^r \frac{1}{s_i k_i^{s_i}}.$$

Proof. Let $x \in \chi$, where $x = (k_1)^{s_1} (k_2)^{s_2} \cdots (k_r)^{s_r}$, with $\text{lcm}\{k_1, k_2, \dots, k_r\} = k$. Then, the number of choices for such x is $\frac{n!}{\left(n - \sum_{i=1}^r k_i s_i\right)!}$ with $\prod_{i=1}^r \frac{1}{s_i k_i^{s_i}}$ repetitions. Thus

$$|\chi| = \frac{n!}{\left(n - \sum_{i=1}^r k_i s_i\right)!} \prod_{i=1}^r \frac{1}{s_i k_i^{s_i}}.$$

□

Example 3. In Example 2, if $\chi = \{(2)^2(3)\}$, then:

$$\begin{aligned} |\chi| &= \frac{9!}{\left(9 - \sum_{i=1}^2 k_i s_i\right)!} \prod_{i=1}^2 \frac{1}{s_i k_i^{s_i}} \\ &= \frac{9!}{(9 - (2(2) + 3))!} \left(\frac{1}{2(2^2)} \times \frac{1}{1(3)^1}\right) \\ &= \frac{9!}{2 \times 8 \times 3} = 7560. \end{aligned}$$

Similarly

$$|\{(2)(3)\}| = 2520, \quad |\{(2)^3(3)\}| = 2520,$$

$$|\{(2)(3)^2\}| = 10080, \quad |\{(2)(6)\}| = 30240, \quad \text{and} \quad |\{(3)(6)\}| = 20160.$$

Then $|N_6| = 73080$. Since S_9 has only one representation for simple permutation of length 6, with $|C_6| = 10080$, then this implies that S_9 has $|N_6| + |C_6| = 73080 + 10080 = 83160$ elements of order 6.

Finally, let $G = S_n$, then $|G| = n! = p_1^{m_1} \times p_2^{m_2} \times \cdots \times p_j^{m_j}$, where p_i are primes, $p_i \leq n$ and $m_i \in \mathbb{N}$ for all $i = 1, 2, \dots, j$. Let $P = \{p_1, p_2, \dots, p_j\}$. Let W be the set of all integers $m > n$ each m can be produced by multiplying elements of P while the sum of these elements is less than or equal to n . Thus:

$$K = \left\{ m > n \mid m = \prod_{p \in P} p \text{ while } \sum_{p \in P} p \leq n \right\}.$$

All of the non-primes $k \in \{2, 3, \dots, n-1\}$, $n \geq 6$ are related to orders for elements which are both simple and none-simple. As an example, in S_6 the order 4, it is the order of simple permutation of the form $x = (4)$ or a non-simple permutation of the form $x = (2)(4)$. Let L denotes all none-primes k in $\{2, 3, \dots, n-1\}$. Then

$$|OC_G(k)| = |C_k| + |N_k| \text{ for all } k \in L. \quad (1)$$

The previous illustrations will give the following result:

Theorem 3. *If $G = S_n$. Then:*

$$O_G = \{1, n\} \cup \{o(x) \mid x \in C_i; i \in \{2, 3, \dots, n\}\} \\ \cup \{o(x) \mid x \in N_k; k \in W \cup L\}.$$

Proof. Let $x \in S_n$, then $x = (k_1)(k_2) \cdots (k_r)$, with $\sum_{i=1}^r k_i \leq n$ and $2 \leq k_i \leq n$ for all $i = 1, 2, \dots, r$. Using Remark 1 it follows that $o(x) = lcm\{k_1, k_2, \dots, k_r\} = m$. To classify the order classes of G there are two cases:

Case 1. If m is prime, then $m = k_s$ for some $s \in \{1, 2, \dots, r\}$, which implies that $m \leq n$. Therefore x is a simple permutation. Hence $x \in C_m$. All of the primes divisors of $n!$ should give rise to simple elements. This implies that $P \subseteq O_G$.

Case 2. If m is not prime. Then:
Either $m \leq n$ or $m > n$. This implies that either $m \in L$ or $m \in W$. Then, $x \in N_m$. That means, there are two types of such element order and it need to verify both: all $x \in N_m$ and all $x \in C_m$. Therefore:

$$O_G(x) = N_m \cup C_m.$$

If $m > n$, then $C_m = \emptyset$. Hence $OC_G(x) = N_m$. □

Example 4. Let $G = S_8$. Then $8! = 2^7 \times 3^2 \times 5 \times 7$, therefore $P = \{2, 3, 5, 7\}$, which gives $W = \{2 \times 5, 2 \times 2 \times 3, 3 \times 5\} = \{10, 12, 15\}$, and $L = \{4, 6\}$.

Using Theorem 3 it follows that:

$$\begin{aligned}
 O_G &= \{1, n\} \cup \{o(x) \mid x \in C_i ; i \in \{2, 3, \dots, n\}\} \\
 &\quad \cup \{o(x) \mid x \in N_k ; k \in W \cup L\} \\
 &= \{1, 8\} \cup \{2, 3, 4, 5, 6, 7, 8\} \cup \{4, 6, 10, 12, 15\} \\
 &= \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 15\}.
 \end{aligned}$$

To count the number of elements for each order class; we should start by counting: all simple elements $x \in C_k$, for $k \in \{2, 3, 4, 5, 6, 7, 8\}$, and all $x \in N_k$, for $k \in \{4, 6, 10, 12, 15\}$. It follows:

$$\begin{aligned}
 |C_2| &= |\{(2), (2)^2, (2)^3, (2)^4\}| \\
 &= \frac{8!}{2(8-2)!} + \frac{8!}{2^2(8-2 \times 2)!} + \frac{8!}{2^3(8-3 \times 2)!} + \frac{8!}{2^4(8-4 \times 2)!} \\
 &= 28 + 210 + 420 + 105 \\
 &= 763.
 \end{aligned}$$

Similarly, $|C_3| = 1232$, $|C_4| = 1680$, $|C_5| = 1344$, $|C_6| = 3360$, $|C_7| = 5760$, $|C_8| = 5040$

$$\begin{aligned}
 |N_4| &= |\{(2)(4), (2)^2(4)\}| \\
 &= \frac{8!}{(8-(2+4))!} \left(\frac{1}{2} \cdot \frac{1}{4}\right) + \frac{8!}{(8-(2 \times 2+4))!} \left(\frac{1}{2!(2)^2} \cdot \frac{1}{4}\right) \\
 &= 2520 + 1260 \\
 &= 3780.
 \end{aligned}$$

Similarly, $|N_6| = 7280$, $|N_{10}| = 4032$, $|N_{12}| = 3360$, $|N_{15}| = 2688$.

Using Equation 1, implies

$|OC_G(4)| = |C_4| + |N_4| = 1680 + 3780 = 5460$ and $|OC_G(6)| = |C_6| + |N_6| = 3360 + 7280 = 10640$.

Therefore:

$$\begin{aligned}
 G &= \{[1, 1], [2, 763], [3, 1232], [4, 5460], [5, 1344], [6, 10640], [7, 5760], \\
 &\quad [8, 5040], [10, 4032], [12, 3360], [15, 2688]\}
 \end{aligned}$$

4. Conclusions

This paper concerns with the classification of the elements orders. The new results do not need to deal with all of the elements in symmetric group. This research classifies all of the elements in term of cycles lengths into two classes: the first class is the class of all simple permutations and the second is the class

of all non-simple permutations. Where each class has a determinant method to count its elements. That makes it easy to classify the order classes of symmetric groups even for a large size.

References

- [1] John R. Durbin, *Modern Algebra. An Introduction*, Wiley, Sixth Edition, 2009.
- [2] P. Erdős, P. Turán, On some problems of a statistical group theory III, *Acta Math. Acad. Sci. Hung.*, **18** (1967), 309-320.
- [3] I.N. Herstein, *Topic In Algebra*, Second Edition, Wiley, 1999.
- [4] Thomas W. Hungerford, *Abstract Algebra*, Second Edition, Thomson Learning, 1997.