CUTTING ANGLE METHOD FOR CONSTRAINED GLOBAL MAXIMIZING OF EXTENDED REAL-VALUED IPH FUNCTIONS

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Abstract: In this paper, we present an algorithm for finding constrained global maximizers of extended real valued increasing and positively homogeneous (IPH) functions, which is a version of the cutting angle method. Also, we discuss the proof of convergency of the algorithm and give some numerical experiments.

1. Introduction

The cutting angle method for the global minimization of non-negative valued IPH functions over the unit simplex $S := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$ was introduced and studied in [1]. In this paper, we present an approach for constrained global maximization of extended real valued IPH functions over $S_A := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n_- : \sum_{i=1}^n a_i x_i = -1\}$ (where $A := (a_1, a_2, \ldots, a_n)$, $0 < a_i \leq 1$, $i = 1, 2, \ldots, n$), which is a version of the cutting

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Received: July 3, 2013   © 2013 Academic Publications

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angle method. The cutting angle method for the global maximization such functions is reduced to the solution of the following auxiliary problem:

$$\max \ h(x) \text{ subject to } \ x \in S_A, \tag{1.1}$$

where

$$h(x) := \min_{k \leq j} (\max_{i \in I_-(y^k)} \frac{-x_i}{y_i^k}), \quad x = (x_1, \cdots, x_n) \in S_A, \quad I_-(y^k) = \{i : y_i^k < 0\},$$

and

$$A := (a_1, a_2, \ldots, a_n), \ 0 < a_i \leq 1, \ i = 1, 2, \ldots, n.$$

The approach is based on a convenient description of all local maxima of the function h, then a global maximizer can be found by sorting out the local maxima of the function h. By using some transformations of variables, we can present the maximization of an extended real valued IPH function subject to the linear constraints as the maximization of a non-positive valued IPH function over S_A.

The structure of the paper is as follows: In Section 2, we provide some definitions and preliminary results on IPH functions. A representation of non-positive valued IPH functions on $\underline{\mathbb{R}^n}$ by max-type functions is given in Section 3. In Section 4, we present an algorithm for finding constrained global maximizers of non-positive valued IPH functions and to prove the convergency of the algorithm. The description of local maxima of the problem (1.1) is given in Section 5. We give a new version of the cutting angle method in Section 6. Finally, the results of numerical experiments are presented in Section 7.

2. Preliminaries and IPH Functions

Consider n-dimensional linear space $\mathbb{R}^n$. We shall use the following notations:

- $I := \{1, \ldots, n\}$.
- x_i is the ith coordinate of a vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$.
- $\mathbb{R}^n_- := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : \ x_i \leq 0, \ \forall \ i \in I\}$.
- $\mathbb{R}^n_{--} := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : \ x_i < 0, \ \forall \ i \in I\}$.
- $\mathbb{R}^n_+ := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : \ x_i \geq 0, \ \forall \ i \in I\}$.
- $S := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$.
- $S_- := \{x \in \mathbb{R}^n_- : \ \sum_{i=1}^n x_i = -1\}$.

- $S_A := \{x = (x_1, \ldots, x_n) \in \mathbb{R}_-^n : \sum_{i=1}^n a_i x_i = -1\}$,
  where $A := (a_1, a_2, \ldots, a_n)$, $0 < a_i \leq 1$, $\forall\, i \in I$.
- If $x,\ y \in \mathbb{R}^n$, then $x \geq y \iff x_i \geq y_i$ for all $i \in I$.
- If $x,\ y \in \mathbb{R}^n$, then $x >> y \iff x_i > y_i$ for all $i \in I$.
- $I(x) := \{i \in I : x_i \neq 0\}$ for each $x \in \mathbb{R}^n$.

We shall consider the following optimization problem:

$$\max\quad p(x) \text{ subject to } x \in S_A, \tag{2.1}$$

where p is an extended real valued IPH (increasing and positively homogeneous of degree one) function defined on $\mathbb{R}^n$. Recall (see [7]) that a function $p : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is called increasing and positively homogeneous of degree one (IPH), if p is increasing $(x \geq y \implies p(x) \geq p(y))$ and p is positively homogeneous of degree one, that is, $p(\lambda x) = \lambda p(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$. In this paper, we shall consider the extended real-valued IPH functions p defined on $\mathbb{R}^n$ such that $0 \in dom p := \{x \in \mathbb{R}^n : -\infty < p(x) < +\infty\}$.

Proposition 2.1. *Let $p : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ be an IPH function. Then*
(1) $p(0) = 0$.
(2) $p(x) \geq 0$ *for all $x \geq 0$, and $p(x) \leq 0$ for all $x \leq 0$.*

Proof. The assertions (1) and (2) follow from the definition of an IPH function. $\square$

The preceding proposition shows that the problem of the global maximization of an extended real valued IPH function over S_A can be reduced to the global maximization of a non-positive valued IPH function over S_A. Therefore, in order to solve the problem (2.1), it suffices to restrict our attention to the non-positive valued IPH functions $p : \mathbb{R}_-^n \longrightarrow [-\infty, 0]$.

In the following, we give a definition of supergradient for a function f (see [7]).

Definition 2.1. Let X be a non-empty set and $H := \{h : X \longrightarrow (-\infty, 0] : h \text{ is a function}\}$ be a set of finite valued functions defined on X. A function $h \in H$ is called an H-supergradient of a function $f : X \longrightarrow [-\infty, 0]$ at a point $x_0 \in X$, if $f(x) - f(x_0) \leq h(x) - h(x_0)$ for all $x \in X$. The set of all H-supergradients of a function f at a point $x_0 \in X$ is called the H-superdifferential of f at the point $x_0 \in X$, and is denoted by $\partial_H^+ f(x_0)$.

Let f be a real valued function defined on a subset $X \subseteq \mathbb{R}^n$, $x \in X$ and let $u \in \mathbb{R}^n$ be such that $x + \alpha u \in X$ for all fairly small $\alpha > 0$. We say that the function f is directional differentiable at the point x in the direction u, if the limit

$$\lim_{\alpha \longrightarrow 0+} \frac{1}{\alpha}[f(x + \alpha u) - f(x)]$$

exists. In this case, the directional derivative of the function f at x in the direction u is denoted by $f'(x, u)$, and we have

$$f'(x, u) = \lim_{\alpha \longrightarrow 0+} \frac{1}{\alpha}[f(x + \alpha u) - f(x)].$$

3. Representation of Non-Positive Valued IPH Functions on $\mathbb{R}^n_-$ by Max-Type Functions

The theory of non-positive valued IPH functions defined on $\mathbb{R}^n_-$ differs from the one of non-negative valued IPH functions defined on $\mathbb{R}^n_+$ (see [1]) from some ways. In this section, we give some characterizations of these functions.

We introduce the coupling function $v : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [-\infty, 0]$ defined by

$$v(x, y) := \inf\{\lambda \leq 0 : \lambda y \leq -x\}, \tag{3.1}$$

(with the convention $\inf \emptyset = 0$).

The characterizations and properties of the coupling function v have been investigated in [3].

Each vector $x \in \mathbb{R}^n$ generates the following sets of indices:

$$I_+(x) := \{i \in I : x_i > 0\}, \quad I_0(x) := \{i \in I : x_i = 0\}, \quad I_-(x) := \{i \in I : x_i < 0\}.$$

Fix $y \in \mathbb{R}^n$. Let us define the function v_y by $v_y(x) := v(x, y)$ for all $x \in \mathbb{R}^n$. In view of (3.1), we obtain

$$v_y(x) = \begin{cases} \max\limits_{i \in I_-(y)} \frac{-x_i}{y_i}, & x \in V_y \\ 0, & x \notin V_y, \end{cases} \tag{3.2}$$

where

$$V_y := \{x \in \mathbb{R}^n : \forall\, i \in I_-(y) \cup I_0(y), \quad x_i \leq 0,$$

$$\text{and } \max_{i \in I_-(y)} \frac{-x_i}{y_i} \leq \min_{i \in I_+(y)} \frac{-x_i}{y_i}\}. \tag{3.3}$$

(Note that we use the convention $\max \emptyset = -\infty$.) It is easy to check that the set V_y $(y \in \mathbb{R}^n)$ is a closed, convex and downward cone. Moreover, the function $v_y : \mathbb{R}^n \longrightarrow [-\infty, 0]$ is an IPH function.

Lemma 3.1. *Let $y \in \mathbb{R}^n_- \setminus \{0\}$. Then, $V_y = \mathbb{R}^n_-$ and the restriction function $v_y : \mathbb{R}^n_- \longrightarrow [-\infty, 0]$ is a continuous, convex and finite valued IPH function.*

Proof. Fix $y \in \mathbb{R}^n_- \setminus \{0\}$. Clearly $I_+(y) = \emptyset$, and $I_-(y) \cup I_0(y) = I$. Thus, we get $V_y = \mathbb{R}^n_-$, and $v_y(x) = \max\limits_{i \in I_-(y)} \frac{-x_i}{y_i}$ for all $x \in \mathbb{R}^n_-$. Note that $I_-(y) \neq \emptyset$, and so, v_y is a finite valued function. Also, it is easy to see that the function $v_y : \mathbb{R}^n_- \longrightarrow [-\infty, 0]$ is an IPH and continuous function. For the proof of the convexity of v_y, consider $x, \ x' \in \mathbb{R}^n_-$. It is clear that $x + x' \in \mathbb{R}^n_-$, and we have

$$\frac{-x_i}{y_i} \leq \max\limits_{i \in I_-(y)} \frac{-x_i}{y_i} = v_y(x), \quad \forall \ i \in I_-(y),$$

and

$$\frac{-x'_i}{y_i} \leq \max\limits_{i \in I_-(y)} \frac{-x'_i}{y_i} = v_y(x'), \quad \forall \ i \in I_-(y).$$

Hence,

$$v_y(x + x') = \max\limits_{i \in I_-(y)} \frac{-(x_i + x'_i)}{y_i} \leq v_y(x) + v_y(x').$$

This, together with the positively homogenity of v_y implies that the function v_y is convex. $\qquad\square$

Theorem 3.1. *A function $p : \mathbb{R}^n_- \longrightarrow [-\infty, 0]$ is IPH if and only if*

$$p(x) \leq -v_y(x)p(y), \quad \forall \ x, \ y \in \mathbb{R}^n_-. \tag{3.4}$$

Theorem 3.2. *(1) A function $p : \mathbb{R}^n_- \longrightarrow [-\infty, \ 0]$ is IPH if and only if*

$$p(x) = \inf\{v_y(x) : \ p(y) \leq -1\}.$$

(2) Let $x^0 \in \mathbb{R}^n_-$ be a vector such that $-\infty < p(x^0) < 0$, and $y := \frac{x^0}{-p(x^0)}$. Then, $0 \geq v_y(x) \geq p(x)$ for all $x \in \mathbb{R}^n_-$, and $v_y(x^0) = p(x^0)$.

Corollary 3.1. *Every IPH function $p : \mathbb{R}^n_- \longrightarrow [-\infty, \ 0]$ is upper semi-continuous.*

Proof. It is an immediate consequence of Lemma 3.1 and Theorem 3.2(1). $\qquad\square$

Remark 3.1. Assume that $H := \{v_y : y \in \mathbb{R}^n_- \setminus \{0\}\}$, and $p : \mathbb{R}^n_- \longrightarrow (-\infty, 0]$ is an IPH function. Consider the point $x^0 \in domp$ such that $-\infty < p(x^0) < 0$, and let $y := \frac{x^0}{-p(x^0)}$. Then, by Theorem 3.2(2), we have $v_y \in \partial^+_H p(x^0)$.

In the sequel, we will use the vector $e^A_m := (0, \cdots, 0, \frac{-1}{a_m}, 0, \cdots, 0) \in \mathbb{R}^n_-$, where $A := (a_1, a_2, \ldots, a_n)$, $0 < a_m \le 1$ for each $m \in I$. Note that $e^A_m \in S_A$ for all $m \in I$.

Clearly, $I_-(e^A_m) = \{m\}$, and for the vector $y := \frac{e^A_m}{-p(e^A_m)}$ we have $v_y(x) = \frac{-1}{a_m} x_m p(e^A_m)$ for each $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_-$.

4. Algorithm

We now present an algorithm for the search for a global maximizer of a finite valued IPH function p over S_A. Recall that a finite valued IPH function p defined on $\mathbb{R}^n$ is non-positive valued, because $p(x) \le p(0) = 0$ for all $x \in \mathbb{R}^n_-$. We assume that $p(x) < 0$ for all $x \in S_A$. It follows from the non-positivity of p that $I_-(y) = I_-(x)$ for all $x \in S_A$, and $y = \frac{x}{-p(x)}$.

Algorithm 1

Step **0:** (initialization)
a) Take points $x^m := e^A_m$ for $m = 1, \cdots, n$, and construct the basis vectors $y^m := \frac{x^m}{-p(x^m)}$ $(m = 1, \cdots, n)$.
b) Define the function $h_n(x) := \min_{j \le n} v_{y^j}(x) = \min_{j \le n} \frac{-1}{a_j} x_j p(e^A_j)$, $\quad x = (x_1, \cdots, x_n) \in S_A$.
c) Set $k := n$.
Step **1:** Find $x^* := arg[\max_{x \in S_A} h_k(x)]$.
Step **2:** Set $k := k + 1$, and $x^k := x^*$.

Step **3:** Compute $y^k := \frac{x^k}{-p(x^k)}$. Define the function

$$
\begin{aligned}
h_k(x) \quad &:= \min_{j \leq k} v_{y^j}(x) \\
&= \min(h_{k-1}(x), \max_{i \in I_-(y^k)} \frac{-x_i}{y_i^k}) \\
&= \min_{j \leq k} \max_{i \in I_-(y^j)} \frac{-x_i}{y_i^j}, \quad x = (x_1, \cdots, x_n) \in S_A.
\end{aligned}
$$

Go to Step **1**.

The Algorithm **1** can be considered as a version of the cutting angle method (see [7]). Let

$$
\lambda_k = \max_{x \in S_A} h_k(x). \tag{4.1}
$$

It follows from Theorem 3.2(2) that

$$
v_{y^j}(x) \geq p(x) \quad \forall\, x \in S_A, \quad j = 1, \cdots, k. \tag{4.2}
$$

Hence

$$
h_k(x) \geq p(x) \quad \forall\, x \in S_A, \tag{4.3}
$$

and

$$
\lambda_k = \max_{x \in S_A} h_k(x) \geq \max_{x \in S_A} p(x). \tag{4.4}
$$

Thus, λ_k is a upper estimate of the global maximum $p^* = \max_{x \in S_A} p(x)$.

Proposition 4.1. Let $k \geq n + 1$. Then $y_i^i \leq y_i^k$ for all $i \in I$.

Proof. In view of (4.3) and the definition of h_k the result follows. $\quad\square$

It is worth noting (by Theorem 5.1, below) that the vectors y^k and x^k belong to $\mathbb{R}^n_{--}$.

Proposition 4.2. Let $k \geq n + 1$. If $y_i^i \geq y_i^k$ for some $i \in I$, where $y^i = \frac{e_i^A}{-p(e_i^A)}$ $(i = 1, \cdots n)$ and $y^k = \frac{x^k}{-p(x^k)}$, then x^k is a global maximizer of the function p over S_A.

Proof. It is obvious. $\quad\square$

Remark 4.1. Let $k \geq n + 1$. If there exists $i \in I$ such that $y_i^i = y_i^k$, then x^k is a global maximizer of the function p over S_A.

Proposition 4.3. Let $k \geq n + 1$. If there exists $n + 1 \leq j < k + 1$ such that $y^{k+1} \leq y^j$, then x^{k+1} is a global maximizer of the function p over S_A.

Proof. This is an immediate consequence of (4.4). □

We will now show that the sequence $\{x^k\}_{k \geq 1}$ generated by Algorithm **1** converges to a global maximizer of IPH function p over S_A. The proof of the convergency of this algorithm differs from the one of given in [1] for non-negative valued IPH functions on $\mathbb{R}_+^n$. In order to prove the convergency of this algorithm, we first state and prove some results.

Proposition 4.4. Let X and H be as in Definition 2.1. Assume that $h \in H$ is an H-supergradient of a function $f : X \longrightarrow [-\infty, 0]$ at a point $x_0 \in X$. If h has a global maximum at the point $x_0 \in X$, then the function f has a global maximum at the point $x_0 \in X$.

Proof. It is obvious. □

Proposition 4.5. Let X and H be as in Definition 2.1. Assume that $A \subseteq X$ and for each $a \in A$, h_a is an H-supergradient of the function $f : X \longrightarrow [-\infty, 0]$ at the point $a \in A$ such that $h_a(a) = f(a)$. Then the function ψ_A is an H-supergradient of the function f on the set A, that is,

$$\psi_A(x) - \psi_A(a) \geq f(x) - f(a), \quad \forall\, x \in X, \forall\, a \in A,$$

where $\psi_A(x) := \inf_{a \in A} h_a(x)$ for all $x \in X$.

Proof. We have $h_a(x) - h_a(a) \geq f(x) - f(a)$ for all $x \in X$ and all $a \in A$. Since $h_a(a) = f(a)$, we get $h_a(x) \geq f(x)$ for all $x \in X$ and all $a \in A$. Then, $\psi_A(x) = \inf_{a \in A} h_a(x) \geq f(x)$ for all $x \in X$. Let $a_0 \in A$ be arbitrary. We have

$$f(a_0) \leq \psi_A(a_0) = \inf_{a \in A} h_a(a_0) \leq h_{a_0}(a_0) = f(a_0).$$

Therefore, since $a_0 \in A$ was arbitrary, we deduce that $f(a) = \psi_A(a)$ for all $a \in A$. Hence, $\psi_A(x) - \psi_A(a) \geq f(x) - f(a)$ for all $x \in A$ and all $a \in X$, which completes the proof. □

In the following, we assume that X is a metric space and

$$H := \{h : X \longrightarrow (-\infty, 0] : h \ is \ a \ function\} \qquad (4.5)$$

is a set of finite valued functions defined on X.

Proposition 4.6. *Let X and H be as in (4.5). Let $A \subseteq X$ and $f : X \longrightarrow [-\infty, 0]$ be an upper semi-continuous function. Assume that $h \in H$ is a lower semi-continuous function such that h is an H-supergradient of the function f on the set A. Then h is an H-supergradient of the function f on the closure of the set A.*

Proof. Let $\hat{a} \in \bar{A}$ and $\varepsilon > 0$ be arbitrary. Since the functions f and h are upper semi-continuous and lower semi-continuous, respectively, it follows that there exists a neighborhood V of $\hat{a}$ such that

$$f(x) \leq f(\hat{a}) + \varepsilon, \quad \forall \ x \in V, \qquad (4.6)$$

and

$$h(\hat{a}) \leq h(x) + \varepsilon, \quad \forall \ x \in V. \qquad (4.7)$$

Let $a_0 \in A \cap V$ (such a_0 exists, because $\hat{a}$ belongs to $\bar{A}$). Since h is an H-supergradient of the function f on the set A, we have

$$h(x) - h(a_0) \geq f(x) - f(a_0), \quad \forall \ x \in X.$$

This, together with (4.6) and (4.7) implies that

$$\begin{aligned}
h(x) - h(\hat{a}) \quad &\geq h(x) - h(a_0) + h(a_0) - h(\hat{a}) \\
&\geq h(x) - h(a_0) - \varepsilon \\
&\geq f(x) - f(a_0) - \varepsilon \\
&\geq f(x) - f(\hat{a}) - 2\varepsilon, \quad \forall \ x \in X.
\end{aligned}$$

Hence, since $\hat{a} \in \bar{A}$ and $\varepsilon > 0$ were arbitrary, we deduce that h is an H-supergradient of the function f on the set $\bar{A}$. $\square$

The following two propositions have been proved in [5] and [4], respectively.

Proposition 4.7. *([5], Theorem 10.6). Let $\{f_i : \mathbb{R}^n_- \longrightarrow \mathbb{R} : i \in I\}$ be a point-wise bounded collection of real valued convex functions defined on $\mathbb{R}^n_-$. Let D be any compact subset of $\mathbb{R}^n_-$. Then the collection $\{f_i\}_{i \in I}$ is equicontinuous and uniformly bounded on D.*

Proposition 4.8. *([4], Proposition 9.1.4). Let X be a metric space and let $\{f_i : X \longrightarrow \mathbb{R} : i \in I\}$ be a family of real valued equicontinuous functions defined on X. Let $\mathcal{A}$ be a family of subsets of the index set I. Then the family $\{\psi_A\}_{A \in \mathcal{A}}$ is equicontinuous, where $\psi_A(x) := \inf_{t \in A} f_t(x)$ for all $x \in X$ and all $A \in \mathcal{A}$.*

Remark 4.2. Note that by Lemma 3.1 and Theorem 3.2 we have the sequence $\{v_{y^i}\}_{i \geq 1}$ in Algorithm **1** is a point-wise bounded sequence of real valued convex functions defined on $\mathbb{R}^n_-$. Hence, by Proposition 4.7 we conclude that the sequence $\{v_{y^i}\}_{i \geq 1}$ is equicontinuous and uniformly bounded on S_A.

Theorem 4.1. *Let X be a metric space and $\{h_n\}_{n \geq 1}$ be a sequence of real valued equicontinuous functions defined on the compact set $D \subset X$. Let $\psi(x) := \inf\{h_1(x), h_2(x), \cdots\}$ and $\psi_k(x) := \min\{h_1(x), h_2(x), \cdots, h_k(x)\}$ $(x \in D,\ k \in \mathbb{N})$. Assume that x_k is a maximizer of the function ψ_k on D $(k = 1, 2, \cdots)$. If the sequence $\{x_k\}_{k \geq 1}$ has a limit point $x^* \in D$, then x^* is a maximizer of the function $\psi(x) = \inf_{k \geq 1} \psi_k(x)$ $(x \in D)$.*

Proof. Since x_k is a maximizer of the function ψ_k $(k = 1, 2, \cdots)$, we conclude from the definition of ψ_k that

$$\psi_k(x_k) \geq \psi_k(x_{k+1}) \geq \psi_{k+1}(x_{k+1}), \quad \forall\, k \in \mathbb{N}.$$

Thus the sequence $\{\psi_k(x_k)\}_{k \geq 1}$ is decreasing. Also, observe that for every $x \in D$ and every $k \in \mathbb{N}$, we have

$$\psi_k(x_k) \geq \psi_k(x) \geq \psi(x).$$

Then the decreasing sequence $\{\psi_k(x_k)\}_{k \geq 1}$ is bounded from below, and so it is convergent. Moreover,

$$\lim_{k \longrightarrow +\infty} \psi_k(x_k) \geq \sup_{x \in D} \psi(x) \geq \psi(x^*). \tag{4.8}$$

We will now show that $\psi(x^*) = \lim_{k \longrightarrow +\infty} \psi_k(x_k)$. Let $\varepsilon > 0$ be given. In view of Proposition 4.8 and the equicontinuity of the sequence $\{h_n\}_{n \geq 1}$ on D, we get the sequence $\{\psi_k\}_{k \geq 1}$ is equicontinuous. Then there exists $\delta > 0$ such that

$$\forall\, x,\ y \in D \text{ with } d(x, y) < \delta \implies |\psi_k(x) - \psi_k(y)| < \frac{\varepsilon}{2}, \quad \forall\, k \geq 1. \tag{4.9}$$

Since x^* is a limit point of the sequence $\{x_k\}_{k\geq 1}$, then there exists a subsequence $\{x_{k_n}\}_{n\geq 1}$ of $\{x_k\}_{k\geq 1}$ such that $x_{k_n} \longrightarrow x^*$. Thus, by (4.9) we have

$$|\psi_{k_n}(x_{k_n}) - \psi_{k_n}(x^*)| < \frac{\varepsilon}{2} \quad \text{for sufficiently large } k_n. \qquad (4.10)$$

On the other hand, since by the definition of ψ and ψ_k we have $\psi_{k_n}(x^*) \longrightarrow \psi(x^*)$, it follows that

$$|\psi_{k_n}(x^*) - \psi(x^*)| < \frac{\varepsilon}{2} \quad \text{for sufficiently large } k_n. \qquad (4.11)$$

Now, in view of (4.10) and (4.11), we get $\psi_{k_n}(x_{k_n}) \longrightarrow \psi(x^*)$. Since

$$\{\psi_{k_n}(x_{k_n})\}_{n\geq 1}$$

is a subsequence of the convergent sequence $\{\psi_k(x_k)\}$, we conclude that

$$\psi_k(x_k) \longrightarrow \psi(x^*).$$

Finally, it follows from (4.8) that $\psi(x^*) = \max\limits_{x\in D} \psi(x)$. □

Now, in the following we prove the convergency of Algorithm **1**.

The proof of the convergency of Algorithm 1: By Lemma 3.1, we have each function v_{y^i} $(i = 1, 2, \cdots)$ is continuous, convex, IPH and finite valued. In view of Remark 4.2 we observe that the sequence $\{v_{y^i}\}_{i\geq 1}$ is equicontinuous and uniformly bounded on S_A, and so the sequence $\{h_k\}_{k\geq 1}$ in Algorithm 1 is uniformly bounded on S_A. Moreover, we conclude from Proposition 4.8 that the sequence $\{h_k\}_{k\geq 1}$ is equicontinuous. Thus, by [[6], Theorem 25.7] there exists a subsequence of $\{h_k\}_{k\geq 1}$, which is uniformly convergent on S_A. Say, converges to the function h on S_A. Then the function h is continuous on S_A. On the other hand, we have $v_{y^i} \in \partial_H^+ p(x^i)$ and $v_{y^i}(x^i) = p(x^i)$ for all $i \geq 1$, where $H := \{h,\ h_k,\ v_{y^i} : i,\ k \in \mathbb{N}\}$ and by the definition of the function h_k, it is easy to see $h_k \in \partial_H^+ p(x^k)$ and $h_k(x^k) = p(x^k)$ for all $k \geq 1$. Then, by Proposition 4.5 we deduce that $h \in \partial_H^+ p$ on the set $B := \{x^i : i \in \mathbb{N}\}$. Since h is continuous on S_A and the function p is upper semi-continuous on S_A, we conclude from Proposition 4.6 that $h \in \partial_H^+ p$ on the closure $\bar{B}$ of B. Hence, $h \in \partial_H^+ p(x^*)$, where $x^* \in S_A$ is a limit point of the sequence $\{x^i\}_{i\geq 1}$ (note that S_A is a compact set, and so the sequence $\{x^i\}_{i\geq 1}$ has always a limit point $x^* \in S_A$). Because of $h(x) = \inf\limits_{k\geq 1} h_k(x)$ for all $x \in S_A$ and x^k is a maximizer of the function h_k on S_A, in view of Theorem 4.1 we have $h(x^*) = \max\limits_{x\in S_A} h(x)$. Therefore, by Proposition 4.4 we get $p(x^*) = \max\limits_{x\in S_A} p(x)$. This completes the proof. □

5. Auxiliary Problem

Step 1 of Algorithm **1** (that is, finding the global maximum of the function h_k on the set S_A) is the most difficult part of Algorithm **1**. This problem can be represented in the following form:

$$\max \ h_k(x) \ \ subject \ to \ \ x \in S_A, \tag{5.1}$$

where

$$h_k(x) := \min_{j \le k} (\max_{i \in I_-(y^j)} \frac{-x_i}{y_i^j}), \quad k \ge n; \quad x = (x_1, \cdots, x_n) \in S_A, \tag{5.2}$$

$y^j := \frac{x^j}{-p(x^j)}$ $(j = 1, \cdots, k)$, and $x^j := e_j^A$ for $j = 1, \cdots, n$.

Theorem 5.1. Let $k > n$, $y^j := \frac{e_j^A}{-p(e_j^A)}$ $(j = 1, \cdots, n)$, and $y^j << 0$ for all $j = n+1, \cdots, k$. Then each local maximizer of the function h_k, defined by (5.2) over S_A, is a strictly negative vector.

Proof. It is sufficient to show that for each non-strictly negative vector $x \in S_A$, and for each $\varepsilon > 0$ there exist $x' \in S_A$ such that $x' << 0$, $||x' - x|| < \varepsilon$ and $h_k(x') > h_k(x)$. For this end, let $x = (x_1, \cdots, x_n) \in S_A$ be an arbitrary non-strictly negative vector and $\varepsilon > 0$ be given. Then, $I_0(x)$ is non-empty, where $I_0(x) = I \setminus I_-(x) = \{i \in I : x_i = 0\}$. Let us calculate the function v_{y^j} which defined by (3.2) at the point x. We have

$$v_{y^j}(x) = \frac{-1}{a_j} x_j p(e_j^A), \quad j = 1, \cdots, n, \tag{5.3}$$

where $A = (a_1, a_2, \ldots, a_n)$, $0 < a_j \le 1$, $j \in I$, and, in particular, we get

$$v_{y^j}(x) = 0 \ \ \text{for each } j \in I_0(x) \ (j = 1, \cdots, n). \tag{5.4}$$

We also have

$$v_{y^j}(x) = \max_{i \in I} \frac{-x_i}{y_i^j} = 0, \quad j = n+1, \cdots, k. \tag{5.5}$$

Therefore, it follows from (5.3), (5.4) and (5.5) that $v_{y^j}(x) < 0$ if and only if $j \le n$ and $j \notin I_0(x)$, that is, $j \in I_-(x)$. Hence,

$$h_k(x) = \min_{j \le k} v_{y^j}(x) = \min_{j \in I_-(x)} v_{y^j}(x) = \min_{j \in I_-(x)} \frac{-1}{a_j} x_j p(e_j^A). \tag{5.6}$$

Let $m = |I_0(x)|$. Thus, $m < n$. Choose $\varepsilon' > 0$ such that $0 < \frac{1}{a_i}\varepsilon' < \varepsilon$, $x_i + \frac{1}{a_i}\varepsilon' < 0$ for all $i \in I_-(x)$, and $\frac{n-m}{m}\varepsilon' < \varepsilon$, where $A = (a_1, a_2, \ldots, a_n)$, $0 < a_i \leq 1$, $\forall\, i \in I$.

Now, define the point $x(\varepsilon')$ by

$$x(\varepsilon')_i := \begin{cases} x_i + \frac{1}{a_i}\varepsilon', & i \notin I_0(x), \\[2mm] \frac{n-m}{m}(-\frac{1}{a_i}\varepsilon'), & i \in I_0(x). \end{cases}$$

Then it is clear that $x(\varepsilon') << 0$. Since $x \in S_A$, one has

$$\sum_{i=1}^{n} a_i x(\epsilon')_i$$

$$= \sum_{i \notin I_0(x)} a_i(x_i + \frac{1}{a_i}\epsilon') + \sum_{i \in I_0(x)} a_i(\frac{n-m}{m}(-\frac{1}{a_i}\epsilon'))$$

$$= \sum_{i \notin I_0(x)} a_i x_i + (n-m)\epsilon' - (n-m)\epsilon'$$

$$= \sum_{i \notin I_0(x)} a_i x_i$$

$$= \sum_{i \notin I_0(x)} a_i x_i + \sum_{i \in I_0(x)} a_i x_i$$

$$= \sum_{i=1}^{n} a_i x_i = -1.$$

Hence, $x(\varepsilon') \in S_A$. Also, we have

$$\|x - x(\varepsilon')\| = \max_{1 \leq i \leq n} |x_i - x(\varepsilon')_i| = \max\{\frac{1}{a_i}\varepsilon', \ \frac{n-m}{m}\frac{1}{a_i}\varepsilon'\} < \varepsilon.$$

Now, we calculate $h_k(x(\varepsilon')) = \min_{j \leq k} v_{y^j}(x(\varepsilon'))$. For $j \geq n+1$ and sufficiently small ε', we have

$$v_{y^j}(x(\varepsilon')) = \max_{i \in I} \frac{-x(\varepsilon')_i}{y_i^j} = \frac{n-m}{m}\varepsilon' \max_{i \in I_0(x)} \frac{1}{a_i}\frac{1}{y_i^j}. \tag{5.7}$$

Also, for $j \in I_0(x)$, we deduce that

$$v_{y^j}(x(\varepsilon')) = -\frac{1}{a_j}x(\varepsilon')_j\, p(e_j^A) = \frac{1}{a_j^2}\frac{n-m}{m}\varepsilon' p(e_j^A). \tag{5.8}$$

Finally, for $j \in I_-(x)$, we get

$$v_{y^j}(x(\varepsilon')) = -\frac{1}{a_j}(x_j + \frac{1}{a_j}\varepsilon')p(e_j^A). \qquad (5.9)$$

Therefore, for sufficiently small $\varepsilon' > 0$, it follows from (5.7), (5.8) and (5.9) that

$$h_k(x(\varepsilon')) = \min_{j \leq k} v_{y^j}(x(\varepsilon')) = \min_{j \in I_-(x)} v_{y^j}(x(\varepsilon'))$$

$$= \min_{j \in I_-(x)}[-\frac{1}{a_j}(x_j + \frac{1}{a_j}\varepsilon')p(e_j^A)]. \qquad (5.10)$$

Consequently, in view of (5.6) and (5.10) we conclude that $h_k(x(\varepsilon')) > h_k(x)$, which completes the proof. $\qquad\square$

Corollary 5.1. *Let $\{x^k\}_{k\geq 1}$ be a sequence generated by Algorithm **1**. Then, $x^k << 0$ for all $k > n$, and hence $y^k << 0$ for all $k > n$.*

Proof. By induction on k, the result follows from Theorem 5.1. $\qquad\square$

It is well-known that the functions v_{y^k} and h (we omit the index k for the sake of simplicity) are directionally differentiable. In order to show it, we can use the well-known results related to the directional derivative of the functions:

$$f(x) := \max_{j \in J} g_j(x), \quad \text{and} \quad g(x) := \min_{j \in J} g_j(x), \quad x \in \mathbb{R}^n,$$

where J is a finite set (see, for example, [2], Corollary 3.2). Now, let

$$R(x) := \{k : v_{y^k}(x) = h(x)\}, \quad \text{and} \quad Q_k(x) := \{i \in I_-(y^k) : v_{y^k}(x) = \frac{-x_i}{y_i^k}\}.$$

We need to introduce some well-known definitions about point-to-set mappings. Consider a point-to-set mapping f defined on $\mathbb{R}_-^n$ which associates a subset of $\mathbb{N}$ to each point of $\mathbb{R}_-^n$. The mapping f is closed at a point x, if the relations

$$x_k \longrightarrow x, \quad y_k \longrightarrow y \text{ and } y_k \in f(x_k) \ (k = 1, 2, \cdots)$$

imply $y \in f(x)$. If f is closed at each point $x \in \mathbb{R}_-^n$, then we say that f is closed on $\mathbb{R}_-^n$.

Lemma 5.1. *The point-to-set mappings $Q_k : \mathbb{R}_-^n \longrightarrow 2^I$ and $R : \mathbb{R}_-^n \longrightarrow 2^{\mathbb{N}}$ defined by*

$$Q_k(x) := \{i \in I_-(y^k) : v_{y^k}(x) = \frac{-x_i}{y_i^k}\} \text{ and } R(x) := \{k \in \mathbb{N} : v_{y^k}(x) = h(x)\}$$

are closed mappings.

Proof. It is obvious. □

In the following proposition, we consider the vectors $u \in \mathbb{R}^n$ such that there exists $\delta > 0$ with $x + \delta u \in \mathbb{R}^n_-$.

Proposition 5.1. *Consider the functions v_{y^k} and h defined by (3.2) and (5.2), respectively. Then, for each $x \in \mathbb{R}^n_-$, one has*

$$v'_{y^k}(x, u) = \max_{i \in Q_k(x)} \frac{-u_i}{y_i^k},$$

and

$$h'(x, u) = \min_{k \in R(x)} v'_{y^k}(x, u) = \min_{k \in R(x)} \max_{i \in Q_k(x)} \frac{-u_i}{y_i^k}.$$

Proof. See [2]. □

In the sequel, we consider the relative interior $\text{ri}\, S_A$ of S_A, which is given as follows:

$$\text{ri}\, S_A = \{x \in S_A : \ x_i < 0 \ for \ all \ i \in I\}.$$

Now, let $x \in S_A$. The cone

$$K(x, S_A) := \{u \in \mathbb{R}^n : \ \exists \ \alpha_0 > 0 \ \ such \ that \ \ x + \alpha u \in S_A, \ \forall \ \alpha \in (0, \alpha_0)\}$$

is called the tangent cone at the point x with respect to S_A.

The following necessary condition for a local maximum is well-known (see, for example, [2]), and therefore we omit its proof.

Proposition 5.2. *Let $x \in S_A$ be a local maximizer of the function h over S_A. Then, $h'(x, u) \leq 0$ for all $u \in K(x, S_A)$.*

Proposition 5.3. *Let $x \in \text{ri}\, S_A$. Then*

$$K(x, S_A) = \{u \in \mathbb{R}^n : \ \sum_{i \in I} a_i u_i = 0\},$$

where $A = (a_1, a_2, \ldots, a_n)$, $0 < a_i \leq 1$, $\forall\, i \in I$.

Proof. It is an immediate consequence of the definition of the tangent cone. □

Applying Proposition 5.1, Proposition 5.2 and Proposition 5.3, we can obtain the following result:

Theorem 5.2. Let $x << 0$ be a local maximizer of the function h_k over the set $ri\, S_A$ such that $h_k(x) < 0$. Then there exists a subset $\{y^{j_1}, \cdots, y^{j_n}\}$ of the set $\{y^1, \cdots, y^k\}$ such that:

(1) $x = (y_1^{j_1}, \cdots, y_n^{j_n})d_A$, where

$$d_A := -h_k(x) = \frac{-1}{\sum\limits_{i \in I} a_i y_i^{j_i}}, \quad \text{and } A = (a_1, a_2, \ldots, a_n),$$

$0 < a_i \le 1, \ \forall\, i \in I.$

(2) $\max\limits_{j \le k} \min\limits_{i \in I_-(y^j)} \dfrac{y_i^{j_i}}{y_i^j} = 1.$

(3) If $j_m \le n$ $(m \in I)$, then $j_m = m$, and if $j_m \ge n+1$, then $y_i^{j_i} < y_i^{j_m}, \ \forall\, i \in I, \ i \ne m.$

Remark 5.1. Consider the set of k vectors $\Lambda^k = \{y^1, ..., y^k\}$ generated by Algorithm **1**. Every local maximizer x of h_k in $ri\, S_A$ corresponds to a combination of n vectors $L = \{y^{j_1}, ..., y^{j_n}\}$ which satisfy the following conditions:

(I) For all $i, \ r \in I, \ i \ne r$, we have $y_i^{j_i} < y_i^{j_r}$.

(II) For each $y^r \in \Lambda^k \setminus L$, there exists $i \in I$ such that $y_i^{j_i} \ge y_i^r$.

To illustrate the above conditions, visualize L as an $n \times n$ matrix, whose rows are $y^{j_1}, y^{j_2}, ..., y^{j_n}$:

$$\begin{pmatrix} y_1^{j_1} & y_2^{j_1} & \cdot & \cdot & y_n^{j_1} \\ y_1^{j_2} & y_2^{j_2} & \cdot & \cdot & y_n^{j_2} \\ \cdot & \cdot & & & \cdot \\ y_1^{j_n} & y_2^{j_n} & \cdot & \cdot & y_n^{j_n} \end{pmatrix}.$$

Condition (I) implies that the diagonal of L is dominated by their columns, and condition (II) implies that the diagonal of L is not dominated by any other vector y^r, not already in L ($diag(L)$ is dominated by y^r, means that $diag(L) < y^r$).

The location of the local maximum x_{max} and its value $d(L) = h_k(x_{max})$ can be found from the diagonal of L:

$$x_{max} = -\frac{diag(L)}{trace(L)},$$

and

$$d(L) = h_k(x_{max}) = \frac{1}{trace(L)}.$$

Recall that the function h_k is continuous on the compact set S_A. In order to find the global maximum of the function h_k at Step **1** of Algorithm **1**, we need to examine all its local maxima, and hence all combinations of L of the n vectors which satisfy the conditions (I) and (II). In view of

$$h_k(x) = \min(h_{k-1}(x), \ v_{y^k}(x)),$$

if we have already computed all combinations of n vectors out of $k - 1$ vectors satisfying the conditions (I) and (II) (i.e. all candidates for local maxima of the auxiliary function $h_{k-1}(x)$), at the previous iteration, we only need to compute those combinations that have been added by aggregation of the last vector y^k, that is, those combinations of L that include vector y^k. Suppose we already know the set V^{k-1} of combinations of $k - 1$ vectors satisfying (I) and (II). We need to update V^{k-1} to V^k (i.e. all possible combinations of n vectors out of k vectors satisfying (I) and (II)). At this stage, two events can take place:

(a) Some of elements of V^{k-1} may be deleted because they fail test (II) (with y^k playing the role of y^r).

(b) New combinations containing y^k may be added to V^k. By [8], Theorem 2, these new combinations containing y^k can be obtained from those that just have been deleted from V^{k-1} because they fail test (II), and the way to do it is to repeat replace each other vector in these deleted combinations with y^k and check condition (I). If it passed, add the new combination to V^k, and otherwise discard it.

Algorithm 2

(Update of the set V^{k-1} to V^k)
Input: the set V^{k-1}; the new vector y^k.
Output: the set V^k.
Step **1:** Set $V^k = \emptyset$.
Step **2:** Test all elements L of V^{k-1} against condition (II), with $y^r = y^k$. Put those L that fail the test into Temp and those that pass into V^k.
Step **3:** For every L in Temp, form n copies of it, and replace row i in the ith copy with y^k. Test condition (I). If test passed, add this modified copy to V^k, otherwise discard it.

Step 4: Calculate $d(L) = \frac{1}{trace(L)}$ for all elements L of V^k and sort V^k with respect to $d(L)$ in ascending order and choose the largest $d := d(L)$.

The cutting angle method frequently calls Algorithm 2 to solve the problem (5.2), it sorts the set V^k for the highest local maximum of $h_k(x)$, evaluate p at this point and adds the newly formed vector to the set Λ^k.

6. A New Version of the Cutting Angle Method

The results of Sections 4 and 5 allow us to give a new version of the cutting angle method for maximization non-positive valued IPH functions over S_A.

Algorithm 3

Step 0:
(a) Evaluate the objective function $p(x)$ in the vertices of S_A and form the matrix $L_{root} = \{y^1, ..., y^n\}$.
(b) Calculate $d_A = -(\sum_{i \in I} a_i y_i^i)^{-1}$, where $A = (a_1, a_2, \ldots, a_n)$, $0 < a_i \leq 1$, $i \in I$.
(c) Set $k = n$, $\Lambda^k = \{y^1, ..., y^n\}$ and $V^k = \{L_{root}\}$.
Step 1:
(a) Select $L = Head(V^k)$ with the biggest d_A (the global maximum of $h_k(x)$ exception case $k = n$).
(b) Form $x^* = -\frac{diag(L)}{trace(L)}$, and evaluate $p_{best} = p(x^*)$.
Step 2: Set $k = k + 1$, form
$y^k = (\frac{x_1^*}{-p(x^*)}, \frac{x_2^*}{-p(x^*)}, \cdots , \frac{x_n^*}{-p(x^*)})$, and
set $\Lambda^k = \Lambda^{k-1} \cup \{y^k\}$.
Test if $y_i^k = y_i^i$ for some $i \in I$, then
print f(x^* is a global maximizer of p) and stop,
else, call Algorithm 2 (V^{k-1}, y^k, V^k).
Step 3: (Stopping Criterion)
If $k < k_{max}$ and $d - p_{best} > \varepsilon$, go to Step **1**.

7. Numerical Experiments

The applicability of our approach was checked by solving a number of test problems with IPH objective functions. In some cases, we can add $-1 = \sum_{i \in I} a_i x_i$ to the objective function to hold the condition $p(x) < 0$ for all $x \in S_A$. The

results of numerical examples with estimates of the precision $\varepsilon = 0.01$ are presented. To describe the results we use the following notations:

- $f = f(x)$ is the objective function;
- k is the number of iterations;
- n is the number of variables.

To carry out numerical experiments, we use the following problems:

Problem 7.1.

$$f = f(x_1,\ x_2) = \min(x_1 + 2x_2,\ 2x_1 + x_2), \qquad (x_1,\ x_2) \in \mathbb{R}^2.$$

Problem 7.2.

$$f = f(x_1,\ x_2,\ x_3) = \sqrt[3]{x_1 x_2 x_3} + \min(x_1 + 2x_3,\ 2x_1 + x_2), \quad (x_1,\ x_2,\ x_3) \in \mathbb{R}^3.$$

Problem 7.3.

$$f = \max\{c_i x_i :\ i = 1, 2, \cdots, n\} + \min\{b_j x_j :\ j = 1, 2, \cdots, n\},$$

where $c_i = 2 + 0.5i$ $(i = 1, 2, \cdots, n)$, $b_j = (j + 2)(n - j + 2)$ $(j = 1, 2, \cdots, n)$ and $x_i \in \mathbb{R}$ $(i = 1, ..., n)$.

Comments. Note that the objective functions of three problems are real valued IPH functions. The version of the cutting angle method described in Algorithm 3 was applied for finding the global maximum of these functions over S_A. The execution time of Algorithm 3 for Problem 7.1 and Problem 7.3 was less than 2 seconds. First, note that the optimal solution of Problem 7.1 is the point $(-1/2, -1/2)$ over S_-, which is obtained during 3 iterations within a precision $\varepsilon = 0.01$. The optimal solution of Problem 7.2 is the point $(0,\ -2/3,\ -1/3)$ over S_-. The best found solution of Problem 7.2 over S_- is

$$x_1 = -0.0000, \quad x_2 = -0.6667, \quad x_3 = -0.3333.$$

The optimal solution of Problem 7.1 over S_A is the point (-0.5 , -0.5) for the coefficient matrix $A := (1, 1)$, which is obtained during 3 iterations within a precision $\epsilon = 0.01$ that is equal to optimal solution for this problem over S_-.

Now, we show that when the coefficient matrix $A := (a_1, a_2)$, $0 < a_i \le 1$ $(i = 1, 2)$ approaches to the coefficient matrix $A_0 = (1, 1)$, the global maximum of the Problem 7.1 over S_A approaches to global maximum of this problem

over S_- when we choose $k = 4$ and $\epsilon = 0.01$. The optimal solutions of this problem over S_A are listed in the following table.

$A = (a_1, a_2)$	$p(x^*)$	x^*
$(0.3, 0.7)$	-2.9630	$(-0.7407, -1.1111)$
$(0.4, 0.8)$	-2.5000	$(-0.6250, -0.9375)$
$(0.5, 0.8)$	-2.3077	$(-0.7692, -0.7692)$
$(0.6, 0.9)$	-2.0000	$(-0.6667, -0.6667)$
$(0.8, 0.8)$	-1.8750	$(-0.6250, -0.6250)$
$(0.8, 0.9)$	-1.7674	$(-0.5882, -0/5882)$
$(0.8, 1.0)$	-1.6667	$(-0.5556, -0.5556)$
$(0.9, 1.0)$	-1.5789	$(-0.5263, -0.5263)$
$(1.0, 1.0)$	-1.5000	$(-0.5000, -0.5000)$

(Table 7.1.1)

For Problem 7.2 we choose $k = 49$ and $\epsilon = 0.01$, we conclude that for $A_0 := (1, 1, 1)$ one has $p(x^*) = -7.7043$ and $x^* = (-0.0002, -0.6665, -0.3330)$ over S_-. The optimal solutions of this problem over S_A are listed in the following table.

$A = (a_1, a_2, a_3)$	$p(x^*)$	x^*
$(0.3, 0.5, 0.8)$	-1.2430	$(-0.0004, -1.1763, -0.5882)$
$(0.4, 0.5.0.8)$	-1.1739	$(-0.0004, -1.1109, -0.5555)$
$(0.5, 0.6, 0.8)$	-1.0565	$(-0.0004. -0.9999, -0.4999)$
$(0.7, 0.7, 0.9)$	-0.9187	$(-0.0003, -0.8694, -0.4347)$
$(0.9, 0.8, 0.9)$	-0.8452	$(-0.0003, -0.7998, -0.3999)$
$(0.9, 0.9, 1.0)$	-0.7546	$(-0.0003, , -0.7141 - 0.3570)$
$(0.9, 1.0, 1.0)$	-0.7043	$(-0.0002, -0.6665, -0.3333)$
$(1.0, 1.0, 1.0)$	-0.7043	$(-0.0002, -0.6665, -0.3333)$

(Table 7.2.1)

For Problem 7.3, consider two cases.

Case 1: For $n = 2$, $k = 10$, $\epsilon = 0.01$ and $A_0 = (1, 1)$, the optimal solution of this problem over the unit simplex S_- is $p(x^*) = -5.1341$ and $x^* = (-0.3595, -0.6404)$. The optimal solutions of this problem over S_A are

listed in the following table.

$A = (a_1, a_2)$	$p(x^*)$	x^*
$(0.3, 0.7)$	-9.6115	$(-0.5342, -1.1996)$
$(0.5, 0.7)$	-8.3441	$(-0.5415, -1.0418)$
$(0.7, 0.8)$	-6.7794	$(-0.4617, -0.8460)$
$(0.8, 0.9)$	-5.9888	$(-0.4092, -0.7474)$
$(0.9, 1.0)$	-5.3630	$(-0.3740, -0.6693)$
$(1.0, 1.0)$	-5.1341	$(-0.3595, -0.6405)$

(Table 7.3.1)

Case 2: For $n = 3$, $k = 4$, $\epsilon = 0.01$ and $A_0 = (1, 1, 1)$, the optimal solution of this problem over the unit simplex S_- is $p(x^*) = -4.5313$ and $x^* = (-0.3125, -0.3125, -0.3125)$. The optimal solutions of this problem over S_A are listed in the following table.

$A = (a_1, a_2, a_3)$	$p(x^*)$	x^*
$(0.2, 0.6, 0.8)$	-8.2386	$(-0.5682, -0.5682, -0.6818)$
$(0.7, 0.8, 0.9)$	-6.7300	$(-0.4630, -0.4630, -0.5556)$
$(0.7, 0.8, 0.9)$	-5.6202	$(-0.3876, -0.3876, -0.4651)$
$(0.7, 0.8, 1.0)$	-5.3704	$(-0.3330, -0.3333, -0.4444)$
$(0.8, 1.0, 1.0)$	-4.8333	$(-0.3330, -0.3333, -0.4000)$
$(1.0, 1.0, 1.0)$	-4.5313	$(-0.3125, -0.3125, -0.3125)$

(Table 7.3.2)

Conclusions

The numerical results show that the suggested algorithm is effective for finding of the approximate global maximizers for real valued IPH functions over S_A. Given the upper bound on the number of iterations, computing time can be very short. From the above problems and their optimal solutions we conclude that when the coefficient matrix A over S_A approaches to the coefficient matrix A_0 over the unit simplex S_-, the global maxima of the objective functions over S_A approaches to global maxima of these functions over the unit simplex S_-.

Acknowledgments

References

[1] A. Bagirov and A.M. Rubinov, Global minimization of increasing positively homogeneous functions over the unit simplex, *Annals of Operations Research,* **98** (2000), 171-187.

[2] V.F. Demyanov and A.M. Rubinov, *Constructive Nonsmooth Analysis*, Peter Lang, Frankfurt am Main, 1995.

[3] H. Mohebi and H. Sadeghi, Monotonic analysis over ordered topological vector spaces: I, *Optimization,* **56**, No 3 (2007), 305-321.

[4] D. Pallaschke and S. Rolewicz, *Foundations of Mathematical Optimization (Convex Analysis Without Linearity)*, Kluwer Acadamic Publishers, Dordrecht, 1997.

[5] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[6] W. Rudin, *Principles of Mathematical Analysis*, New York, McGraw-Hill, 1964.

[7] A.M. Rubinov, *Abstract Convexity and Global Optimization*, Kluwer Acadamic Publishers, Boston, Dordrecht, London, 2000.

[8] L.M. Batten and G. Beliakov, Fast algorithm for the cutting angle method of global optimization, *Journal of Global Optimization,* **24** (2002), 149-161.