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ON DECAY ESTIMATES OF SOLUTIONS FOR SOME DEGENERATE NONLINEAR KIRCHHOFF STRINGS WITH DISSIPATION

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Abstract: We consider the initial-boundary value problem for the degenerate nonlinear dissipative wave equation of Kirchhoff type:

$$u_{tt} - \left(\int_0^1 |u_x(x,t)|^2 dx \right)^\gamma u_{xx} + u_t + f(u) = 0,$$

where $f(u)$ is like as $|u|^p u$. If the initial energy is appropriately small and $\gamma \geq 1$ and $p > 2\gamma$, then we obtain some optimal time decay estimates of the solution $u = u(t)$.

AMS Subject Classification: 35L80, 35B40, 35L20
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1. Introduction

In this paper we investigate the decay properties of solutions to the initial-boundary value problem for the following degenerate nonlinear dissipative wave equation of Kirchhoff type: for $0 < x < 1$, $0 < t < \infty$,

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$$u_{tt} - \left(\int_0^1 |u_x(x,t)|^2 dx \right)^\gamma u_{xx} + u_t + f(u) = 0\,, \tag{1}$$
$$u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)\,, \quad 0 < x < 1\,,$$
$$u(0,t) = u(1,t) = 0\,, \quad 0 < t < \infty\,,$$

where $u = u(x,t)$ is an unknown real value function and $\gamma \geq 1$ and $f(u)$ is a C^1-function satisfying

$$0 \leq F(u) \leq k_0 f(u) u\,, \quad F(u) \equiv \int_0^u f(\eta)\, d\eta\,, \tag{2}$$
$$|f(u)| \leq k_1 |u|^{p+1}\,, \quad 0 \leq f'(u) \leq k_2 |u|^p\,, \quad p > 0$$

with positive constants k_0, k_1, $k_2 > 0$.

Equation (1) describes small amplitude vibrations of an elastic string (see Kirchhoff [6] for the original equation).

Through this paper, we will use the following energy $E(u, u_t)$ and functional $H(u, u_t)$ associated with (1):

$$E(u, u_t) \equiv \frac{1}{2}\|u_t\|^2 + \frac{1}{2(\gamma+1)}\|u_x\|^{2(\gamma+1)} + \int_\Omega F(u)\, dx\,, \tag{3}$$
$$H(u, u_t) \equiv \frac{\|u_{xt}\|^2}{\|u_x\|^{2\gamma}} + \|u_{xx}\|^2\,, \tag{4}$$

where the symbol $\|\cdot\|$ is the usual norm of $L^2 = L^2(\Omega)$ with $\Omega = (0,1)$. We often denote $E(t) \equiv E(u(t), u_t(t))$ and $H(t) \equiv H(u(t), u_t(t))$ for simplicity. In particular, we will use the following notations related with the initial data $\{u_0, u_1\}$:

$$E(0) \equiv \frac{1}{2}\|u_1\|^2 + \frac{1}{2(\gamma+1)}\|u_{0,\,x}\|^{2(\gamma+1)} + \int_\Omega F(u_0)\, dx\,, \tag{5}$$
$$H(0) \equiv \frac{\|u_{1,\,x}\|^2}{\|u_{0,\,x}\|^{2\gamma}} + \|u_{0,\,xx}\|^2\,. \tag{6}$$

When the initial data belong to usual Sobolev spaces, Arosio & Garabaldi [1] have studied on the unique local weak solutions for the Kirchhoff type wave equations (also see [2], [3] and the references cited therein).

In the non-degenerate case, Hosoya and Yamada [5] have proved global existence theorem and they have derived that the energy has an exponential decay estimate under some small data conditions.

On the other hand, in the degenerate case and $f(u) \equiv 0$, many authors have studied the global existence theorem and the decay estimates of solutions (see [9], [11] and the references cited therein).

In the previous paper [10], we have shown the existence of the unique global solution of (1) and we have derived the decay estimate of the energy $E(t)$ associated with (1).

Theorem 1.1. *Suppose that*

$$\gamma \geq 1 \quad \text{and} \quad p+1 > 2\gamma \tag{7}$$

and the initial data $\{u_0, u_1\}$ *belong to* $H^2 \cap H_0^1 \times H_0^1$ *with* $u_0 \neq 0$ *and* $\{u_0, u_1\}$ *are appropriately small in the sense of* (19). *Then, the problem* (1) *admits a unique global solution* $u(t)$ *in the class* $C^0([0,\infty); H^2 \cap H_0^1) \cap C^1([0,\infty); H_0^1) \cap C^2([0,\infty); L^2)$, $\|u_x(t)\| \neq 0$, *and the energy* $E(t)$ *satisfies* $E(t) \leq C(1+t)^{-1-\frac{1}{\gamma}}$, *that is,*

$$0 < \|u_x(t)\|^2 \leq C(1+t)^{-\frac{1}{\gamma}}\,, \tag{8}$$

$$\|u_t(t)\|^2 \leq C(1+t)^{-1-\frac{1}{\gamma}} \qquad \text{for} \quad t \geq 0\,, \tag{9}$$

where C *is some positive constant.*

The purpose of this paper is to derive the decay estimates which are more detailed the previous estimates (8) and (9). In particular, we will show the lower decay estimates of $\|u_x(t)\|$ and $\|u_{xx}(t)\|$, and we will improve the decay estimate of $\|u_t(t)\|$. In order to obtain these estimates, we use the new identity (33) associated with the H^2-norm of the solution $u(t)$ and the function $\alpha(t)$ given by (29).

Our main result is as follows.

Theorem 1.2. *Suppose that the assumption of Theorem 1.1 is fulfilled and*

$$\gamma \geq 1 \quad \text{and} \quad p > 2\gamma \tag{10}$$

and the initial energy $E(0)$ *is appropriately small in the sense of* (37). *Then, the solution* $u(t)$ *of* (1) *satisfies that*

$$C^{-1}(1+t)^{-\frac{1}{\gamma}} \leq \|u_x(t)\|^2\,,\ \|u_{xx}(t)\|^2 \leq C(1+t)^{-\frac{1}{\gamma}}\,, \tag{11}$$

$$\|u_t(t)\|^2 \leq C(1+t)^{-2-\frac{1}{\gamma}} \qquad \text{for} \quad t \geq 0\,, \tag{12}$$

where C *is some positive constant.*

The notations we use in this paper are standard. The symbol $(\cdot,\cdot)$ means the inner product in $L^2 = L^2(\Omega)$, $\Omega = (0,1)$, or sometimes duality between the space X and its dual X'. We denote the Sobolev-Poincaré constant by c_*, that is, $\|v\|_p \le c_*\|v_x\|$ for $1 \le p \le \infty$, where $\|\cdot\|_p$ is the usual L^p-norm ($\|\cdot\| = \|\cdot\|_2$ if $p = 2$). We denote $(a)^+ = \max\{0, a\}$.

2. Preliminaries

By standard arguments, we have the following local existence theorem (see [1], [3], [10] and the references cited therein). We omit the proof.

Proposition 2.1. *Suppose that the initial data* $\{u_0, u_1\}$ *belong to* $H^2 \cap H_0^1 \times H_0^1$ *and* $u_0 \neq 0$. *Then, the problem* (1) *admits a unique local solution* $u(t)$ *in the class* $C([0,T); H^2 \cap H_0^1) \cap C^1([0,T); H_0^1) \cap C^2([0,T); L^2)$ *for some* $T \equiv T(\|u_0\|_{H^2}, \|u_1\|_{H^1}) > 0$. *Moreover, if* $\|u_x(t)\| > 0$ *and* $\|u(t)\|_{H^2} + \|u_t(t)\|_{H^1} < \infty$ *for* $t \ge 0$, *then we can take* $T = \infty$.

In what follows, we denote $\|u_x(t)\|^2$ by $M(t)$ for simplicity. From the definition (3) of the energy $E(t) \equiv E(u(t), u_t(t))$, it is easy to see that

$$M(t) \equiv \|u_x(t)\|^2 \le (2(\gamma+1)E(t))^{\frac{1}{\gamma+1}} \,. \tag{13}$$

By simple calculation, we see the energy $E(t)$ has the energy identity

$$\frac{d}{dt}E(t) + \|u_t(t)\|^2 = 0 \tag{14}$$

or

$$E(t) + \int_0^t \|u_t(s)\|^2 \, ds = E(0) \,. \tag{15}$$

Indeed, multiplying (1) by u_t and integrating over Ω or $\Omega \times (0,t)$, we obtain (14) or (15). Moreover, applying energy method together with the Nakao's inequality (see citeNa78 and [8]), we have the following decay estimate of the energy $E(t)$ (see [7] and [10] for the proof).

Proposition 2.2. *Let* $u(t)$ *be a solution of* (1). *Then the energy* $E(t)$ *satisfies*

$$E(t) \le \left(E(0)^{-\frac{\gamma}{\gamma+1}} + d_1^{-1}(E(0)^{\frac{\gamma}{\gamma+1}} + 1)^{-1}(t-1)^+\right)^{-\frac{\gamma+1}{\gamma}} \tag{16}$$

for $t \ge 0$, *where* d_1 *is some positive constant.*

Moreover, we see immediately that the inequality (13) and the energy decay (16) yield the following estimates.

Corollary 2.3. *Let $u(t)$ be a solution of* (1) *and* $E(0) \le 1$.
(i) *If $q > \gamma$, then*

$$\int_0^t M(s)^q \, ds \le d_2 E(0)^{\frac{q-\gamma}{\gamma+1}} . \tag{17}$$

(ii) *If $q > \gamma/2$, then*

$$\int_0^t (1+s)^{-\frac{1}{2}} M(s)^q \, ds \le d_3 E(0)^{\frac{1}{\gamma+1}(q-\frac{\gamma}{2})} , \tag{18}$$

where d_2 and d_3 are some positive constants.

3. Global Existence

In this section we will prove Theorem 1.1.

Proposition 3.1. *Let $u(t)$ be a solution and $M(t) \equiv \|u_x(t)\|^2 > 0$. Suppose that $\gamma \ge 1$ and $p+1 > 2\gamma$ and $u_0 \ne 0$ and*

$$I(0) \equiv 2\left(2(\gamma+1)E(0)\right)^{\frac{\gamma-1}{2(\gamma+1)}} \left(H(0) + d_4 E(0)^{\frac{p+1-2\gamma}{\gamma+1}}\right)^{\frac{1}{2}} < \frac{2}{\gamma+2} , \tag{19}$$

where d_4 is some positive constant given by (27)*. Then, it holds that*

$$H(t) \le H(0) + d_3 E(0)^{\frac{p+1-2\gamma}{\gamma+1}} , \tag{20}$$

$$\frac{|M'(t)|}{M(t)} \le I(0) < \frac{2}{\gamma+2} . \tag{21}$$

Proof. Multiplying (1) by $(-2u_{xxt}/M(t)^\gamma)$ and integrating it over Ω, we have

$$\frac{d}{dt} H(t) + 2\left(1 + \frac{\gamma}{2}\frac{M'(t)}{M(t)}\right) \frac{\|u_{xt}(t)\|^2}{M(t)^\gamma} = -\frac{2}{M(t)^\gamma}(f(u), u_{xxt})$$

$$\le \frac{2k_2}{M(t)^\gamma}\|u(t)\|_\infty^p \|u_x(t)\| \|u_{xt}(t)\| \le 2k_2 c_*^p \left(M(t)^{p+1-\gamma}\frac{\|u_{xt}(t)\|^2}{M(t)^\gamma}\right)^{\frac{1}{2}} . \tag{22}$$

We observe from (4) and (15) that

$$\frac{|M'(t)|}{M(t)} \le 2\frac{\|u_{xt}(t)\|}{\|u_x(t)\|} = 2\left(M(t)^{\gamma-1}\frac{\|u_{xt}(t)\|^2}{M(t)^{\gamma}}\right)^{\frac{1}{2}} \tag{23}$$

$$\le 2\left(2(\gamma+1)E(0)\right)^{\frac{\gamma-1}{2(\gamma+1)}} H(t)^{\frac{1}{2}} . \tag{24}$$

In what follows, we may assume that $E(0) \le 1$.

Under the assumption (19), putting

$$T \equiv \sup\{t \mid 2\left(2(\gamma+1)E(0)\right)^{\frac{\gamma-1}{2(\gamma+1)}} H(s)^{\frac{1}{2}} < \frac{2}{\gamma+2}\, ,\, 0 \le s < t\}\, , \tag{25}$$

then we see that $T > 0$.

If $T < \infty$, then

$$2\left(2(\gamma+1)E(0)\right)^{\frac{\gamma-1}{2(\gamma+1)}} H(T)^{\frac{1}{2}} = \frac{2}{\gamma+2}\, . \tag{26}$$

For $0 \le t \le T$, we observe from (23)–(26) that

$$1 + \frac{\gamma}{2}\frac{|M'(t)|}{M(t)} \ge 1 - \frac{\gamma}{2}\frac{2}{\gamma+2} = \frac{2}{\gamma+2}\, ,$$

and from (22) that

$$\frac{d}{dt}H(t) + \frac{4}{\gamma+2}\frac{\|u_{xt}(t)\|^2}{M(t)^{\gamma}} \le 2k_2c_*^p\left(M(t)^{p+1-\gamma}\frac{\|u_{xt}(t)\|^2}{M(t)^{\gamma}}\right)^{\frac{1}{2}}\, ,$$

and from the Young inequality that

$$\frac{d}{dt}H(t) \le \frac{\gamma+2}{4}(k_2c_*^p)^2 M(t)^{p+1-\gamma}\, .$$

Moreover, if $p+1 > 2\gamma$, we have from (17) in Corollary 2.3 that

$$H(t) \le H(0) + d_4 E(0)^{\frac{p+1-2\gamma}{\gamma+1}}\, , \qquad d_4 = \frac{\gamma+2}{4}(k_2c_*^p)^2 d_2\, . \tag{27}$$

Then, we have from (19) that

$$2\left(2(\gamma+1)E(0)\right)^{\frac{\gamma-1}{2(\gamma+1)}} H(t)^{\frac{1}{2}} \le I(0) < \frac{2}{\gamma+2} \qquad \text{for} \quad 0 \le t < T\, , \tag{28}$$

which is a contradiction to (26), and hence, we see that $T = \infty$, and we conclude that (27) and (28) hold true for $t \ge 0$. □

Proof of Theorem 1.1. By $u_0 \neq 0$, we see $M(0) > 0$. If there exists $T > 0$ such that $M(t) > 0$ for $0 \le t < T$ and $M(T) = 0$, then since $H(t) \le C < \infty$ by (20), we have that $\lim_{t\to T} \|u_{xt}(t)\| = 0$, and hence, we see that $\{u(T), u_t(T)\} = \{0, 0\}$.

On the other hand, by the backward uniqueness to (1) with $\{u(T), u_t(T)\} = \{0, 0\}$ (see [9] and [10]), we observe that $u \equiv 0$ on $[0, T]$, which is a contradiction to the assumption $u_0 \neq 0$. Thus, we conclude that $M(t) > 0$ for $t \ge 0$, and moreover, from Proposition 3.1 we obtain the a-priori estimate $\|u(t)\|_{H^2} + \|u_t(t)\|_{H^1} \le C < \infty$ for $t \ge 0$. Therefore, the local solution $u(t)$ of (1) in the sense of Proposition 2.1 can be continued globally in time. Also, from Proposition 2.2 we obtain the decay estimates (7) and (8) for $t \ge 0$. □

4. Proof of Theorem 1.2

In this section, let $u(t)$ be the global solution of (1) given by Theorem 1.1. In order to get the sharp decay estimates of $\|u(t)\|_{H^2}$ and $\|u_t(t)\|$, we use the following function $\alpha(t)$ defined by

$$\alpha(t) \equiv \sup_{0\le s\le t} \left\{ (1+s)\frac{\|u_t(s)\|^2}{M(s)^{\gamma+1}} \right\} . \tag{29}$$

Proposition 4.1. *Suppose that $p > 2\gamma$. Then, it holds that*

$$\frac{\|u_{xx}(t)\|^2}{M(t)} \le G(t) \le 2G(0) + d_5 E(0)^{\frac{p-\gamma}{\gamma+1}} \alpha(t) , \tag{30}$$

where d_5 is some positive constant given by (35) and

$$G(t) \equiv \frac{\|u_{xx}(t)\|^2}{M(t)} + \frac{2}{M(t)^{\gamma+2}}((f(u))_x, u_x) + Q(t) \quad (\ge 0) , \tag{31}$$

$$Q(t) \equiv \frac{1}{M(t)^{\gamma+2}} \left(M(t)\|u_{xx}(t)\|^2 - \left(\frac{1}{2}M'(t)\right)^2 \right) \quad (\ge 0) . \tag{32}$$

Proof. Since we observe from (1) that

$$\begin{aligned}
M(t)^\gamma \frac{d}{dt}\|u_{xx}(t)\|^2 &= 2\left(M(t)^\gamma u_{xx}, u_{xxt}\right) \\
&= -2\|u_{xt}(t)\|^2 - 2(u_{xtt}, u_{xt}) + 2\frac{d}{dt}(f(u), u_{xx}) - 2((f(u))_t, u_{xx})\,, \\
M(t)^\gamma \|u_{xx}(t)\|^2 &= 2(M(t)^\gamma u_{xx}, u_{xx}) \\
&= -\frac{1}{2}M'(t) + \|u_{xt}(t)\|^2 - \frac{1}{2}M''(t) + (f(u), u_{xx})\,,
\end{aligned}$$

we have

$$\frac{d}{dt}\left(\frac{\|u_{xx}(t)\|^2}{M(t)} + \frac{2}{M(t)^{\gamma+1}}((f(u))_x, u_x)\right) = -2Q(t) - R(t) + S(t)\,, \tag{33}$$

where $Q(t)$ is given by (32) and

$$\begin{aligned}
R(t) &\equiv \frac{1}{M(t)^{\gamma+2}}\left(M(t)\frac{d}{dt}\|u_{xt}(t)\|^2 + M'(t)\left(\|u_{xt}(t)\|^2 - \frac{1}{2}M''(t)\right)\right)\,, \\
S(t) &\equiv \frac{1}{M(t)^{\gamma+2}}\left((2\gamma+1)M'(t)(f(u), u_{xx}) - 2M(t)((f(u))_t, u_{xx})\right)\,.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{d}{dt}Q(t) &= -(\gamma+2)\frac{M'(t)}{M(t)}Q(t) \\
&\quad + \frac{1}{M(t)^{\gamma+2}}\left(M(t)\frac{d}{dt}\|u_{xt}(t)\|^2 + M'(t)\|u_{xt}(t)\|^2 - \frac{1}{2}M'(t)M''(t)\right) \\
&= -(\gamma+2)\frac{M'(t)}{M(t)}Q(t) + R(t)\,.
\end{aligned} \tag{34}$$

Adding (33) to (34), we have

$$\frac{d}{dt}G(t) = -2\left(1 + \frac{\gamma+2}{2}\frac{M'(t)}{M(t)}\right)Q(t) + S(t)\,.$$

Moreover, since we observe from (21) that

$$1 + \frac{\gamma+2}{2}\frac{|M'(t)|}{M(t)} \geq 0\,,\ \ Q(t) \geq 0\,,\ \ ((f(u))_x, u_x) = \int_\Omega f'(u)|u_x|^2\,dx \geq 0\,,$$

and

$$|S(t)| \le \frac{2(2\gamma+1)k_2}{M(t)^{\gamma+2}}\|u_t(t)\|\|u_{xx}(t)\|\|u(t)\|_\infty^p\|u_x(t)\|^2 + \frac{2k_2}{M(t)^{\gamma+1}}\|u(t)\|_\infty^p\|u_t(t)\|\|u_{xx}(t)\| \le c_2\left(\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}}\frac{\|u_{xx}(t)\|^2}{M(t)}M(t)^{p-\gamma}\right)^{\frac{1}{2}}$$

with $c_2 = 4(\gamma+1)k_2c_*^p$, we have

$$\frac{d}{dt}G(t) \le c_2\left(\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}}G(t)M(t)^{p-\gamma}\right)^{\frac{1}{2}}$$

or

$$2\frac{d}{dt}G(t)^{\frac{1}{2}} \le c_2\left((1+t)^{-1}M(t)^{p-\gamma}\right)^{\frac{1}{2}}\left((1+t)\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}}\right)^{\frac{1}{2}}.$$

Thus, we obtain from (29) and (18) that if $p > 2\gamma$,

$$2G(t)^{\frac{1}{2}} \le 2G(0)^{\frac{1}{2}} + c_2\alpha(t)^{\frac{1}{2}}\int_0^t(1+s)^{-\frac{1}{2}}M(s)^{\frac{p-\gamma}{2}}\,ds \le 2G(0)^{\frac{1}{2}} + c_2d_2E(0)^{\frac{1}{\gamma+1}(\frac{p}{2}-\gamma)}\alpha(t)^{\frac{1}{2}},$$

and hence,

$$G(t) \le 2G(0) + d_5E(0)^{\frac{p-2\gamma}{\gamma+1}}\alpha(t), \qquad d_5 = \frac{c_2^2d_2^2}{2}, \tag{35}$$

which implies the desired estimate (30). □

Proposition 4.2. *Suppose that $p > 2\gamma$ and $E(0)$ is appropriate small like as (37). Then, it holds that*

$$\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} \le C(1+t)^{-1}, \qquad \|u_t(t)\|^2 \le C(1+t)^{-2-\frac{1}{\gamma}}. \tag{36}$$

Proof. Multiplying (1) by $(2u_t/M(t)^{\gamma+1})$ and integrating it over Ω, we have

$$
\begin{aligned}
&\frac{d}{dt}\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} + 2\left(1+\frac{\gamma+1}{2}\frac{M'(t)}{M(t)}\right)\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} \\
&= -\frac{M'(t)}{M(t)} - \frac{2}{M(t)^{\gamma+1}}(f(u), u_t) \\
&\leq 2\frac{\|u_t(t)\|}{M(t)^{\frac{\gamma+1}{2}}}\frac{\|u_{xx}(t)\|}{M(t)^{\frac{1}{2}}}M(t)^{\frac{\gamma}{2}} + 2c_*^{p+1}k_1\frac{\|u_t(t)\|}{M(t)^{\frac{\gamma+1}{2}}}M(t)^{\frac{p-\gamma}{2}},
\end{aligned}
$$

and from (21) that

$$
\begin{aligned}
&\frac{d}{dt}\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} + \frac{2}{\gamma+2}\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} \\
&\quad \leq 2\frac{\|u_t(t)\|}{M(t)^{\frac{\gamma+1}{2}}}\left(\frac{\|u_{xx}(t)\|}{M(t)^{\frac{1}{2}}} + c_*^{p+1}k_1 M(t)^{\frac{p-2\gamma}{2}}\right)M(t)^{\frac{\gamma}{2}},
\end{aligned}
$$

and from the Young inequality and (15) and (30) that

$$
\begin{aligned}
&\frac{d}{dt}\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} + b\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} \\
&\quad \leq 2(\gamma+2)\left(\frac{\|u_{xx}(t)\|^2}{M(t)} + c_*^{2(p+1)}k_1^2 M(t)^{p-2\gamma}\right)M(t)^{\gamma} \\
&\quad \leq \left(c_3 + 2(\gamma+2)d_5 E(0)^{\frac{p-2\gamma}{\gamma+1}}\alpha(t)\right)M(t)^{\gamma}
\end{aligned}
$$

with $b = 1/(\gamma+2)$ and $c_3 = 2(\gamma+2)(2G(0) + c_*^{2(p+1)}k_1^2(2(\gamma+1)E(0))^{\frac{p-2\gamma}{\gamma+1}})$, where we used (15) and (30) at the last inequality. We observe that

$$
\begin{aligned}
&\int_0^t e^{-b(t-s)}M(s)^{\gamma}\,ds = \left(\int_0^{t/2} + \int_{t/2}^t\right)e^{-b(t-s)}M(s)^{\gamma}\,ds \\
&\quad \leq e^{-\frac{b}{2}t}\,(2(\gamma+1)E(0))^{\frac{\gamma}{\gamma+1}}\,\frac{t}{2} + \frac{1}{b}\left(2(\gamma+1)E(\frac{t}{2})\right)^{\frac{\gamma}{\gamma+1}} \\
&\quad \leq d_6(1+t)^{-1},
\end{aligned}
$$

where d_6 is some positive constant independent of $E(0)$ when $E(0) \leq 1$. Thus, we have

$$
\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} \leq \frac{\|u_1\|^2}{M(0)^{\gamma+1}}e^{-bt} + d_6\left(c_3 + 2(\gamma+2)d_5 E(0)^{\frac{p-2\gamma}{\gamma+1}}\alpha(t)\right)(1+t)^{-1}
$$

and

$$\alpha(t) \le c_4 \frac{\|u_1\|^2}{M(0)^{\gamma+1}} + d_6 \left(c_3 + 2(\gamma+2) d_5 E(0)^{\frac{p-2\gamma}{\gamma+1}} \alpha(t) \right)$$

with $c_4 = \sup_{t\ge 0}(1+t)e^{-bt}$. If $E(0)$ is small like as

$$2(\gamma+2) d_5 d_6 E(0)^{\frac{p-2\gamma}{\gamma+1}} < 1, \tag{37}$$

then we obtain

$$\alpha(t) \le C \qquad \text{or} \qquad \frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} \le C(1+t)^{-1}$$

which gives the desired estimate (36) with (8). □

As a corollary of Proposition 4.1 and Proposition 4.2, we obtain the following estimate.

Proposition 4.3. *Under the assumption of Proposition 4.2, it holds that*

$$\frac{\|u_{xx}(t)\|^2}{M(t)} \le C\,, \qquad \|u_{xx}(t)\|^2 \le C(1+t)^{-\frac{1}{\gamma}}\,. \tag{38}$$

We will derive the lower decay estimate of the function $M(t)$.

Proposition 4.4. *Under the assumption of Proposition 4.2, it holds that*

$$M(t) \equiv \|u_x(t)\|^2 \ge C'(1+t)^{-\frac{1}{\gamma}} \tag{39}$$

with some positive constant C'.

Proof. Multiplying (1) by $(2u_t/M(t)^{\gamma+2})$ and integrating it over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\|u_t(t)\|^2}{M(t)^{\gamma+2}} + \frac{1}{M(t)} \right) + 2\left(1 + \frac{\gamma+2}{2}\frac{M'(t)}{M(t)}\right) \frac{\|u_t(t)\|^2}{M(t)^{\gamma+2}} \\
&= -2\frac{M'(t)}{M(t)^2} - \frac{2}{M(t)^{\gamma+2}}(f(u), u_t) \\
&\le 2\left(\frac{\|u_t(t)\|^2}{M(t)^{\gamma+2}} \right)^{\frac{1}{2}} \left(\frac{\|u_{xx}(t)\|^2}{M(t)} M(t)^{\gamma-1} \right)^{\frac{1}{2}} \\
&\quad + 2c_*^{p+2} \left(\frac{\|u_t(t)\|^2}{M(t)^{\gamma+2}} \right)^{\frac{1}{2}} \left(M(t)^{p-2\gamma} M(t)^{\gamma-1} \right)^{\frac{1}{2}}
\end{aligned}$$

and from (21) and the Young inequality that

$$\frac{d}{dt}\left(\frac{\|u_t(t)\|^2}{M(t)^{\gamma+2}}+\frac{1}{M(t)}\right)\le C\frac{\|u_{xx}(t)\|^2}{M(t)}M(t)^{\gamma-1}+CM(t)^{p-2\gamma}M(t)^{\gamma-1}$$
$$\le CM(t)^{\gamma-1}\le C(1+t)^{-1+\frac{1}{\gamma}},$$

where we used the estimates (16) and (38) at the last inequality.

Thus, we obtain

$$\frac{\|u_t(t)\|^2}{M(t)^{\gamma+2}}+\frac{1}{M(t)}\le C+C\int_0^t(1+s)^{-1+\frac{1}{\gamma}}\,ds\le C(1+t)^{\frac{1}{\gamma}},$$

and hence, we see $M(t)\ge C'(1+t)^{-\frac{1}{\gamma}}$ for $t\ge 0$. □

Proof of Theorem 1.2. The upper decay estimate (11) follows from (38) in Proposition 4.3. The lower decay estimate 11 follows from (39) in Proposition 4.4. The sharp decay estimate (12) follows from (36) in Proposition 4.2.

The proof of Theorem 1.2 is now completed. □

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