

EXISTENCE AND NONEXISTENCE RESULTS FOR  
A CLASS OF HAMILTONIAN ELLIPTIC  
SYSTEMS WITH WEIGHTS

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**Abstract:** In this paper we investigate the existence and non existence of solutions to the following singular semilinear elliptic system:

$$\begin{cases} -\Delta u = \lambda v + \frac{v^p}{|x|^\alpha} & \text{in } \Omega \\ -\Delta v = \lambda u + \frac{u^q}{|x|^\beta} & \text{in } \Omega \\ u \geq 0, v \geq 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^N$  ( $N \geq 3$ ),  $0 \in \Omega$ ,  $\lambda > 0$ ,  $0 < \alpha, \beta < N$  and  $p, q > 1$  satisfy the condition  $\frac{N-\alpha}{p+1} + \frac{N-\beta}{q+1} > N - 2$ . The existence of a nontrivial solution is obtained by variational methods.

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## 1. Introduction

In this paper we are concerned with the following system of singular elliptic

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equations

$$\begin{cases} -\Delta u = \lambda v + \frac{v^p}{|x|^\alpha} & \text{in } \Omega \\ -\Delta v = \lambda u + \frac{u^q}{|x|^\beta} & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^N$  ( $N \geq 3$ ),  $0 \in \Omega$ ,  $\lambda > 0$ ,  $0 < \alpha, \beta < N$  and  $p, q > 1$  satisfy the condition  $\frac{N-\alpha}{p+1} + \frac{N-\beta}{q+1} > N-2$ .

Hulshof, Mitidieri and Van der Vorst [8] (see also [7]) considered the problem (1) in the case  $\alpha = \beta = 0$ , and proved the existence of at least one nontrivial solution with positive components  $(u, v) \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2$ . A nonexistence result is obtained in [11] by Mitidieri.

Recently, Figueiredo, Peral and Rossi [4] treated the problem (1) where  $\lambda = 0$  and proved the following result:

Let us assume that  $p, q, \alpha, \beta$  verify

$$\frac{N-\alpha}{p+1} + \frac{N-\beta}{q+1} > N-2, \quad (2)$$

$$\frac{1}{p+1} + \frac{1}{q+1} < 1 \quad (3)$$

and

$$q+1 < \frac{2(N-\beta)}{N-4} \quad \text{and} \quad p+1 < \frac{2(N-\alpha)}{N-4} \quad \text{if } N \geq 5. \quad (4)$$

Then, there exist infinitely many strong solutions and at least one positive strong solution of (1)

In the present paper, we shall prove that if (2) holds, the system (1) has at least one positive strong solution for any  $\lambda \in (0, \lambda_0)$  and has no positive solutions for any  $\lambda \geq \lambda_1$ , where  $\lambda_0$  is a positive constant to be specified later, and  $\lambda_1$  is the first eigenvalue of  $-\Delta$ .

The main result of the paper is stated in the following theorem:

**Theorem 1.1.** *Let  $0 < \alpha, \beta < N$  and  $p, q > 1$  satisfy*

$$\frac{N-\alpha}{p+1} + \frac{N-\beta}{q+1} > N-2.$$

*Then, there exists  $\lambda_0 > 0$  such that for all  $\lambda \in (0, \lambda_0)$ , problem (1) has at least one positive strong solution.*

**Theorem 1.2.** *Under the hypothesis of Theorem 1.1, problem (1) has no nontrivial positive solutions for any  $\lambda > \lambda_1$ .*

The rest of the paper is organized as follows: in Section 2 we recall some preliminaries results. In Section 3 we give the proof of Theorem 1.1 and in Section 4 we give a non existence result.

## 2. Preliminaries

Our approach is based on a linking theorem due to Li and Willem [10]. Let  $E$  be a Hilbert space and  $I : E \rightarrow \mathbb{R}$  be a strongly indefinite functional near zero in the sense that there exist two subspaces  $E^+$  and  $E^-$  with  $E = E^+ \oplus E^-$  such that the functional  $I$  is positive definite on  $E^+$  and negative definite on  $E^-$  (near zero). We assume also that there are sequences of subspaces of finite dimensions  $E_n^\pm$  such that

$$E_1^\pm \subset E_2^\pm \subset E_3^\pm \dots, \quad \text{and} \quad \overline{\bigcup_{n=1}^\infty E_n^\pm} = E^\pm.$$

Denote

$$E_n = E_n^+ \oplus E_n^-, \quad \text{and} \quad I_n = I|_{E_n}.$$

We have

$$E_1 \subset E_2 \subset E_3 \dots, \quad \text{and} \quad \overline{\bigcup_{n=1}^\infty E_n} = E.$$

**Definition 2.1.** We say that  $I$  satisfies the  $(PS^*)$  condition with respect to the scale of subspaces  $(E_n)_n$  if every sequence  $(z_n)_n$  such that

$$z_n \in E_n, \quad |I_n(z_n)| \leq C, \quad \left| \left\langle I'_n(z_k), \eta \right\rangle \right| \leq \varepsilon_n \|\eta\|_E \quad \forall \eta \in E_n, \quad \varepsilon_n \rightarrow 0,$$

contains a subsequence which converges to a critical point of  $I$ .

We need the following result of Li and Willem [10]:

**Theorem 2.2.** *Let  $I \in C^1(E, \mathbb{R})$  such that:*

**A1)**  *$I$  has a local linking at the origin, i.e. for some  $r > 0$*

$$I(z) \geq 0 \text{ for } z \in E^+, \text{ and } I(z) \leq 0 \text{ for } z \in E^-, \text{ with } \|z\|_E \leq r,$$

**A2)**  *$I$  maps bounded sets into bounded sets,*

**A3)**  *$I(z) \rightarrow -\infty$  as  $\|z\| \rightarrow \infty$ ,  $z \in E_n^+ \oplus E^-$ , for every  $n \in \mathbb{N}$ ,*

**A4)** *I satisfies the  $(PS^*)$  condition with respect to the scale of subspaces  $(E_n)_n$ .*

*Then, I has a nontrivial critical point.*

Solutions of (1) will be found as critical points of the corresponding functional in suitable spaces obtained us the domains of fractional powers of the Laplace operator [2]. For this purpose, let  $A^s = (-\Delta)^{s/2}$  for  $0 \leq s \leq 2$ , and let  $E^s = D\left((-\Delta)^{s/2}\right)$ .  $E^s$  is a Hilbert space with the inner product

$$\langle u, v \rangle_{E^s} = \int_{\Omega} A^s u A^s v \, dx.$$

Its associated norm is denoted by  $\|u\|_{E^s}$ . The Poincaré's inequality for the operator  $A^s$  is

$$\|A^s u\|_{L^2(\Omega)} \geq \lambda_1^{s/2} \|u\|_{L^2(\Omega)} \quad \text{for all } u \in E^s. \quad (5)$$

The Sobolev embedding theorem for spaces  $E^s$  says that

$$E^s \hookrightarrow L^r(\Omega) \quad \text{if } \frac{1}{r} \geq \frac{1}{2} - \frac{s}{N},$$

and the embedding is compact if the previous inequality is strict.

We recall the following result from [4]:

**Proposition 2.3.** *Let  $q > 1$ ,  $\beta > 0$  and  $s > 0$  such that*

$$q + 1 < \frac{2(N - \beta)}{N - 2s}. \quad (6)$$

*Then, the inclusion map  $i : E^s \rightarrow L^{q+1}\left(\Omega, |x|^{-\beta}\right)$  is well defined and compact.*

For numbers  $s > 0$  and  $t > 0$  with  $s + t = 2$  we define the Hilbert space  $E = E^s \times E^t$  and the bilinear form  $B : E \times E \rightarrow \mathbb{R}$  by the formula

$$B((u, v), (\phi, \psi)) = \int_{\Omega} (A^s u A^t \psi + A^s \phi A^t v) \, dx.$$

The form  $B$  is continuous and symmetric and there exists a selfadjoint bounded linear operator  $L : E \rightarrow E$  so that

$$B(z, \eta) = (Lz, \eta)_E \quad \text{for all } z, \eta \in E.$$

Here  $(\cdot, \cdot)_E$  denotes the natural inner product on  $E$  induced by  $E^s$  and  $E^t$ . We can also define the quadratic form  $Q : E \rightarrow \mathbb{R}$  associated to  $B$  and  $L$  as

$$Q(z) = \frac{1}{2} (Lz, \eta)_E = \int_{\Omega} A^s u A^t v \, dx, \text{ for all } (u, v) \in E.$$

Following De Figueiredo and Felmer [2], we can define the subspaces

$$E^+ = \{(u, A^{s-t}u) \mid u \in E^s\}, \quad E^- = \{(u, -A^{s-t}u) \mid u \in E^s\} \quad (7)$$

which give the natural splitting  $E = E^+ \oplus E^-$ . The spaces  $E^+$  and  $E^-$  are the positive and negative eigenspaces of  $L$ , they are consequently orthogonal with respect to the bilinear form  $B$ , and we also have

$$\frac{1}{2} \|z\|_E^2 = Q(z^+) - Q(z^-),$$

where  $z = z^+ + z^-$ ,  $z^+ \in E^+$  and  $z^- \in E^-$ .

**Proposition 2.4.** *Suppose  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ ,  $r > 1$ ,  $\sigma > 0$  and  $u \in L^r(\Omega, |x|^{-\sigma})$ . Then,*

$$\left( \int_{\Omega} |u|^2 \, dx \right)^{\frac{1}{2}} \leq C \left( \int_{\Omega} \frac{|u|^{r+1}}{|x|^{\sigma}} \, dx \right)^{\frac{1}{r+1}}. \quad (8)$$

*Proof.* We have

$$\left( \int_{\Omega} |u|^2 \, dx \right)^{\frac{1}{2}} = \left( \int_{\Omega} \frac{|u|^2}{|x|^{\frac{2\sigma}{r+1}}} |x|^{\frac{2\sigma}{r+1}} \, dx \right)^{\frac{1}{2}}.$$

By the Hölder's inequality, we have

$$\left( \int_{\Omega} |u|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} \frac{|u|^{r+1}}{|x|^{\sigma}} \, dx \right)^{\frac{1}{r+1}} \left( \int_{\Omega} |x|^{\frac{2\sigma}{r-1}} \, dx \right)^{\frac{r-1}{2(r+1)}},$$

since  $N + \frac{2\sigma}{r-1} > 0$ , we get

$$\left( \int_{\Omega} |u|^2 \, dx \right)^{\frac{1}{2}} \leq C \left( \int_{\Omega} \frac{|u|^{r+1}}{|x|^{\sigma}} \, dx \right)^{\frac{1}{r+1}}. \quad \square$$

**Remark 2.5.**  $C$  is a generic positive constant.

### 3. The Existence Result

In order to set up our problem variationally, we shall use fractional Sobolev spaces introduced in the previous section. First, we define the functional associated to the Hamiltonian. We will choose the numbers  $s$  and  $t$  defining the orders of the involved Sobolev spaces. From the fact that  $p, q > 1$  and the inequality (2), we can choose  $s, t > 0$  such that  $s + t = 2$ ,  $s > t$ , and

$$q + 1 < \frac{2(N - \beta)}{N - 2s}, \quad p + 1 < \frac{2(N - \alpha)}{N - 2t} \quad (9)$$

and

$$\frac{1}{q+1} > \frac{1}{2} - \frac{s}{N}, \quad \frac{1}{p+1} > \frac{1}{2} - \frac{t}{N}. \quad (10)$$

These last inequalities and Sobolev embedding theorem yield the compact inclusions:

$$E^s \hookrightarrow L^{q+1}(\Omega, |x|^{-\beta}), \quad E^t \hookrightarrow L^{p+1}(\Omega, |x|^{-\alpha}),$$

and

$$E^s \hookrightarrow L^2(\Omega), \quad E^t \hookrightarrow L^2(\Omega).$$

Second, for  $z = (u, v) \in E$ , we define the functional  $I : E \rightarrow \mathbb{R}$  as

$$\begin{aligned} I(z) &= \int_{\Omega} A^s u A^t v dx - \frac{\lambda}{2} \int_{\Omega} (|u|^2 + |v|^2) dx \\ &\quad - \frac{1}{p+1} \int_{\Omega} \frac{v^{p+1}}{|x|^{\alpha}} dx - \frac{1}{q+1} \int_{\Omega} \frac{u^{q+1}}{|x|^{\beta}} dx. \end{aligned} \quad (11)$$

The functional  $I$  is of class  $C^1$  and

$$\begin{aligned} (I'(z), \eta) &= \int_{\Omega} (A^s u A^t \psi + A^s \phi A^t v) dx - \lambda \int_{\Omega} (u\phi + v\psi) dx \\ &\quad - \int_{\Omega} \left( \frac{v^p \psi}{|x|^{\alpha}} + \frac{u^q \phi}{|x|^{\beta}} \right) dx \end{aligned} \quad (12)$$

for  $z = (u, v) \in E$  and  $\eta = (\phi, \psi) \in E$ .

**Definition 3.1.** We say that  $z = (u, v) \in E$  is a  $(s, t)$ -weak solution of (1) if  $z$  is a critical point of  $I$ , i.e. for every  $(\varphi, \psi) \in E$  we have

$$\begin{cases} \int_{\Omega} A^s u A^t \psi dx = \lambda \int_{\Omega} v \psi dx + \int_{\Omega} \left( \frac{v^p \psi}{|x|^{\alpha}} \right) dx \quad \forall \psi \in E^t \\ \int_{\Omega} A^s \varphi A^t v dx = \lambda \int_{\Omega} u \varphi dx + \int_{\Omega} \left( \frac{u^q \varphi}{|x|^{\beta}} \right) dx \quad \forall \varphi \in E^s. \end{cases} \quad (13)$$

We have the following regularity result. It is an adapted version of Theorem 1.2 in [4]:

**Proposition 3.2.** *Suppose that  $(u, v) \in E$  is a weak solution of (1). Then  $u \in W^{2,a}(\Omega)$  and  $v \in W^{2,b}(\Omega)$  for every*

$$1 < a < \frac{2N}{p(N-2t)+2\alpha} \text{ and } 1 < b < \frac{2N}{q(N-2s)+2\beta}.$$

Hence,  $(u, v)$  is in fact a strong solution of (1).

*Proof.* Since  $(u, v) \in E$  is a weak solution, we have for every  $(\varphi, \psi) \in E$ :

$$\int_{\Omega} \left( A^s u A^t \psi + A^s \varphi A^t v - \lambda u \varphi - \lambda v \psi - \frac{v^p \psi}{|x|^\alpha} - \frac{u^q \phi}{|x|^\beta} \right) dx = 0.$$

Put  $\psi = 0$  in this last equality. Then we have

$$\int_{\Omega} \left( A^s \varphi A^t v - \lambda u \varphi - \frac{u^q \varphi}{|x|^\beta} \right) dx = 0 \text{ for all } \varphi \in E^s.$$

If we take  $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ , then we have:

$$\int_{\Omega} A^s \varphi A^t v dx = - \int_{\Omega} \Delta \varphi v dx.$$

Let  $b > 1$  and let  $r$  be such that  $\frac{1}{r} > \frac{N-4s}{2N}$ . By the Hölder inequality we have

$$\begin{aligned} \int_{\Omega} \left| \frac{u^q}{|x|^\beta} \right|^b dx &= \int_{\Omega} \frac{|u|^{qb}}{|x|^{\beta b}} dx \\ &\leq \left( \int_{\Omega} \left( u^{qb} \right)^{\frac{r}{qb}} dx \right)^{\frac{qb}{r}} \left( \int_{\Omega} |x|^{-\frac{\beta br}{r-qb}} dx \right)^{\frac{r-qb}{r}} \\ &\leq C \|u\|_{L^r(\Omega)}^{qb} \text{ iff } \frac{\beta br}{r-qb} < N, \end{aligned}$$

$$\begin{aligned} \frac{\beta br}{r-qb} < N &\implies \frac{\beta br}{N} < r-qb \\ &\implies \frac{\beta b}{N} + \frac{qb}{r} < 1. \end{aligned}$$

This last inequality combined with the inequality  $\frac{1}{r} > \frac{N-4s}{2N}$  yields:

$$1 < b < \frac{2N}{q(N-2s) + 2\beta}.$$

Then,  $\frac{u^q}{|x|^\beta} \in L^b(\Omega)$  if  $1 < b < \frac{2N}{q(N-2s) + 2\beta}$ .

From the basic elliptic theory (see for example [6]), there exists a function  $w \in W^{2,b}(\Omega)$  such that

$$\begin{cases} -\Delta w = \lambda u + \frac{u^q}{|x|^\beta} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

which, by integrating by parts, yields to:

$$\begin{aligned} - \int_{\Omega} \Delta w \varphi \, dx - \int_{\Omega} \left( \lambda u + \frac{u^q}{|x|^\beta} \right) \varphi \, dx &= - \int_{\Omega} \Delta \varphi w \, dx \\ &\quad - \int_{\Omega} \left( \lambda u + \frac{u^q}{|x|^\beta} \right) \varphi \, dx \\ &= 0. \end{aligned}$$

Hence,  $\int_{\Omega} (v - w) \Delta \varphi \, dx = 0$  with  $v = w = 0$  on  $\partial\Omega$ , which means that  $v \equiv w$ .

We conclude then, that  $v \in W^{2,b}(\Omega)$ . The proof for  $u$  is similar.  $\square$

Now, we prove that the assumptions of Theorem 2.2 are satisfied:

**Lemma 3.3.** *Assume (2) and let  $p, q > 1$  and  $\alpha, \beta < N$ . Then there exists a  $\lambda_* > 0$  such that for all  $\lambda \in (0, \lambda_*)$  the functional  $I$  has a local linking at the origin.*

*Proof.* For  $z = (u, v) \in E^+$  we have

$$I(z) = \frac{1}{2} \|z\|_E^2 - \frac{\lambda}{2} \int_{\Omega} (|u|^2 + |v|^2) dx - \frac{1}{p+1} \int_{\Omega} \frac{v^{p+1}}{|x|^\alpha} dx - \frac{1}{q+1} \int_{\Omega} \frac{u^{q+1}}{|x|^\beta} dx.$$

Using Sobolev imbedding and (5), we obtain

$$\begin{aligned} I(z) &\geq \frac{1}{2} \|z\|_E^2 - \frac{\lambda}{2} \left( \frac{1}{\lambda_1^s} \|u\|_{E^s}^2 + \frac{1}{\lambda_1^t} \|v\|_{E^t}^2 \right) - C \left( \|v\|_{E^t}^{p+1} + \|u\|_{E^s}^{q+1} \right) \\ &\geq \left( \frac{1}{2} - \frac{\lambda}{2 \min \{\lambda_1^s, \lambda_1^t\}} \right) \|z\|_E^2 - C \|z\|_E^\theta, \text{ for some } \theta > 2. \end{aligned} \quad (14)$$



Put  $\lambda_* = \min \{\lambda_1^s, \lambda_1^t\}$ , then there is an  $r > 0$  such that  $I(z) \geq 0$  for all  $\lambda \in (0, \lambda_*)$  and  $z \in E^+$  with  $\|z\|_E \leq r$ .

Next, for  $z = (u, v) \in E^-$ , we have

$$\begin{aligned} I(z) &= -\frac{1}{2} \|z\|_E^2 - \frac{\lambda}{2} \int_{\Omega} (|u|^2 + |v|^2) dx \\ &\quad - \frac{1}{p+1} \int_{\Omega} \frac{|v|^{p+1}}{|x|^\alpha} dx - \frac{1}{q+1} \int_{\Omega} \frac{|u|^{q+1}}{|x|^\beta} dx \\ &\leq -\frac{1}{2} \|z\|_E^2. \end{aligned} \quad (15)$$

Hence,  $I(z) \leq 0$  if  $z \in E^-$  and  $\|z\|_E \leq r$ .  $\square$

**Lemma 3.4.**  *$I$  maps bounded sets into bounded sets.*

*Proof.* Let  $B \subset E^s \times E^t$  be a bounded set, i.e. there exists  $C > 0$  such that

$$\|u\|_{E^s} \leq C \quad \text{and} \quad \|v\|_{E^t} \leq C, \quad \text{for all } z = (u, v) \in B. \quad (16)$$

Now, for  $z = (u, v) \in B$  we have

$$\begin{aligned} |I(z)| &\leq \int_{\Omega} |A^s u A^t v| dx + \frac{\lambda}{2} \int_{\Omega} (|u|^2 + |v|^2) dx \\ &\quad + \frac{1}{p+1} \int_{\Omega} \frac{|v|^{p+1}}{|x|^\alpha} dx + \frac{1}{q+1} \int_{\Omega} \frac{|u|^{q+1}}{|x|^\beta} dx. \end{aligned}$$

By the Hölder inequality and the embedding theorem, we obtain

$$\begin{aligned} |I(z)| &\leq \|A^s u\|_{L^2(\Omega)} \|A^t v\|_{L^2(\Omega)} + \frac{\lambda}{2} (\|u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2) \\ &\quad + \frac{1}{p+1} \|v\|_{L^{p+1}(\Omega, |x|^{-\alpha})}^2 + \frac{1}{q+1} \|u\|_{L^{q+1}(\Omega, |x|^{-\beta})}^{q+1} \\ &\leq \|u\|_{E^s} \|v\|_{E^t} + C \left( \|u\|_{E^s} + \|v\|_{E^t} + \|v\|_{E^t}^{p+1} + \|u\|_{E^s}^{q+1} \right). \end{aligned} \quad (17)$$

From (16) and (17), we get

$$|I(z)| \leq C \quad \text{for all } z \in B. \quad \square$$

**Lemma 3.5.** *Let  $n \in \mathbb{N}$  be fixed and let  $z_k \in E_n^+ \oplus E^-$ , where  $E_n^+$  denotes an  $n$ -dimensional subspace of  $E^+$ . Then*

$$I(z_k) \rightarrow -\infty \quad \text{if} \quad \|z_k\|_E \rightarrow \infty.$$

*Proof.* By (7),  $z_k$  may be written as

$$z_k = (u_k, A^{s-t}u_k) + (v_k, -A^{s-t}v_k) \quad \text{with } u_k \in E_n^s \text{ and } v_k \in E_n^s.$$

Thus,

$$\begin{aligned} I(z_k) &= \int_{\Omega} |A^s u_k|^2 dx - \int_{\Omega} |A^s v_k|^2 dx \\ &\quad - \frac{\lambda}{2} \int_{\Omega} (|u_k + v_k|^2 + |A^{s-t}(u_k - v_k)|^2) dx \\ &\quad - \frac{1}{p+1} \int_{\Omega} \frac{|A^{s-t}(u_k - v_k)|^{p+1}}{|x|^\alpha} dx - \frac{1}{q+1} \int_{\Omega} \frac{|u_k + v_k|^{q+1}}{|x|^\beta} dx \\ &= \|u_k\|_{E^s}^2 - \|v_k\|_{E^s}^2 - \frac{\lambda}{2} \int_{\Omega} (|u_k + v_k|^2 + |A^{s-t}(u_k - v_k)|^2) dx \\ &\quad - \frac{1}{p+1} \int_{\Omega} \frac{|A^{s-t}(u_k - v_k)|^{p+1}}{|x|^\alpha} dx - \frac{1}{q+1} \int_{\Omega} \frac{|u_k + v_k|^{q+1}}{|x|^\beta} dx. \end{aligned}$$

Note that

$$\|z_k\|_E \rightarrow \infty \iff \|u_k\|_{E^s}^2 + \|v_k\|_{E^s}^2 \rightarrow \infty. \quad (18)$$

Now, we deduce that:

1) If  $\|u_k\|_{E^s} \leq C$ , then  $\|v_k\|_{E^s} \rightarrow \infty$  and then it is easy to see that  $I(z_k) \rightarrow -\infty$ .

2) If  $\|u_k\|_{E^s} \rightarrow \infty$ , then from (8) we estimate

$$\begin{aligned} \int_{\Omega} \frac{|u_k + v_k|^{q+1}}{|x|^\beta} dx &\geq C \left( \int_{\Omega} |u_k + v_k|^2 dx \right)^{\frac{q+1}{2}} \\ &\geq C \|u_k + v_k\|_{L^2(\Omega)}^{q+1}, \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} \frac{|A^{s-t}(u_k - v_k)|^{p+1}}{|x|^\alpha} dx &\geq C \left( \int_{\Omega} |A^{s-t}(u_k - v_k)|^2 dx \right)^{\frac{p+1}{2}} \\ &\geq C \|A^{s-t}(u_k - v_k)\|_{L^2(\Omega)}^{p+1}, \end{aligned}$$

and since  $s > t$ , by (5) we have

$$\int_{\Omega} \frac{|A^{s-t}(u_k - v_k)|^{p+1}}{|x|^\alpha} dx \geq C \|u_k - v_k\|_{L^2(\Omega)}^{p+1}.$$

Hence, for some  $\theta > 2$ , we obtain the estimate

$$I(z_k) \leq \|u_k\|_{E^s}^2 - C \left( \|u_k + v_k\|_{L^2(\Omega)}^\theta + \|u_k - v_k\|_{L^2(\Omega)}^\theta \right). \quad (19)$$

The function  $\varphi(x) = x^\theta$  is convex, then  $\frac{1}{2}(\varphi(x) + \varphi(y)) \geq \varphi(\frac{1}{2}(x+y))$  and hence

$$\begin{aligned} I(z_k) &\leq \|u_k\|_{E^s}^2 - C \left( \|u_k + v_k\|_{L^2(\Omega)}^\theta + \|u_k - v_k\|_{L^2(\Omega)}^\theta \right)^\theta \\ &\leq \|u_k\|_{E^s}^2 - C \|u_k\|_{L^2(\Omega)}^\theta. \end{aligned}$$

We know that the norms  $\|\cdot\|_{E^s}$  and  $\|\cdot\|_{L^2(\Omega)}$  are equivalent on  $E_n^s$ . Thus, we conclude that also in this case  $I(z_k) \rightarrow -\infty$ .  $\square$

**Lemma 3.6.** *There exists  $\lambda_{**} > 0$  such that for all  $\lambda \in (0, \lambda_{**})$  the functional  $I$  satisfies the  $(PS^*)$  condition.*

*Proof.* Let  $(z_n)$  be a sequence of  $E$  such that

$$z_n \in E_n, \quad |I_n(z_n)| \leq C,$$

and

$$\left| \left\langle I'_n(z_k), \eta \right\rangle \right| \leq \varepsilon_n \|\eta\|_E, \quad \text{for all } \eta \in E_n, \text{ and } \varepsilon_n \rightarrow 0. \quad (20)$$

As we did in [9], and following the spirit of [5, 3], we base our proof on the fact that  $z_n \in E$ . We first show that  $(z_n)$  is uniformly bounded in  $E$ . Taking  $\eta = z_n$  we have for  $z_n = (u_n, v_n) \in E$

$$\begin{aligned} I(z_n) - \frac{1}{2} (I'(z_n), z_n) &= \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\Omega} \frac{|v_n|^{p+1}}{|x|^\alpha} dx \\ &\quad + \left( \frac{1}{2} - \frac{1}{q+1} \right) \int_{\Omega} \frac{|u_n|^{q+1}}{|x|^\beta} dx \\ &\leq C + \varepsilon_n \|z_n\|_E. \end{aligned}$$

Now, since  $p, q > 1$  we have

$$\int_{\Omega} \frac{|v_n|^{p+1}}{|x|^\alpha} dx \leq C + \varepsilon_n (\|u_n\|_{E^s} + \|v_n\|_{E^t}) \quad (21)$$

and

$$\int_{\Omega} \frac{|u_n|^{q+1}}{|x|^\beta} dx \leq C + \varepsilon_n (\|u_n\|_{E^s} + \|v_n\|_{E^t}). \quad (22)$$

Note that  $A^{s-t}(u_n) \in E^t$ . Thus, choosing  $\eta = (0, A^{s-t}(u_n))$  in (20) we get

$$\begin{aligned} \int_{\Omega} |A^s u_n|^2 dx &\leq \lambda \int_{\Omega} |v_n A^{s-t}(u_n)| dx + \int_{\Omega} \frac{|v_n|^p |A^{s-t}(u_n)|}{|x|^\alpha} dx \\ &\quad + \varepsilon_n \|A^{s-t}(u_n)\|_{E^t}, \end{aligned}$$

and hence, by the Hölder inequality

$$\begin{aligned} \|u_n\|_{E^s}^2 &\leq \lambda \|v_n\|_{L^2(\Omega)} \|A^{s-t}(u_n)\|_{L^2(\Omega)} \\ &\quad + \left( \int_{\Omega} \frac{|v_n|^{p+1}}{|x|^\alpha} dx \right)^{\frac{p}{p+1}} \left( \int_{\Omega} \frac{|A^{s-t}(u_n)|^{p+1}}{|x|^\alpha} dx \right)^{\frac{1}{p+1}} + \varepsilon_n \|u_n\|_{E^s}. \end{aligned}$$

Using the Sobolev embedding theorem, (5) and (21) we obtain

$$\begin{aligned} \|u_n\|_{E^s}^2 &\leq \frac{\lambda}{\lambda_1^2} \|v_n\|_{E^t} \|u_n\|_{E^s} + (C + \varepsilon_n (\|u_n\|_{E^s} + \|v_n\|_{E^t}))^{\frac{p}{p+1}} \|u_n\|_{E^s} \\ &\quad + \varepsilon_n \|u_n\|_{E^s}, \end{aligned}$$

and thus

$$\|u_n\|_{E^s} \leq \frac{\lambda}{\lambda_1^2} \|v_n\|_{E^t} + \varepsilon_n (\|u_n\|_{E^s} + \|v_n\|_{E^t})^{\frac{p}{p+1}} + C. \quad (23)$$

Similarly,  $\eta = (A^{t-s}(v_n), 0)$  in (20) we obtain as above

$$\|v_n\|_{E^t} \leq \frac{\lambda}{\lambda_1^2} \|u_n\|_{E^s} + \varepsilon_n (\|u_n\|_{E^s} + \|v_n\|_{E^t})^{\frac{q}{q+1}} + C. \quad (24)$$

Joining (23) and (24), we finally get, for some  $\tau < 1$ ,

$$\left(1 - \frac{\lambda}{\lambda_1^2}\right) (\|u_n\|_{E^s} + \|v_n\|_{E^t}) \leq \varepsilon_n (\|u_n\|_{E^s} + \|v_n\|_{E^t})^\tau + C.$$

Taking  $\lambda_{**} = \lambda_1^2$ , thus  $\|u_n\|_{E^s} + \|v_n\|_{E^t}$  is bounded for all  $\lambda \in (0, \lambda_{**})$ .

Now, we prove that  $(z_n)$  converges strongly in  $E$ . Since  $(z_n) = (u_n, v_n)$  is bounded in  $E = E^s \times E^t$ , there exists a subsequence denoted again by  $(u_n, v_n)$  which converges weakly to  $(u, v)$  in  $E^s \times E^t$ . The mappings  $A^s : E^s \rightarrow L^2(\Omega)$  and  $A^{-t} : L^2(\Omega) \rightarrow E^t$  are continuous isomorphisms, thus we get  $A^s(u_n - u) \rightharpoonup 0$  in  $L^2(\Omega)$  and  $A^{s-t}(u_n - u) \rightharpoonup 0$  in  $E^t$ . Since  $E^t \hookrightarrow L^{p+1}(\Omega, |x|^{-\alpha})$  and  $E^t \hookrightarrow L^2(\Omega)$  compactly, we conclude that  $A^{s-t}(u_n - u) \rightarrow 0$  strongly in

$L^{p+1}(\Omega, |x|^{-\alpha})$  and  $L^2(\Omega)$ . Choosing  $\eta = (0, A^{s-t}(u_n - u)) \in E^s \times E^t$  in (20) we obtain

$$\begin{aligned} \left| \int_{\Omega} (|A^s u_n|^2 - A^s u_n A^s u) dx \right| &\leq \lambda \int_{\Omega} |v_n A^{s-t}(u_n - u)| dx \\ &+ \int_{\Omega} \frac{|v_n|^p |A^{s-t}(u_n - u)|}{|x|^\alpha} dx + \varepsilon_n \|A^{s-t}(u_n - u)\|_{E^t} \\ &\leq \lambda \|v_n\|_2 \|A^{s-t}(u_n - u)\|_2 \\ &+ \|v_n\|_{L^{p+1}(\Omega, |x|^{-\alpha})}^p \|A^{s-t}(u_n - u)\|_{L^{p+1}(\Omega, |x|^{-\alpha})} \\ &+ \varepsilon_n \|(u_n - u)\|_{E^s}. \end{aligned}$$

Observe that the right hand-side of the above inequality converges to 0, thus

$$\int_{\Omega} |A^s u_n|^2 dx \longrightarrow \int_{\Omega} |A^s u|^2 dx.$$

Similarly, we prove that the sequence  $(v_n)$  converges strongly in  $E^t$ .  $\square$

*Proof of Theorem 1.2.* Put  $\lambda_0 = \min \{\lambda_{**}, \lambda_*\}$ . Conditions of Theorem 2.2 are satisfied for all  $\lambda \in (0, \lambda_0)$ . Then, we find a critical point  $(u, v)$  for the functional  $I$  which yields a weak solution, and by Proposition 3.2 we conclude that this solution is strong. Finally, by the maximum principle, it follows that  $u, v$  are strictly positive in  $\Omega$ .  $\square$

#### 4. The Nonexistence Result

Let us now denote by  $\varphi_1$  the first eigenfunction of  $-\Delta$  on  $\Omega$  with Dirichlet boundary conditions.

*Proof of Theorem 1.2.* We argue by contradiction. Suppose that  $(u, v)$  is a nontrivial positive solution of problem (1) if

$$\lambda > \lambda_1. \quad (25)$$

Multiplying equations of the system (1) by  $\varphi_1$  and integrating by part we find

$$\lambda_1 \int_{\Omega} v \varphi_1 dx = - \int_{\Omega} \Delta v \varphi_1 dx = \lambda \int_{\Omega} u \varphi_1 dx + \int_{\Omega} \frac{u^q \varphi_1}{|x|^\beta} dx \quad (26)$$

and

$$\lambda_1 \int_{\Omega} u \varphi_1 dx = - \int_{\Omega} \Delta u \varphi_1 dx = \lambda \int_{\Omega} v \varphi_1 dx + \int_{\Omega} \frac{v^p \varphi_1}{|x|^\alpha} dx. \quad (27)$$

Since  $v > 0$ , from (27) we obtain

$$\int_{\Omega} v \varphi_1 dx \leq \frac{\lambda_1}{\lambda} \int_{\Omega} u \varphi_1 dx. \quad (28)$$

Inserting (28) into (26), we get

$$\lambda \int_{\Omega} u \varphi_1 dx + \int_{\Omega} \frac{u^q \varphi_1}{|x|^{\beta}} dx \leq \frac{\lambda_1^2}{\lambda} \int_{\Omega} u \varphi_1 dx.$$

Therefore,

$$\left( \frac{\lambda_1^2 - \lambda^2}{\lambda} \right) \int_{\Omega} u \varphi_1 dx \geq 0. \quad (29)$$

A contradiction with (25).  $\square$

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